Fu and Liu *Journal of Inequalities and Applications* (2016) 2016:262
DOI 10.1186/s13660-016-1208-8

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH | Open Access

Some determinantal inequalities for Hadamard and Fan products of matrices

Xiaohui Fu[1,2*] and Yang Liu[2]

*Correspondence: fxh6662@sina.com
[1]School of Mathematics and Statistics, Hainan Normal University, Haikou, 571158, P.R. China
[2]College of Mathematics and Statistics, Chongqing University, Chongqing, 401331, P.R. China

Abstract

In this note, we generalize some determinantal inequalities which are due to Lynn (Proc. Camb. Philos. 60:425-431, 1964), Chen (Linear Algebra Appl. 368:99-106, 2003) and Ando (Linear Multilinear Algebra 8:291-316, 1980).

MSC: 47A63; 47A30

Keywords: determinantal inequalities; *H*-matrix; *M*-matrix; Hadamard product; Fan product

1 Introduction

Let $C^{m\times n}$ ($R^{m\times n}$) be the set of all complex (real) matrices and let $\mathbb{M}_n^+$ be the positive definite Hermitian matrices. Let $Z^{n\times n} = \{A = (a_{ij}) \in R^{n\times n} : a_{ij} \leq 0, i \neq j, i, j \in \{1,2,\dots,n\}\}$. For any $A = (a_{ij}) \in C^{n\times n}$, its associated matrix is defined by $A' = (\alpha_{ij})$, where $\alpha_{ii} = |a_{ii}|$, $\alpha_{ij} = -|a_{ij}|$ ($i \neq j$). For $A = (a_{ij})$, $B = (b_{ij}) \in C^{m\times n}$, the Hadamard product of A and B is $A \circ B = (a_{ij}b_{ij}) \in C^{m\times n}$ while their Fan product $A * B = (c_{ij})$ is defined by $c_{ii} = a_{ii}b_{ii}$ and $c_{ij} = -a_{ij}b_{ij}$ for $i \neq j$.

If $A = (a_{ij}) \in C^{n\times n}$, then the $k \times k$ leading principal submatrix of A is denoted by A_k ($k \in \{1,2,\dots,n\}$). A_α denotes the principal submatrix of A, with indices in $\alpha \subseteq \{1,2,\dots,n\}$. $A \in R^{n\times n}$ is called an M-matrix if $A \in Z^{n\times n}$ and $\det A_k > 0$ ($\forall k \in \{1,2,\dots,n\}$), and we denote it by $A \in M_n$. A matrix $A \in C^{n\times n}$ is called an H-matrix if A' is an M-matrix, and we denote it by $A \in H_n$.

Lynn [1], Theorem 3.1, proved the following determinantal inequality for H-matrices: if $A, B \in H_n$, then

$$\det(A \circ B)' + \det A' \det B' \geq \prod_{i=1}^{n} |b_{ii}| \det A' + \prod_{i=1}^{n} |a_{ii}| \det B',$$

i.e.

$$\det(A \circ B)' \geq \det A' \det B' \left(\frac{\prod_{i=1}^{n} |a_{ii}|}{\det A'} + \frac{\prod_{i=1}^{n} |b_{ii}|}{\det B'} - 1 \right). \tag{1.1}$$

Chen [2], Theorem 2.7, obtained a determinantal inequality for positive definite matrices: if $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{M}_n^+$, then

$$\det(A \circ B) \geq \det A \det B \prod_{k=2}^{n} \left(\frac{a_{kk} \det A_{k-1}}{\det A_k} + \frac{b_{kk} \det B_{k-1}}{\det B_k} - 1 \right). \tag{1.2}$$

Lin [3] recently proved that a similar result to the block positive definite matrices holds for the block Hadamard product.

Ando [4], Theorem 5.3, has given the following result: if $A = (a_{ij})$, $B = (b_{ij})$ are M-matrices, then

$$\det(A * B) + \det A \cdot \det B \geq \left(\prod_{i=1}^{n} a_{ii}\right) \cdot \det B + \det A \cdot \left(\prod_{i=1}^{n} b_{ii}\right),$$

i.e.

$$\det(A * B) \geq \det A \det B \left(\frac{\prod_{i=1}^{n} a_{ii}}{\det A} + \frac{\prod_{i=1}^{n} b_{ii}}{\det B} - 1\right). \tag{1.3}$$

In this paper, we will present some determinantal inequalities for matrices which are generalizations of (1.1), (1.2), and (1.3).

2 Main results and some remarks

We give some lemmas before we present the main theorems of this paper.

Lemma 1 ([4], Corollary 4.1.2) *Let $A = (a_{ij}) \in R^{n\times n}$ be an M-matrix. If $\alpha_i \subseteq \{1,2,\ldots,n\}$ $(i = 1,2,3,\ldots,N)$ satisfies $\alpha_i \cap \alpha_j = \phi$ for $i \neq j$ and $\bigcup_{j=1} \alpha_j = \{1,2,\ldots,n\}$, then*

$$\det A \leq \prod_{i=1}^{N} \det(A_{\alpha_i}).$$

In particular,

$$\det A \leq \prod_{i=1}^{n} a_{ii}. \tag{2.1}$$

Lemma 2 ([1], Theorem 3.1) *If A, B are H-matrices and $C = A \circ B$, then C is H-matrix.*

Lemma 3 ([5], Theorem 5.2.1) *If A, B are positive definite matrices and $C = A \circ B$, then C is positive definite matrix.*

Lemma 4 ([6]) *If A, B are M-matrices and $C = A * B$, then C is M-matrix.*

Now we present the main results.

First of all, we give a determinantal inequality for the Hadamard product of finite number of H-matrices as follows:

Theorem 5 *If $A_1 = (a_1^{kl}), A_2 = (a_2^{kl}), \ldots, A_m = (a_m^{kl})$ $(k,l = 1,\ldots,n)$ are H-matrices, then*

$$\det(A_1 \circ \cdots \circ A_m)' \geq \det(A_1 \cdots A_m) \times \left(\frac{\prod_{i=1}^{n} |a_1^{ii}|}{\det A_1'} + \cdots + \frac{\prod_{i=1}^{n} |a_m^{ii}|}{\det A_m'} - (m-1)\right). \tag{2.2}$$

Proof By Lemma 2, it is straightforward to observe that the Hadamard product $A_1 \circ \cdots \circ A_m$ is an H-matrix. Use induction on k. When $k = 2$, the result is (1.1). Suppose that (2.2) holds when $k = m - 1$

$$\det(A_1 \circ \cdots \circ A_{m-1})' \geq \det(A_1' \cdots A_{m-1}') \times \left(\frac{\prod_{i=1}^n |a_1^{ii}|}{\det A_1'} + \cdots + \frac{\prod_{i=1}^n |a_{m-1}^{ii}|}{\det A_{m-1}'} - (m-2) \right).$$

When $k = m$, we need to show

$$\det(A_1 \circ \cdots \circ A_m)' \geq \det(A_1' \cdots A_m') \times \left(\frac{\prod_{i=1}^n |a_1^{ii}|}{\det A_1'} + \cdots + \frac{\prod_{i=1}^n |a_m^{ii}|}{\det A_m'} - (m-1) \right).$$

By (1.1), we have

$$\det\bigl((A_1 \circ \cdots \circ A_{m-1}) \circ A_m\bigr)' \geq \det\bigl((A_1' \circ \cdots \circ A_{m-1}')A_m'\bigr) \times \left(\frac{\prod_{i=1}^n (|a_1^{ii}| \cdots |a_{m-1}^{ii}|)}{\det(A_1' \circ \cdots \circ A_{m-1}')} + \frac{\prod_{i=1}^n |a_m^{ii}|}{\det A_m'} - 1 \right).$$

By the inductive assumption, the above inequality is

$$\begin{aligned} \det(A_1 \circ \cdots \circ A_m)' \geq {} & \det(A_1' \cdots A_{m-1}') \det A_m' \\ & \times \left(\frac{\prod_{i=1}^n |a_1^{ii}|}{\det A_1'} + \cdots + \frac{\prod_{i=1}^n |a_{m-1}^{ii}|}{\det A_{m-1}'} - (m-2) \right) \\ & \times \left(\frac{\prod_{i=1}^n (|a_1^{ii}| \cdots |a_{m-1}^{ii}|)}{\det(A_1' \circ \cdots \circ A_{m-1}')} + \frac{\prod_{i=1}^n |a_m^{ii}|}{\det A_m'} - 1 \right). \end{aligned} \tag{2.3}$$

Let

$$a = \left(\frac{\prod_{i=1}^n |a_1^{ii}|}{\det A_1'} + \cdots + \frac{\prod_{i=1}^n |a_{m-1}^{ii}|}{\det A_{m-1}'} - (m-2) \right),$$
$$b = \left(\frac{\prod_{i=1}^n (|a_1^{ii}| \cdots |a_{m-1}^{ii}|)}{\det(A_1' \circ \cdots \circ A_{m-1}')} + \frac{\prod_{i=1}^n |a_m^{ii}|}{\det A_m'} - 1 \right).$$

By (2.1), we have

$$\frac{\prod_{i=1}^n |a_j^{ii}|}{\det A_j'} \geq 1, \quad j = 1, \ldots, m,$$
$$\frac{\prod_{i=1}^n (|a_1^{ii}| \cdots |a_{m-1}^{ii}|)}{\det(A_1' \circ \cdots \circ A_{m-1}')} \geq 1,$$

and so

$$a, b \geq 1.$$

Thus by $ab \geq a + b - 1$ for $a, b \geq 1$, the above inequality (2.3) is

$$\begin{aligned}\det(A_1 \circ \cdots \circ A_m)' &\geq \det\bigl(A_1' \cdots A_{m-1}'\bigr) \det A_m' \times a \times b \\ &\geq \det\bigl(A_1' \cdots A_m'\bigr) \times (a + b - 1) \\ &\geq \det\bigl(A_1' \cdots A_m'\bigr) \\ &\quad \times \left(\frac{\prod_{i=1}^n |a_1^{ii}|}{\det A_1'} + \cdots + \frac{\prod_{i=1}^n |a_m^{ii}|}{\det A_m'} - (m-1) \right).\end{aligned}$$

This completes the proof. □

Remark 6 The above inequality in Theorem 5 is a generalization of the inequality (1.1).

Second, we achieve a determinantal inequality for the Hadamard product of positive definite matrices as follows:

Theorem 7 *If A_i $(i = 1, \ldots, m)$ $(m \geq 2)$ are $n \times n$ positive definite matrices, the Hadamard product of $A_i = (a_i^{lt})$ and $A_j = (a_j^{lt})$ $(i \neq j)$ is denoted by $A_i \circ A_j$, and $A_i^{(k)}$ is the $k \times k$ $(k = 1, 2, \ldots, n)$ leading principal submatrix of A_i, then*

$$\begin{aligned}\det(A_1 \circ \cdots \circ A_m) &\geq \det(A_1 \cdots A_m) \\ &\quad \times \prod_{\mu=2}^{n} \left(\frac{a_1^{\mu\mu} \det A_1^{(\mu-1)}}{\det A_1^{(\mu)}} + \cdots + \frac{a_m^{\mu\mu} \det A_m^{(\mu-1)}}{\det A_m^{(\mu)}} - (m-1) \right). \qquad (2.4)\end{aligned}$$

Proof By Lemma 3, it is straightforward to see that the Hadamard product $A_1 \circ \cdots \circ A_m$ is a positive definite matrix. Use induction on m. When $k = 2$, the result is (1.2). Suppose that (2.4) holds when $k = m - 1$. We have

$$\begin{aligned}\det(A_1 \circ \cdots \circ A_{m-1}) &\geq \det(A_1 \cdots A_{m-1}) \\ &\quad \times \prod_{\mu=2}^{n} \left(\frac{a_1^{\mu\mu} \det A_1^{(\mu-1)}}{\det A_1^{(\mu)}} + \cdots + \frac{a_{m-1}^{\mu\mu} \det A_{m-1}^{(\mu-1)}}{\det A_{m-1}^{(\mu)}} - (m-2) \right).\end{aligned}$$

When $k = m$, we need to show

$$\begin{aligned}\det(A_1 \circ \cdots \circ A_m) &\geq \det(A_1 \cdots A_m) \\ &\quad \times \prod_{\mu=2}^{n} \left(\frac{a_1^{\mu\mu} \det A_1^{(\mu-1)}}{\det A_1^{(\mu)}} + \cdots + \frac{a_m^{\mu\mu} \det A_m^{(\mu-1)}}{\det A_m^{(\mu)}} - (m-1) \right).\end{aligned}$$

By (1.2), we have

$$\begin{aligned}&\det\bigl((A_1 \circ \cdots \circ A_{m-1}) \circ A_m\bigr) \\ &\quad \geq \det\bigl((A_1 \circ \cdots \circ A_{m-1}) A_m\bigr) \\ &\quad \times \prod_{\mu=2}^{n} \left(\frac{(a_1^{\mu\mu} \cdots a_{m-1}^{\mu\mu}) \det(A_1 \circ \cdots \circ A_{m-1})^{(\mu-1)}}{\det(A_1 \circ \cdots \circ A_{m-1})^{(\mu)}} + \frac{a_m^{\mu\mu} \det A_m^{(\mu-1)}}{\det A_m^{(\mu)}} - 1 \right).\end{aligned}$$

By the inductive assumption, the above inequality is such that

$$\begin{aligned}
&\det(A_1 \circ \cdots \circ A_{m-1}) \det A_m \\
&\quad \times \prod_{\mu=2}^{n} \left(\frac{(a_1^{\mu\mu} \cdots a_{m-1}^{\mu\mu}) \det(A_1 \circ \cdots \circ A_{m-1})^{(\mu-1)}}{\det(A_1 \circ \cdots \circ A_{m-1})^{(\mu)}} + \frac{a_m^{\mu\mu} \det A_m^{(\mu-1)}}{\det A_m^{(\mu)}} - 1 \right) \\
&\geq \det(A_1 \cdots A_{m-1}) \det A_m \\
&\quad \times \prod_{\mu=2}^{n} \left(\frac{a_1^{\mu\mu} \det A_1^{(\mu-1)}}{\det A_1^{(\mu)}} + \frac{a_2^{\mu\mu} \det A_2^{(\mu-1)}}{\det A_2^{(\mu)}} + \cdots + \frac{a_m^{\mu\mu} \det A_{m-1}^{(\mu-1)}}{\det A_{m-1}^{(\mu)}} - (m-2) \right) \\
&\quad \times \prod_{\mu=2}^{n} \left(\frac{(a_1^{\mu\mu} \cdots a_{m-1}^{\mu\mu}) \det(A_1 \circ \cdots \circ A_{m-1})^{(\mu-1)}}{\det(A_1 \circ \cdots \circ A_{m-1})^{(\mu)}} + \frac{a_m^{\mu\mu} \det A_m^{(\mu-1)}}{\det A_m^{(\mu)}} - 1 \right).
\end{aligned} \tag{2.5}$$

Let

$$\begin{aligned}
a_\mu &= \frac{a_1^{\mu\mu} \det A_1^{(\mu-1)}}{\det A_1^{(\mu)}} + \frac{a_2^{\mu\mu} \det A_2^{(\mu-1)}}{\det A_2^{(\mu)}} + \cdots + \frac{a_n^{\mu\mu} \det A_{m-1}^{(\mu-1)}}{\det A_{m-1}^{(\mu)}} - (m-2), \\
b_\mu &= \frac{(a_1^{\mu\mu} \cdots a_{m-1}^{\mu\mu}) \det(A_1 \circ \cdots \circ A_{m-1})^{(\mu-1)}}{\det(A_1 \circ \cdots \circ A_{m-1})^{(\mu)}} + \frac{a_m^{\mu\mu} \det A_m^{(\mu-1)}}{\det A_m^{(\mu)}} - 1.
\end{aligned}$$

By Fischer's inequality [5], p.506, we have

$$\begin{aligned}
&\frac{a_i^{\mu\mu} \det A_i^{(\mu-1)}}{\det A_i^{(\mu)}} \geq 1, \quad i = 1, \ldots, m, \\
&\frac{(a_1^{\mu\mu} \cdots a_{m-1}^{\mu\mu}) \det(A_1 \circ \cdots \circ A_{m-1})^{(\mu-1)}}{\det(A_1 \circ \cdots \circ A_{m-1})^{(\mu)}} - 1 \geq 0,
\end{aligned}$$

and so

$$a_\mu, b_\mu \geq 1.$$

Thus by $a_\mu b_\mu \geq a_\mu + b_\mu - 1$ for $a_\mu, b_\mu \geq 1$, the above inequality (2.5) is

$$\begin{aligned}
&\det(A_1 \circ \cdots \circ A_m) \\
&\quad \geq \det(A_1 \cdots A_{m-1}) \det A_m \times \prod_{\mu=2}^{n} a_\mu b_\mu \\
&\quad \geq \det(A_1 \cdots A_m) \times \prod_{\mu=2}^{n} (a_\mu + b_\mu - 1) \\
&\quad \geq \det(A_1 \cdots A_m) \prod_{\mu=2}^{n} \left(\frac{a_1^{\mu\mu} \det A_1^{(\mu-1)}}{\det A_1^{(\mu)}} + \cdots + \frac{a_m^{\mu\mu} \det A_m^{(\mu-1)}}{\det A_m^{(\mu)}} - (m-1) \right).
\end{aligned}$$

This completes the proof. □

Remark 8 The inequality in Theorem 7 is a generalization of the inequality (1.2).

Finally, a result on Fan product of M-matrices is obtained in the following theorem.

Theorem 9 *If $A_1 = (a_1^{kl}), A_2 = (a_2^{kl}), \dots, A_m = (a_m^{kl})$ $(k, l = 1, \dots, n)$ are M-matrices, then*

$$\det(A_1 * \cdots * A_m) \geq \det(A_1 \cdots A_m) \times \left(\frac{\prod_{i=1}^n a_1^{ii}}{\det A_1} + \cdots + \frac{\prod_{i=1}^n a_m^{ii}}{\det A_m} - (m-1) \right). \tag{2.6}$$

Proof By Lemma 4, it is straightforward to see that the Hadamard product $A_1 * \cdots * A_m$ is an M-matrix. Use induction on k. When $k = 2$, the result is (1.3). Let $k = m - 1$, (2.6) holds:

$$\det(A_1 \circ \cdots \circ A_{m-1}) \geq \det(A_1 \cdots A_{m-1}) \times \left(\frac{\prod_{i=1}^n a_1^{ii}}{\det A_1} + \cdots + \frac{\prod_{i=1}^n a_{m-1}^{ii}}{\det A_{m-1}} - (m-2) \right).$$

When $k = m$, we need to show

$$\det(A_1 * \cdots * A_m) \geq \det(A_1 \cdots A_m) \times \left(\frac{\prod_{i=1}^n a_1^{ii}}{\det A_1} + \cdots + \frac{\prod_{i=1}^n a_m^{ii}}{\det A_m} - (m-1) \right).$$

By (1.3), we have

$$\det\bigl((A_1 * \cdots * A_{m-1}) * A_m\bigr) \geq \det\bigl((A_1 * \cdots * A_{m-1}) A_m\bigr) \times \left(\frac{\prod_{i=1}^n (a_1^{ii} \cdots a_{m-1}^{ii})}{\det(A_1 * \cdots * A_{m-1})} + \frac{\prod_{i=1}^n a_m^{ii}}{\det A_m} - 1 \right).$$

By the inductive assumption, the above inequality is

$$\begin{aligned}\det(A_1 * \cdots * A_m) \geq{}& \det(A_1 \cdots A_{m-1}) \det A_m \\ &\times \left(\frac{\prod_{i=1}^n a_1^{ii}}{\det A_1} + \cdots + \frac{\prod_{i=1}^n a_{m-1}^{ii}}{\det A_{m-1}} - (m-2) \right) \\ &\times \left(\frac{\prod_{i=1}^n (a_1^{ii} \cdots a_{m-1}^{ii})}{\det(A_1 * \cdots * A_{m-1})} + \frac{\prod_{i=1}^n a_m^{ii}}{\det A_m} - 1 \right).\end{aligned} \tag{2.7}$$

Let

$$a = \left(\frac{\prod_{i=1}^n a_1^{ii}}{\det A_1} + \cdots + \frac{\prod_{i=1}^n a_{m-1}^{ii}}{\det A_{m-1}} - (m-2) \right),$$

$$b = \left(\frac{\prod_{i=1}^n (a_1^{ii} \cdots a_{m-1}^{ii})}{\det(A_1 \circ \cdots \circ A_{m-1})} + \frac{\prod_{i=1}^n a_m^{ii}}{\det A_m} - 1 \right).$$

By (2.1), we have

$$\frac{\prod_{i=1}^n a_j^{ii}}{\det A_j} \geq 1, \quad j = 1, \dots, m,$$

$$\frac{\prod_{i=1}^n (a_1^{ii} \cdots a_{m-1}^{ii})}{\det(A_1 \circ \cdots \circ A_{m-1})} \geq 1,$$

and so

$$a, b \geq 1.$$

So by $ab \geq a + b - 1$ for $a, b \geq 1$, the above inequality (2.7) is

$$\det(A_1 * \cdots * A_m) \geq \det(A_1 \cdots A_m) \times \left(\frac{\prod_{i=1}^{n} a_1^{ii}}{\det A_1} + \cdots + \frac{\prod_{i=1}^{n} a_m^{ii}}{\det A_m} - (m-1) \right).$$

This completes the proof. □

Remark 10 The inequality in Theorem 9 is a generalization of the inequality (1.3).

Competing interests
The authors declare to have no competing interests.

Authors' contributions
Xiaohui Fu carried out all the proofs of the results and gave the generalizations of Fan product. Yang Liu participated in the design of the study and drafted the manuscript. All authors read and approved the final manuscript.

Acknowledgements
We are grateful to Dr. Limin Zou for fruitful discussions. This research was supported by the key project of the applied mathematics of Hainan Normal University, the natural science foundation of Hainan Province (No. 20161005), the Chongqing Graduate Student Research Innovation Project (No. CYS14020) and the Doctoral scientific research foundation of Hainan Normal University.

Received: 22 June 2016 Accepted: 13 October 2016 Published online: 24 October 2016

References
1. Lynn, MS: On the Schur product of the *H*-matrices and non-negative matrices and related inequalities. Proc. Camb. Philos. **60**, 425-431 (1964)
2. Chen, S: Some determinantal inequalities for Hadamard product of matrices. Linear Algebra Appl. **368**, 99-106 (2003)
3. Lin, M: An Oppenheim type inequality for a block Hadamard product. Linear Algebra Appl. **452**, 1-6 (2014)
4. Ando, T: Inequalities for *M*-matrices. Linear Multilinear Algebra **8**, 291-316 (1980)
5. Horn, RA, Johnson, CR: Matrix Analysis, 2nd edn. Cambridge University Press, Cambridge (2013)
6. Fan, K: Note on *M*-matrices. Indag. Math. **67**, 43-49 (1964)