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On homogeneous second order linear general quantum difference equations

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Abstract

In this paper, we prove the existence and uniqueness of solutions of the β-Cauchy problem of second order β-difference equations

$$a_0(t)D_\beta^2 y(t) + a_1(t)D_\beta y(t) + a_2(t)y(t) = b(t), \quad t \in I,$$

$a_0(t) \neq 0$, in a neighborhood of the unique fixed point s_0 of the strictly increasing continuous function β, defined on an interval $I \subseteq \mathbb{R}$. These equations are based on the general quantum difference operator D_β, which is defined by $D_\beta f(t) = (f(\beta(t)) - f(t))/(\beta(t) - t)$, $\beta(t) \neq t$. We also construct a fundamental set of solutions for the second order linear homogeneous β-difference equations when the coefficients are constants and study the different cases of the roots of their characteristic equations. Finally, we drive the Euler-Cauchy β-difference equation.

MSC: 39A10; 39A13; 39A70; 47B39

Keywords: a general quantum difference operator; general quantum difference equations; Euler-Cauchy general quantum difference equation

1 Introduction

Quantum calculus allows us to deal with sets of non-differentiable functions by substituting the classical derivative by a difference operator. Non-differentiable functions are used to describe many important physical phenomena. Quantum calculus has a lot of applications in different mathematical areas such as the calculus of variations, orthogonal polynomials, basic hyper-geometric functions, economical problems with a dynamic nature, quantum mechanics and the theory of scale relativity; see, *e.g.*, [1–9]. The general quantum difference operator D_β is defined, in [10, p.6], by

$$D_\beta f(t) = \begin{cases} \frac{f(\beta(t))-f(t)}{\beta(t)-t}, & t \neq s_0, \\ f'(s_0), & t = s_0, \end{cases}$$

where $f : I \to \mathbb{X}$ is a function defined on an interval $I \subseteq \mathbb{R}$, $\mathbb{X}$ is a Banach space and $\beta : I \to I$ is a strictly increasing continuous function defined on I, which has only one fixed point $s_0 \in I$ and satisfies the inequality: $(t - s_0)(\beta(t) - t) \leq 0$ for all $t \in I$. The function f is said to be β-differentiable on I, if the ordinary derivative f' exists at s_0. The β-difference operator yields the Hahn difference operator when $\beta(t) = qt + \omega$, $\omega > 0$, $q \in (0, 1)$, and the Jackson

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q-difference operator when $\beta(t) = qt$, $q \in (0,1)$; see [11–16]. In [10], [17, Chapter 2], the definition of the β-derivative, the β-integral, the fundamental theorem of β-calculus, the chain rule, Leibniz's formula and the mean value theorem were introduced. In [18], the β-exponential, β-trigonometric and β-hyperbolic functions were presented. In [19], the existence and uniqueness of solutions of the β-initial value problem of the first order were established. In addition, an expansion form of the β-exponential function was deduced.

This paper is devoted for deducing some results of the solutions of the homogeneous second order linear β-difference equations which are based on D_β. In Section 2, we introduce the needed preliminaries of the β-calculus from [10, 17–19]. In Section 3, we prove the existence and uniqueness of solutions of the β-Cauchy problem of second order β-difference equations in a neighborhood of s_0. We also construct a fundamental set of solutions for the second order linear homogeneous β-difference equations when the coefficients are constants and study the different cases of the roots of their characteristic equations. Finally, we drive the Euler-Cauchy β-difference equation. Throughout this paper, J is a neighborhood of the unique fixed point s_0 of β and $\mathbb{X}$ is a Banach space. If f is β-differentiable two times over I, then the second order derivative of f is denoted by $D_\beta^2 f = D_\beta(D_\beta f)$. Furthermore, $S(y_0, b) = \{y \in \mathbb{X} : \|y - y_0\| \leq b\}$ and the rectangle $R = \{(t,y) \in I \times \mathbb{X} : |t - s_0| \leq a, \|y - y_0\| \leq b\}$, where a, b are fixed positive real numbers.

2 Preliminaries

In this section, we present some needed results associated with the β-calculus from [10, 17–19].

Lemma 2.1 *The following statements are true*:

(i) *The sequence of functions* $\{\beta^k(t)\}_{k=0}^{\infty}$ *converges uniformly to the constant function* $\hat{\beta}(t) := s_0$ *on every compact interval* $V \subseteq I$ *containing* s_0.

(ii) *The series* $\sum_{k=0}^{\infty} |\beta^k(t) - \beta^{k+1}(t)|$ *is uniformly convergent to* $|t - s_0|$ *on every compact interval* $V \subseteq I$ *containing* s_0.

Lemma 2.2 *If* $f : I \to \mathbb{X}$ *is a continuous function at* s_0, *then the sequence* $\{f(\beta^k(t))\}_{k=0}^{\infty}$ *converges uniformly to* $f(s_0)$ *on every compact interval* $V \subseteq I$ *containing* s_0.

Theorem 2.3 *If* $f : I \to \mathbb{X}$ *is continuous at* s_0, *then the series* $\sum_{k=0}^{\infty} \|(\beta^k(t) - \beta^{k+1}(t)) \times f(\beta^k(t))\|$ *is uniformly convergent on every compact interval* $V \subseteq I$ *containing* s_0.

Lemma 2.4 *Let* $f : I \to \mathbb{X}$ *be* β-*differentiable and* $D_\beta f(t) = 0$ *for all* $t \in I$. *Then* $f(t) = f(s_0)$ *for all* $t \in I$.

Theorem 2.5 *Assume that* $f : I \to \mathbb{X}$ *and* $g : I \to \mathbb{R}$ *are* β-*differentiable functions on* I. *Then*:

(i) *the product* $fg : I \to \mathbb{X}$ *is* β-*differentiable on* I *and*

$$\begin{aligned} D_\beta(fg)(t) &= \bigl(D_\beta f(t)\bigr)g(t) + f\bigl(\beta(t)\bigr)D_\beta g(t) \\ &= \bigl(D_\beta f(t)\bigr)g\bigl(\beta(t)\bigr) + f(t)D_\beta g(t), \end{aligned}$$

(ii) *f/g is β-differentiable at t and*

$$D_\beta(f/g)(t) = \frac{(D_\beta f(t))g(t) - f(t)D_\beta g(t)}{g(t)g(\beta(t))},$$

provided that $g(t)g(\beta(t)) \neq 0$.

Theorem 2.6 *Assume $f : I \to \mathbb{X}$ is continuous at s_0. The function F defined by*

$$F(t) = \sum_{k=0}^{\infty}\bigl(\beta^k(t) - \beta^{k+1}(t)\bigr)f\bigl(\beta^k(t)\bigr), \quad t \in I \tag{2.1}$$

is a β-antiderivative of f with $F(s_0) = 0$. *Conversely, a β-antiderivative F of f vanishing at s_0 is given by* (2.1).

Definition 2.7 Let $f : I \to \mathbb{X}$ and $a, b \in I$. The β-integral of f from a to b is

$$\int_a^b f(t)\,d_\beta t = \int_{s_0}^b f(t)\,d_\beta t - \int_{s_0}^a f(t)\,d_\beta t,$$

where

$$\int_{s_0}^x f(t)\,d_\beta t = \sum_{k=0}^{\infty}\bigl(\beta^k(x) - \beta^{k+1}(x)\bigr)f\bigl(\beta^k(x)\bigr), \quad x \in I,$$

provided that the series converges at $x = a$ and $x = b$. f is called β-integrable on I if the series converges at a and b for all $a, b \in I$. Clearly, if f is continuous at $s_0 \in I$, then f is β-integrable on I.

Definition 2.8 The β-exponential functions $e_{p,\beta}(t)$ and $E_{p,\beta}(t)$ are defined by

$$e_{p,\beta}(t) = \frac{1}{\prod_{k=0}^{\infty}[1 - p(\beta^k(t))(\beta^k(t) - \beta^{k+1}(t))]} \tag{2.2}$$

and

$$E_{p,\beta}(t) = \prod_{k=0}^{\infty}\bigl[1 + p\bigl(\beta^k(t)\bigr)\bigl(\beta^k(t) - \beta^{k+1}(t)\bigr)\bigr], \tag{2.3}$$

where $p : I \to \mathbb{C}$ is a continuous function at s_0 and both infinite products are convergent to a non-zero number for every $t \in I$ and $e_{p,\beta}(t) = \frac{1}{E_{p,\beta}(t)}$.

It is worth mentioning that both products in (2.2) and (2.3) are convergent since $\sum_{k=0}^{\infty} |p(\beta^k(t))(\beta^k(t) - \beta^{k+1}(t))|$ is uniformly convergent. See [18, Definition 2.1].

Theorem 2.9 *The β-exponential functions $e_{p,\beta}(t)$ and $E_{-p,\beta}(t)$ are, respectively, the unique solutions of the β-initial value problems*:

$$D_\beta y(t) = p(t)y(t), \quad y(s_0) = 1,$$

$$D_\beta y(t) = -p(t)y\bigl(\beta(t)\bigr), \quad y(s_0) = 1.$$

Theorem 2.10 *Assume that $p, q : I \to \mathbb{C}$ are continuous functions at $s_0 \in I$. The following properties are true*:

(i) $\frac{1}{e_{p,\beta}(t)} = e_{-p/[1+(\beta(t)-t)p]}(t)$,

(ii) $e_{p,\beta}(t)e_{q,\beta}(t) = e_{p+q+(\beta(t)-t)pq}(t)$,

(iii) $e_{p,\beta}(t)/e_{q,\beta}(t) = e_{(p-q)/[1+(\beta(t)-t)q]}(t)$.

Definition 2.11 The β-trigonometric functions are defined by

$$\cos_{p,\beta}(t) = \frac{e_{ip,\beta}(t) + e_{-ip,\beta}(t)}{2},$$
$$\sin_{p,\beta}(t) = \frac{e_{ip,\beta}(t) - e_{-ip,\beta}(t)}{2i}.$$

Theorem 2.12 *For all $t \in I$. The following relation holds true*:

$$e_{ip,\beta}(t) = \cos_{p,\beta}(t) + i \sin_{p,\beta}(t).$$

Theorem 2.13 *Assume that the function $f : R \to \mathbb{X}$ is continuous at $(s_0, y_0) \in R$ and satisfies the Lipschtiz condition* (*with respect to y*)

$$\|f(t,y_1) - f(t,y_2)\| \le L\|y_1 - y_2\|, \quad \text{for all } (t,y_1),(t,y_2) \in R.$$

Then the β-initial value problem $D_\beta y(t) = f(t,y)$, $y(s_0) = y_0$, $t \in I$ has a unique solution on $[s_0 - \delta, s_0 + \delta]$, where L is a positive constant and $\delta = \min\{a, \frac{b}{Lb+M}, \frac{\rho}{L}\}$ with $M = \sup_{(t,y)\in R} \|f(t,y)\| < \infty$, $\rho \in (0,1)$.

3 Main results

In this section, we prove the existence and uniqueness of solutions of the β-Cauchy problem of second order β-difference equations in a neighborhood of s_0. Furthermore, we construct a fundamental set of solutions for the second order linear homogeneous β-difference equations when the coefficients are constants and study the different cases of the roots of their characteristic equations. Finally, we derive the Euler-Cauchy β-difference equation.

3.1 Existence and uniqueness of solutions

Theorem 3.1 *Let $f_i(t,y_1,y_2) : I \times \prod_{i=1}^{2} S_i(x_i,b_i) \to \mathbb{X}$, $s_0 \in I$, such that the following conditions are satisfied*:

(i) *for $y_i \in S_i(x_i,b_i)$, $i = 1,2$, $f_i(t,y_1,y_2)$ are continuous at $t = s_0$*,

(ii) *there is a positive constant A such that, for $t \in I$, $y_i, \tilde{y}_i \in S_i(x_i,b_i)$, $i = 1,2$, the following Lipschitz condition is satisfied*:

$$\|f_i(t,y_1,y_2) - f_i(t,\tilde{y}_1,\tilde{y}_2)\| \le A \sum_{i=1}^{2} \|y_i - \tilde{y}_i\|.$$

Then there exists a unique solution of the β-initial value problem, β-IVP,

$$D_\beta y_i(t) = f_i\bigl(t, y_1(t), y_2(t)\bigr), \quad y_i(s_0) = x_i \in \mathbb{X}, i = 1,2, t \in I. \tag{3.1}$$

Proof Let $y_0 = (x_1, x_2)^T$ and $b = (b_1, b_2)^T$, where $(\cdot,\cdot)^T$ stands for vector transpose. Define the function $f : I \times \prod_{i=1}^{2} S_i(x_i, b_i) \to \mathbb{X} \times \mathbb{X}$ by $f(t, y_1, y_2) = (f_1(t, y_1, y_2), f_2(t, y_1, y_2))^T$. It is easy to show that system (3.1) is equivalent to the β-IVP

$$D_\beta y(t) = f\bigl(t, y(t)\bigr), \quad y(s_0) = y_0. \tag{3.2}$$

Since each f_i is continuous at $t = s_0$, f is continuous at $t = s_0$. The function f satisfies the Lipschitz condition because for $y, \tilde{y} \in \prod_{i=1}^{2} S_i(x_i, b_i)$,

$$\begin{aligned}\bigl\|f(t,y) - f(t,\tilde{y})\bigr\| &= \bigl\|f(t, y_1, y_2) - f(t, \tilde{y}_1, \tilde{y}_2)\bigr\| \\ &= \sum_{i=1}^{2} \bigl\|f_i(t, y_1, y_2) - f_i(t, \tilde{y}_1, \tilde{y}_2,)\bigr\| \\ &\le A \sum_{i=1}^{2} \|y_i - \tilde{y}_i\| = A\|y - \tilde{y}\|.\end{aligned}$$

Applying Theorem 2.13, see the proof in [19], there exists $\delta > 0$ such that (3.2) has a unique solution on $[s_0, s_0 + \delta]$. Hence, the β-IVP (3.1) has a unique solution on $[s_0, s_0 + \delta]$. □

Corollary 3.2 *Let $f(t, y_1, y_2)$ be a function defined on $I \times \prod_{i=1}^{2} S_i(x_i, b_i)$ such that the following conditions are satisfied*:

(i) *for any values of $y_i \in S_i(x_i, b_i)$, $i = 1, 2$, f is continuous at $t = s_0$*,

(ii) *f satisfies the Lipschitz condition*

$$\bigl\|f(t, y_1, y_2) - f(t, \tilde{y}_1, \tilde{y}_2)\bigr\| \le A \sum_{i=1}^{2} \|y_i - \tilde{y}_i\|,$$

where $A > 0$, $y_i, \tilde{y}_i \in S_i(x_i, b_i)$, $i = 1, 2$ and $t \in I$. Then

$$D_\beta^2 y(t) = f\bigl(t, y(t), D_\beta y(t)\bigr), \quad D_\beta^{i-1} y(s_0) = x_i, i = 1, 2 \tag{3.3}$$

has a unique solution on $[s_0, s_0 + \delta]$.

Proof Consider equation (3.3). It is equivalent to (3.1), where $\{\phi_i(t)\}_{i=1}^{2}$ is a solution of (3.1) if and only if $\phi_1(t)$ is a solution of (3.3). Here,

$$f_i(t, y_1, y_2) = \begin{cases} y_2, & i = 1, \\ f(t, y_1, y_2), & i = 2. \end{cases}$$

Hence, by Theorem 3.1, there exists $\delta > 0$ such that system (3.1) has a unique solution on $[s_0, s_0 + \delta]$. □

The following corollary gives us the sufficient conditions for the existence and uniqueness of the solutions of the β-Cauchy problem (3.3).

Corollary 3.3 *Assume the functions $a_j(t) : I \to \mathbb{C}$, $j = 0, 1, 2$, and $b(t) : I \to \mathbb{X}$ satisfy the following conditions*:

(i) $a_j(t), j = 0, 1, 2$ *and* $b(t)$ *are continuous at* s_0 *with* $a_0(t) \neq 0$ *for all* $t \in I$,
(ii) $a_j(t)/a_0(t)$ *is bounded on* I, $j = 1, 2$. *Then*

$$\begin{aligned} &a_0(t)D_\beta^2 y(t) + a_1(t)D_\beta y(t) + a_2(t)y(t) = b(t), \\ &D_\beta^{i-1} y(s_0) = x_i, \quad x_i \in \mathbb{X}, i = 1, 2, \end{aligned} \tag{3.4}$$

has a unique solution on subinterval $J \subseteq I$, $s_0 \in J$.

Proof Dividing by $a_0(t)$, we get

$$D_\beta^2 y(t) = A_1(t)D_\beta y(t) + A_2(t)y(t) + B(t), \tag{3.5}$$

where $A_j(t) = -a_j(t)/a_0(t)$ and $B(t) = b(t)/a_0(t)$. Since $A_j(t)$ and $B(t)$ are continuous at $t = s_0$, the function $f(t, y_1, y_2)$, defined by

$$f(t, y_1, y_2) = A_1(t)y_2 + A_2(t)y_1 + B(t),$$

is continuous at $t = s_0$. Furthermore, $A_j(t)$ is bounded on I. Consequently, there is $A > 0$ such that $|A_j(t)| \leq A$ for all $t \in I$. We can see that f satisfies the Lipschitz condition with Lipschitz constant A. Thus, $f(t, y_1, y_2)$ satisfies the conditions of Corollary 3.2. Hence, there exists a unique solution of (3.5) on J. □

3.2 Fundamental solutions of linear homogeneous β-difference equations

The second order homogeneous linear β-difference equation has the form

$$a_0(t)D_\beta^2 y(t) + a_1(t)D_\beta y(t) + a_2(t)y(t) = 0, \quad t \in I, \tag{3.6}$$

where the coefficients $a_0(t) \neq 0, a_j(t), j = 1, 2$ are assumed to satisfy the conditions of Corollary 3.3.

Lemma 3.4 *If the function* y *is a solution of the homogeneous equation* (3.6), *such that* $y(s_0) = 0$ *and* $D_\beta y(s_0) = 0$, $s_0 \in I$, *then* $y(t) = 0$, *for all* $t \in J$.

Proof By Corollary 3.3, if $x_i = 0$, $i = 1, 2$ in the β-IVP (3.4), which has a unique solution on J, then y such that $y(t) = 0$ for all $t \in J$ is a unique solution of the β-difference equation (3.6), which satisfies the given initial conditions $y(s_0) = 0$, $D_\beta y(s_0) = 0$. Hence we have the desired result. □

Theorem 3.5 *The linear combination* $c_1y_1 + c_2y_2$ *of any two solutions* y_1 *and* y_2 *of the homogeneous linear* β*-difference equation* (3.6) *is also a solution of it in* J, *where* c_1 *and* c_2 *are arbitrary constants.*

Proof The proof is straightforward. □

Theorem 3.6 *Let* y_1 *and* y_2 *be any two linearly independent solutions of the* β*-difference equation* (3.6) *in* J. *Then every solution* y *of* (3.6) *can be expressed as a linear combination* $y = c_1y_1 + c_2y_2$.

Proof Let

$$\phi = \begin{pmatrix} y \\ D_\beta y \end{pmatrix}, \qquad \phi_1 = \begin{pmatrix} y_1 \\ D_\beta y_1 \end{pmatrix}, \qquad \phi_2 = \begin{pmatrix} y_2 \\ D_\beta y_2 \end{pmatrix},$$

be the solutions of the linear system $D_\beta y_i(t) = a_i(t)y_i(t)$, $i = 1, 2$, corresponding, respectively, to the solutions y_1, y_2 of homogeneous linear β-difference equation (3.6). Since y_1, y_2 are linearly independent in J, then ϕ_1, ϕ_2 are linearly independent in J. Then there exist two constants c_1, c_2 such that $\phi = c_1\phi_1 + c_2\phi_2$. The first component of this is $y = c_1y_1 + c_2y_2$. Thus the results hold. □

Definition 3.7 A set of two linearly independent solutions of the second order homogeneous linear β-difference equation (3.6) is called a fundamental set of it.

Theorem 3.8 *There exists a fundamental set of solutions of the second order homogeneous linear β-difference equation* (3.6).

Proof By Corollary 3.3, there exist unique solutions y_1 and y_2 of equation (3.6), such that $y_1(s_0) = 1$, $D_\beta y_1(s_0) = 0$ and $y_2(s_0) = 0$, $D_\beta y_2(s_0) = 1$.

Suppose that y_1 and y_2 are linear dependent, so there exist constants c_1 and c_2 not both zero, such that

$$c_1y_1(t) + c_2y_2(t) = 0, \quad \text{for all } t \in J,$$
$$c_1D_\beta y_1(t) + c_2D_\beta y_2(t) = 0, \quad \text{for all } t \in J.$$

We have $c_1 = c_2 = 0$ at $t = s_0$, which is a contradiction. Thus the solutions y_1 and y_2 are linearly independent in J. Then there exists a fundamental set of the two solutions y_1 and y_2 of equation (3.6). □

Definition 3.9 Let y_1, y_2 be β-differentiable functions. Then we define the β-Wronskian of the functions y_1, y_2, defined on I, by

$$W_\beta(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ D_\beta y_1(t) & D_\beta y_2(t) \end{vmatrix}, \quad t \in I.$$

Lemma 3.10 *Let $y_1(t)$, $y_2(t)$ be functions defined on I. Then, for any $t \in I$, $t \neq s_0$,*

$$D_\beta W_\beta(y_1, y_2)(t) = \begin{vmatrix} y_1(\beta(t)) & y_2(\beta(t)) \\ D^2_\beta y_1(t) & D^2_\beta y_2(t). \end{vmatrix}. \tag{3.7}$$

Proof Since $W_\beta(y_1, y_2)(t) = y_1(t)D_\beta y_2(t) - y_2(t)D_\beta y_1(t)$, then

$$D_\beta W_\beta(y_1, y_2)(t) = y_1\bigl(\beta(t)\bigr)D^2_\beta y_2(t) - y_2\bigl(\beta(t)\bigr)D^2_\beta y_1(t),$$

which is the desired result. □

Theorem 3.11 *Assume that $y_1(t)$ and $y_2(t)$ are two solutions of equation* (3.6). *Then their β-Wronskian, W_β,*

$$W_\beta(y_1,y_2)(t) = e_{-r_1(t)+r_2(t)(\beta(t)-t),\beta}\, W_\beta(y_1,y_2)(s_0), \quad t \in I,$$

where $r_1(t) = \frac{a_1(t)}{a_0(t)}$ and $r_2(t) = \frac{a_2(t)}{a_0(t)}$ satisfy the conditions of Corollary 3.3.

Proof Since y_1 and y_2 are solutions of equation (3.6), from (3.7) we have

$$\begin{aligned} D_\beta W_\beta(y_1,y_2)(t) &= \begin{vmatrix} y_1(\beta(t)) & y_2(\beta(t)) \\ -\frac{a_1(t)}{a_0(t)} D_\beta y_1(t) & -\frac{a_1(t)}{a_0(t)} D_\beta y_2(t) \end{vmatrix} + \begin{vmatrix} y_1(\beta(t)) & y_2(\beta(t)) \\ -\frac{a_2(t)}{a_0(t)} y_1(t) & -\frac{a_2(t)}{a_0(t)} y_2(t) \end{vmatrix} \\ &= -\frac{a_1(t)}{a_0(t)} \begin{vmatrix} y_1(t) & y_2(t) \\ D_\beta y_1(t) & D_\beta y_2(t) \end{vmatrix} + \frac{a_2(t)}{a_0(t)}\bigl(\beta(t)-t\bigr) \begin{vmatrix} y_1(t) & y_2(t) \\ D_\beta y_1(t) & D_\beta y_2(t) \end{vmatrix} \\ &= \bigl[-r_1(t) + r_2(t)\bigl(\beta(t)-t\bigr)\bigr] W_\beta(y_1,y_2)(t), \end{aligned}$$

which has the solution

$$W_\beta(y_1,y_2)(t) = W_\beta(y_1,y_2)(s_0) e_{-r_1(t)+r_2(t)(\beta(t)-t),\beta}, \quad t \in I. \qquad \square$$

Using Theorem 3.11 and Lemma 3.4, we can prove the following corollaries.

Corollary 3.12 *Two solutions y_1 and y_2 of β-difference equation* (3.6) *are linearly dependent in J if and only if $W_\beta(y_1,y_2)(t) = 0$, for all $t \in J$.*

Corollary 3.13 *The value of $W_\beta(y_1,y_2)(t)$ of β-difference equation* (3.6) *either is zero or unequal to zero for all $t \in J$.*

3.3 Homogeneous equations with constant coefficients

Equation (3.6) can be written as

$$Ly(t) = aD_\beta^2 y(t) + bD_\beta y(t) + cy(t) = 0, \tag{3.8}$$

where a, b, and c are constants. The characteristic polynomial of equation (3.8) is

$$P(\lambda) = a\lambda^2 + b\lambda + c = 0, \tag{3.9}$$

where $y(t) = e_{\lambda,\beta}(t)$ is a solution of equation (3.8). Since equation (3.9) is a quadratic equation with real coefficients, it has two roots, which may be real and different, real but repeated, or complex conjugates.

Case 1: real and different roots of the characteristic equation (3.9).

Let λ_1 and λ_2 be real roots with $\lambda_1 \neq \lambda_2$, then $y_1(t) = e_{\lambda_1,\beta}(t)$ and $y_2(t) = e_{\lambda_2,\beta}(t)$ are two solutions of equation (3.8). Therefore,

$$y(t) = c_1 e_{\lambda_1,\beta}(t) + c_2 e_{\lambda_2,\beta}(t)$$

is a general solution of equation (3.8), with

$$c_1 = \frac{D_\beta y_0 - y_0\lambda_2}{\lambda_1 - \lambda_2} e_{-\lambda_1,\beta}(s_0) \quad \text{and} \quad c_2 = \frac{y_0\lambda_1 - D_\beta y_0}{\lambda_1 - \lambda_2} e_{-\lambda_2,\beta}(s_0).$$

Example 3.14 Find the solution of the β-initial value problem

$$D_\beta^2 y(t) + 5D_\beta y(t) + 6y(t) = 0, \quad y(s_0) = 2, D_\beta y(s_0) = 3.$$

By assuming that $y(t) = e_{\lambda,\beta}(t)$, we obtain the solution

$$y(t) = 9e_{-2,\beta}(t) - 7e_{-3,\beta}(t).$$

Case 2: complex roots of the characteristic equation (3.9).

Let $\lambda_1 = \nu + i\mu$ and $\lambda_2 = \nu - i\mu$, where ν and μ are real numbers. Then $y_1(t) = e_{(\nu+i\mu),\beta}(t)$ and $y_2(t) = e_{(\nu-i\mu),\beta}(t)$ are two solutions of equation (3.8). By Theorems 2.10, 2.12, $e_{(\nu+i\mu),\beta}(t) = e_{\nu,\beta}(t)e_{\frac{i\mu}{1+\nu(\beta(t)-t)},\beta}(t)$. So,

$$e_{(\nu+i\mu),\beta}(t) = e_{\nu,\beta}(t)\big(\cos_{\frac{\mu}{1+\nu(\beta(t)-t)},\beta}(t) + i\sin_{\frac{\mu}{1+\nu(\beta(t)-t)},\beta}(t)\big).$$

We have

$$y_1(t) + y_2(t) = 2e_{\nu,\beta}(t)\cos_{\frac{\mu}{1+\nu(\beta(t)-t)},\beta}(t)$$

and

$$y_1(t) - y_2(t) = 2ie_{\nu,\beta}(t)\sin_{\frac{\mu}{1+\nu(\beta(t)-t)},\beta}(t).$$

Therefore,

$$u(t) = e_{\nu,\beta}(t)\cos_{\frac{\mu}{1+\nu(\beta(t)-t)},\beta}(t) \quad \text{and} \quad v(t) = e_{\nu,\beta}(t)\sin_{\frac{\mu}{1+\nu(\beta(t)-t)},\beta}(t)$$

are two solutions of equation (3.8). If the β-Wronskian of u and v is not zero, then u and v form a fundamental set of solutions. The general solution of equation (3.8) is

$$y(t) = c_1 e_{\nu,\beta}(t)\cos_{\frac{\mu}{1+\nu(\beta(t)-t)},\beta}(t) + c_2 e_{\nu,\beta}(t)\sin_{\frac{\mu}{1+\nu(\beta(t)-t)},\beta}(t),$$

where c_1 and c_2 are arbitrary constants.

Example 3.15 Find the general solution of

$$D_\beta^2 y(t) + D_\beta y(t) + y(t) = 0. \tag{3.10}$$

The characteristic equation is $\lambda^2 + \lambda + 1 = 0$, and its roots are

$$\lambda_{1,2} = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}.$$

Thus, the general solution of equation (3.10) is

$$y(t) = c_1 e_{-1/2,\beta}(t) \cos_{\frac{\sqrt{3}/2}{1-1/2(\beta(t)-t)},\beta}(t) + c_2 e_{-1/2,\beta}(t) \sin_{\frac{\sqrt{3}/2}{1-1/2(\beta(t)-t)},\beta}(t).$$

Case 3: repeated roots.

Consider the case that the two roots λ_1 and λ_2 are equal, so

$$\lambda_1 = \lambda_2 = -b/2a.$$

Therefore, the solution $y_1(t) = e_{-b/2a,\beta}(t)$ is one solution of the β-difference equation (3.8), and we give the second solution by the following example:

Example 3.16 Solve the β-difference equation

$$D_\beta^2 y(t) + 4D_\beta y(t) + 4y(t) = 0. \tag{3.11}$$

The characteristic equation is $(\lambda + 2)^2 = 0$, so $\lambda_1 = \lambda_2 = -2$. Therefore, $y_1(t) = e_{-2,\beta}(t)$ is a solution of equation (3.11). To find the second solution, let $y(t) = v(t)e_{-2,\beta}(t)$. Then $D_\beta^2 v(t) = 0$. Therefore, $v(t) = c_1 t + c_2$, where c_1 and c_2 are arbitrary constants. Then the general solution is

$$y(t) = c_1 t e_{-2,\beta}(t) + c_2 e_{-2,\beta}(t),$$

where the two solutions $y_1(t) = e_{-2,\beta}(t)$ and $y_2(t) = te_{-2,\beta}(t)$ form a fundamental set of solutions of equation (3.11).

3.4 Euler-Cauchy β-difference equation

The Euler-Cauchy β-difference equation takes the form

$$t\beta(t)D_\beta^2 y(t) + atD_\beta y(t) + by(t) = 0, \quad t \in I, t \neq s_0, \tag{3.12}$$

where a, b are constants. The characteristic equation of (3.12) is given by

$$\lambda^2 + (a-1)\lambda + b = 0. \tag{3.13}$$

Theorem 3.17 *If the characteristic equation* (3.13) *has two distinct roots* λ_1 *and* λ_2, *then a fundamental set of solutions of* (3.12) *is given by* $e_{\lambda_1/t,\beta}(t)$ *and* $e_{\lambda_2/t,\beta}(t)$.

Proof Let $y(t) = e_{\lambda/t,\beta}(t)$, where λ is a root of equation (3.13). It follows that

$$D_\beta y(t) = \frac{\lambda}{t} y(t), \qquad D_\beta^2 y(t) = \frac{\lambda^2 - \lambda}{t\beta(t)} y(t).$$

Consequently, we have

$$t\beta(t)D_\beta^2 y(t) + atD_\beta y(t) + by(t) = \big(\lambda^2 + (a-1)\lambda + b\big)y(t) = 0.$$

Assume that λ_1 and λ_2 are distinct roots of the characteristic equation (3.13). Then, we have

$$\lambda_1 + \lambda_2 = 1 - a, \qquad \lambda_1\lambda_2 = b.$$

Moreover, $W_\beta(e_{\lambda_1/t,\beta}, e_{\lambda_2/t,\beta})(t) \neq 0$, since $\lambda_1 \neq \lambda_2$. Hence, $e_{\lambda_1/t,\beta}(t)$ and $e_{\lambda_2/t,\beta}(t)$ form a fundamental set of solutions of (3.12). □

The following theorem gives us the general solution of the Euler-Cauchy β-difference equation in the double root case.

Theorem 3.18 *Assume that $1/\beta(t)$ is bounded on I and $0 \notin I$. Then the general solution of the Euler-Cauchy β-difference equation*

$$t\beta(t)D_\beta^2 y(t) + (1-2\gamma)tD_\beta y(t) + \gamma^2 y(t) = 0, \quad t \in I, \tag{3.14}$$

is given by

$$y(t) = c_1 e_{\frac{\gamma}{t},\beta}(t) + c_2 e_{\frac{\gamma}{t},\beta}(t)\int_{s_0}^{t} \frac{e_{\frac{-1}{\beta(\tau)},\beta}}{1+\frac{\gamma}{\tau}(\beta(\tau)-\tau)}\,d_\beta\tau.$$

Proof The characteristic equation of (3.14) is

$$\lambda^2 - 2\gamma\lambda + \gamma^2 = 0.$$

Then the characteristic roots are $\lambda_1 = \lambda_2 = \gamma$. Hence one linearly independent solution of equation (3.14) is $y_1(t) = e_{\frac{\gamma}{t},\beta}(t)$. To obtain the second linearly independent solution, we can rewrite equation (3.14) in the form

$$D_\beta^2 y(t) + r_1(t)D_\beta y(t) + r_2(t)y(t) = 0, \tag{3.15}$$

where $r_1(t) = \frac{1-2\gamma}{\beta(t)}$ and $r_2(t) = \frac{\gamma^2}{t\beta(t)}$. Consequently,

$$-r_1(t) + r_2(t)\bigl(\beta(t)-t\bigr) = \frac{\gamma^2}{t} - \frac{(\gamma-1)^2}{\beta(t)}.$$

Let u be a solution of equation (3.15) such that $u(s_0) = 0$, $D_\beta u(s_0) = 1$. Then

$$W_\beta(e_{\frac{\gamma}{t},\beta}, u)(t) = e_{-r_1(t)+r_2(t)(\beta(t)-t),\beta}(t) = e_{\frac{\gamma^2}{t}-\frac{(\gamma-1)^2}{\beta(t)},\beta}(t).$$

By Theorem 2.5, we find that u satisfies the following β-difference equation:

$$\begin{aligned} D_\beta\left(\frac{u}{e_{\frac{\gamma}{t},\beta}}\right)(t) &= \frac{W_\beta(e_{\frac{\gamma}{t},\beta}, u)(t)}{e_{\frac{\gamma}{t},\beta}(t)e_{\frac{\gamma}{\beta(t)},\beta}(\beta(t))} \\ &= \frac{e_{\frac{\gamma^2}{t}-\frac{(\gamma-1)^2}{\beta(t)},\beta}(t)}{e^2_{\frac{\gamma}{t},\beta}(t)(1+\frac{\gamma}{t}(\beta(t)-t))}. \end{aligned}$$

Then

$$u(t) = e_{\frac{\gamma}{t}}(t) \int_{s_0}^{t} \frac{e_{\frac{\alpha^2}{\tau} - \frac{(\gamma-1)^2}{\beta(\tau)}, \beta}(\tau)}{e^2_{\frac{\gamma}{\tau}, \beta}(\tau)(1 + \frac{\gamma}{\tau}(\beta(\tau) - \tau))} \, d_\beta \tau.$$

Also,

$$\frac{e_{\frac{\gamma^2}{t} - \frac{(\gamma-1)^2}{\beta(t)}, \beta}(t)}{e^2_{\frac{\gamma}{t}, \beta}(t)} = e_{\frac{-1}{\beta(t)}, \beta}(t).$$

Therefore,

$$y(t) = c_1 e_{\frac{\gamma}{t}, \beta}(t) + c_2 e_{\frac{\gamma}{t}, \beta}(t) \int_{s_0}^{t} \frac{e_{\frac{-1}{\beta(\tau)}, \beta}(\tau)}{1 + \frac{\gamma}{\tau}(\beta(\tau) - \tau)} \, d_\beta \tau$$

is the general solution of equation (3.14). □

4 Conclusion

In this paper, the existence and uniqueness of solutions of the β-Cauchy problem of second order β-difference equations were proved. Moreover, a fundamental set of solutions for second order linear homogeneous β-difference equations when the coefficients are constants was constructed. Also, the different cases of the roots of the characteristic equations of these equations were studied. Finally, the Euler-Cauchy β-difference equation was derived.

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Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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