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Decay estimates for fractional wave equations on H-type groups

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Abstract

The aim of this paper is to establish the decay estimate for the fractional wave equation semigroup on H-type groups given by $e^{it\Delta^\alpha}$, $0 < \alpha < 1$. Combining the dispersive estimate and a standard duality argument, we also derive the corresponding Strichartz inequalities.

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1 Introduction

In this paper, we study the decay estimate for a class of dispersive equations:

$$i\partial_t u + \Delta^\alpha u = f, \qquad u(0) = u_0, \tag{1}$$

where Δ is the sub-Laplacian on H-type groups G, $\alpha > 0$.

The partial differential equation in (1) is significantly interesting in mathematics. When $\alpha = \frac{1}{2}$, it is reduced to the wave equation; when $\alpha = 1$, it is reduced to the Schrödinger equation. The two equations are most important fundamental types of partial differential equations.

In 2000, Bahouri *et al.* [1] derived the Strichartz inequalities for the wave equation on the Heisenberg group via a sharp dispersive estimate and a standard duality argument (see [2] and [3]). The dispersive estimate

$$\left\| e^{it\Delta^\alpha} \varphi \right\|_{L^\infty} \le C|t|^{-\theta} \tag{2}$$

plays a crucial role, where φ is the kernel function on the Heisenberg group related to a Littlewood-Paley decomposition introduced in Section 2 and $\theta > 0$. Such an estimate does not exist for the Schrödinger equation (see [1]). The sharp dispersive estimate is also generalized to H-type groups for the wave equation and the Schrödinger equation (see [4–8]). Motivated by Guo *et al.* [9] on the Euclidean space, we consider the fractional wave equation (1) on H-type groups and will prove a sharp dispersive estimate.

Theorem 1.1 *Let N be the homogeneous dimension of the H-type group G, and p the dimension of its center. For $0 < \alpha < 1$, we have*

$$\left\| e^{it\Delta^{\alpha}} u_0 \right\|_{\infty} \le C_{\alpha} |t|^{-p/2} \| u_0 \|_{\dot{B}_{1,1}^{N-p/2}},$$

and the result is sharp in time. Here, the constant $C_\alpha > 0$ does not depend on u_0, t, and $\dot{B}^{\rho}_{q,r}$ is the homogeneous Besov space associated to the sublaplacian Δ introduced in the next section.

Following the work by Keel and Tao [3] or by Ginibre and Velo [2], we also get a useful estimate on the solution of the fractional wave equation.

Corollary 1.1 *If $0 < \alpha < 1$ and u is the solution of the fractional wave equation* (1), *then for $q \in [(2N-p)/p, +\infty)$ and r such that*

$$1/q + N/r = N/2 - 1,$$

we have the estimate

$$\|u\|_{L^q((0,T),L^r)} \le C_\alpha \left(\|u_0\|_{\dot{H}^1} + \|f\|_{L^1((0,T),\dot{H}^1)} \right),$$

where the constant $C_\alpha > 0$ does not depend on u_0, f or T.

Remark 1.1 In this article, we assume $0 < \alpha < 1$. For $\alpha = 1$, the decay estimate has been proved (see [4]). For other cases, we could investigate the problem in a similar way to $0 < \alpha \le 1$.

2 Preliminaries

2.1 *H*-Type groups

Let $\mathfrak{g}$ be a two step nilpotent Lie algebra endowed with an inner product $\langle \cdot, \cdot \rangle$. Its center is denoted by $\mathfrak{z}$. $\mathfrak{g}$ is said to be of H-type if $[\mathfrak{z}^\perp, \mathfrak{z}^\perp] = \mathfrak{z}$, and, for every $s \in \mathfrak{z}$, the map $J_s : \mathfrak{z}^\perp \to \mathfrak{z}^\perp$ defined by

$$\langle J_s u, w \rangle := \langle s, [u, w] \rangle, \quad \forall u, w \in \mathfrak{z}^\perp,$$

is an orthogonal map whenever $|s| = 1$.

An H-type group is a connected and simply connected Lie group G whose Lie algebra is of H-type.

Given $0 \ne a \in \mathfrak{z}^*$, the dual of $\mathfrak{z}$, we can define a skew-symmetric mapping $B(a)$ on $\mathfrak{z}^\perp$ by

$$\langle B(a)u, w \rangle = a([u, w]), \quad \forall u, w \in \mathfrak{z}^\perp.$$

We denote by z_a the element of $\mathfrak{z}$ determined by

$$\langle B(a)u, w \rangle = a([u, w]) = \langle J_{z_a} u, w \rangle.$$

Since $B(a)$ is skew symmetric and non-degenerate, the dimension of $\mathfrak{z}^\perp$ is even, *i.e.* $\dim\mathfrak{z}^\perp = 2d$.

We can choose an orthonormal basis

$$\{E_1(a), E_2(a), \dots, E_d(a), \overline{E}_1(a), \overline{E}_2(a), \dots, \overline{E}_d(a)\}$$

of $\mathfrak{z}^\perp$ such that

$$B(a)E_i(a) = |z_a| J_{\frac{z_a}{|z_a|}} E_i(a) = |a|\overline{E}_i(a)$$

and

$$B(a)\overline{E}_i(a) = -|a|E_i(a).$$

We set $p = \dim\mathfrak{z}$. We can choose an orthonormal basis $\{\epsilon_1, \epsilon_2, \dots, \epsilon_p\}$ of $\mathfrak{z}$ such that $a(\epsilon_1) = |a|, a(\epsilon_j) = 0, j = 2, 3, \dots, p$. Then we can denote the element of $\mathfrak{g}$ by

$$(z,t) = (x,y,t) = \sum_{i=1}^{d}(x_iE_i + y_i\overline{E}_i) + \sum_{j=1}^{p} s_j\epsilon_j.$$

We identify G with its Lie algebra $\mathfrak{g}$ by the exponential map. The group law on H-type group G has the form

$$(z,s)(z',s') = \left(z + z', s + s' + \frac{1}{2}[z,z']\right), \tag{3}$$

where $[z,z']_j = \langle z, U^j z'\rangle$ for a suitable skew-symmetric matrix $U^j, j = 1, 2, \dots, p$.

Theorem 2.1 *G is an H-type group with underlying manifold $\mathbb{R}^{2d+p}$, with the group law* (3) *and the matrix $U^j, j = 1, 2, \dots, p$, satisfies the following conditions*:

(i) *U^j is a $2d \times 2d$ skew-symmetric and orthogonal matrix, $j = 1, 2, \dots, p$.*

(ii) *$U^iU^j + U^jU^i = 0, i, j = 1, 2, \dots, p$ with $i \neq j$.*

Proof See [10]. □

Remark 2.1 It is well known that H-type algebras are closely related to Clifford modules (see [11]). H-type algebras can be classified by the standard theory of Clifford algebras. Specially, on H-type group G, there is a relation between the dimension of the center and its orthogonal complement space. That is $p + 1 \leq 2d$ (see [12]).

Remark 2.2 We identify G with $\mathbb{R}^{2d} \times \mathbb{R}^p$ and denote by $n = 2d + p$ its topological dimension. Following Folland and Stein (see [13]), we will exploit the canonical homogeneous structure, given by the family of dilations $\{\delta_r\}_{r>0}$,

$$\delta_r(z,s) = (rz, r^2s).$$

We then define the homogeneous dimension of G by $N = 2d + 2p$.

The left invariant vector fields which agree, respectively, with $\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}$ at the origin are given by

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2}\sum_{k=1}^{p}\left(\sum_{l=1}^{2d} z_l U_{l,j}^k\right)\frac{\partial}{\partial s_k},$$

$$Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2}\sum_{k=1}^{p}\left(\sum_{l=1}^{2d} z_l U_{l,j+d}^k\right)\frac{\partial}{\partial s_k},$$

where $z_l = x_l, z_{l+d} = y_l, l = 1, 2, \dots, d$. In terms of these vector fields we introduce the sublaplacian Δ by

$$\Delta = -\sum_{j=1}^{d}\left(X_j^2 + Y_j^2\right).$$

2.2 Spherical Fourier transform

Korányi [14], Damek, and Ricci [15] have computed the spherical functions associated to the Gelfand pair $(G, O(d))$ (we identify $O(d)$ with $O(d) \otimes \mathrm{Id}_p$). They involve, as on the Heisenberg group, the Laguerre functions

$$\mathfrak{L}_m^{(\gamma)}(\tau) = L_m^{(\gamma)}(\tau)e^{-\tau/2}, \quad \tau \in \mathbb{R}, m, \gamma \in \mathbb{N},$$

where $L_m^{(\gamma)}$ is the Laguerre polynomial of type γ and degree m.

We say a function f on G is radial if the value of $f(z,s)$ depends only on $|z|$ and s. We denote, respectively, by $\mathscr{S}_{\mathrm{rad}}(G)$ and $L_{\mathrm{rad}}^q(G)$, $1 \le q \le \infty$, the spaces of radial functions in $\mathscr{S}(G)$ and $L^p(G)$. In particular, the set of $L_{\mathrm{rad}}^1(G)$ endowed with the convolution product

$$f_1 * f_2(g) = \int_G f_1\left(gg'^{-1}\right)f_2\left(g'\right)dg', \quad g \in G,$$

is a commutative algebra.

Let $f \in L_{\mathrm{rad}}^1(G)$. We define the spherical Fourier transform, $m \in \mathbb{N}, \lambda \in \mathbb{R}^p$,

$$\hat{f}(\lambda, m) = \binom{m+d-1}{m}^{-1}\int_{\mathbb{R}^{2d+p}} e^{i\lambda s}f(z,s)\mathfrak{L}_m^{(d-1)}\left(\frac{|\lambda|}{2}|z|^2\right)dz\,ds.$$

By a direct computation, we have $\widehat{f_1 * f_2} = \hat{f}_1 \cdot \hat{f}_2$. Thanks to a partial integration on the sphere $\mathbb{S}^{p-1}$, we deduce from the Plancherel theorem on the Heisenberg group its analog for the H-type groups.

Proposition 2.1 *For all $f \in \mathscr{S}_{\mathrm{rad}}(G)$ such that*

$$\sum_{m\in\mathbb{N}}\binom{m+d-1}{m}\int_{\mathbb{R}^p}\left|\hat{f}(\lambda, m)\right||\lambda|^d\,d\lambda < \infty,$$

we have

$$f(z,s) = \left(\frac{1}{2\pi}\right)^{d+p} \sum_{m\in\mathbb{N}} \int_{\mathbb{R}^p} e^{-i\lambda\cdot s} \hat{f}(\lambda,m) \mathfrak{L}_m^{(d-1)}\left(\frac{|\lambda|}{2}|z|^2\right) |\lambda|^d \, d\lambda, \tag{4}$$

the sum being convergent in L^∞ norm.

Moreover, if $f \in \mathscr{S}_{\mathrm{rad}}(G)$, the functions Δf is also in $\mathscr{S}_{\mathrm{rad}}(G)$ and its spherical Fourier transform is given by

$$\widehat{\Delta f}(\lambda,m) = (2m+d)|\lambda|\hat{f}(\lambda,m).$$

The sublaplacian Δ is a positive self-adjoint operator densely defined on $L^2(G)$. So by the spectral theorem, for any bounded Borel function h on $\mathbb{R}$, we have

$$\widehat{h(\Delta)f}(\lambda,m) = h\big((2m+d)|\lambda|\big)\hat{f}(\lambda,m).$$

2.3 Homogeneous Besov spaces

We shall recall the homogeneous Besov spaces given in [4]. Let R be a non-negative, even function in $C_c^\infty(\mathbb{R})$ such that $\operatorname{supp} R \subseteq \{\tau \in \mathbb{R} : \frac{1}{2} \leq |\tau| \leq 4\}$ and

$$\sum_{j\in\mathbb{Z}} R\big(2^{-2j}\tau\big) = 1, \quad \forall \tau \neq 0.$$

For $j \in \mathbb{Z}$, we denote by φ and φ_j, respectively, the kernel of the operator $R(\Delta)$ and $R(2^{-2j}\Delta)$. As $R \in C_c^\infty(\mathbb{R})$, Hulanicki [16] proved that $\varphi \in \mathscr{S}_{\mathrm{rad}}(G)$ and obviously $\varphi_j(z,s) = 2^{Nj}\varphi(\delta_{2^j}(z,s))$. For any $f \in \mathscr{S}'(G)$, we set $\Delta_j f = f * \varphi_j$.

By the spectral theorem, for any $f \in L^2(G)$, the following homogeneous Littlewood-Paley decomposition holds:

$$f = \sum_{j\in\mathbb{Z}} \Delta_j f \quad \text{in } L^2(G).$$

So

$$\|f\|_{L^\infty(G)} \leq \sum_{j\in\mathbb{Z}} \|\Delta_j f\|_{L^\infty(G)}, \quad f \in L^2(G), \tag{5}$$

where both sides of (5) are allowed to be infinite.

Let $1 \leq q, r \leq \infty$, $\rho < N/q$, we define the homogeneous Besov space $\dot{B}^\rho_{q,r}$ as the set of distributions $f \in \mathscr{S}'(G)$ such that

$$\|f\|_{\dot{B}^\rho_{q,r}} = \left(\sum_{j\in\mathbb{Z}} 2^{j\rho r} \|\Delta_j f\|_q^r\right)^{\frac{1}{r}} < \infty,$$

and $f = \sum_{j\in\mathbb{Z}} \Delta_j f$ in $\mathscr{S}'(G)$.

Let $\rho < N/q$. The homogeneous Sobolev space $\dot{H}^\rho$ is

$$\dot{H}^\rho = \dot{B}^0_{2,2},$$

which is equivalent to

$$u \in \dot{H}^{\rho} \quad \Leftrightarrow \quad \Delta^{\rho/2} u \in L^2.$$

Analogous to Proposition 6 of [5] on the Heisenberg group, we list some properties of the spaces $\dot{B}^{\rho}_{q,r}$ in the following proposition.

Proposition 2.2 *Let $q, r \in [1, \infty]$ and $\rho < N/q$.*

(i) *The space $\dot{B}^{\rho}_{q,r}$ is a Banach space with the norm $\|\cdot\|_{\dot{B}^{\rho}_{q,r}}$;*
(ii) *the definition of $\dot{B}^{\rho}_{q,r}$ does not depend on the choice of the function R in the Littlewood-Paley decomposition;*
(iii) *for $-\frac{N}{q'} < \rho < \frac{N}{q}$ the dual space of $\dot{B}^{\rho}_{q,r}$ is $\dot{B}^{-\rho}_{q',r'}$;*
(iv) *for any $u \in \mathscr{S}'(G)$ and $\sigma > 0$, then $u \in \dot{B}^{\rho}_{q,r}$ if and only if $L^{\sigma/2} u \in \dot{B}^{\rho-\sigma}_{q,r}$;*
(v) *for any $q_1, q_2 \in [1, \infty]$, the continuous inclusion holds:*

$$\dot{B}^{\rho_1}_{q_1,r} \subseteq \dot{B}^{\rho_2}_{q_2,r}, \quad \frac{1}{q_1} - \frac{\rho_1}{N} = \frac{1}{q_2} - \frac{\rho_2}{N}, \quad \rho_1 \geq \rho_2;$$

(vi) *for all $q \in [2, \infty]$ we have the continuous inclusion $\dot{B}^{0}_{q,2} \subseteq L^q$;*
(vii) $\dot{B}^{0}_{2,2} = L^2$.

3 Technical lemmas

By the inversion Fourier formula (4), we may write $e^{it\Delta^{\alpha}}\varphi$ explicitly into a sum of a list of oscillatory integrals. In order to estimate the oscillatory integrals, we recall the stationary phase lemma.

Lemma 3.1 (see [17]) *Let $g \in C^{\infty}([a, b])$ be real-valued such that*

$$\left|g''(x)\right| \geq \delta$$

for any $x \in [a, b]$ with $\delta > 0$. Then for any function $h \in C^{\infty}([a, b])$, there exists a constant C which does not depend on δ, a, b, g or h, such that

$$\left|\int_a^b e^{ig(x)} h(x)\, dx\right| \leq C\delta^{-1/2}\left(\|h\|_{\infty} + \left\|h'\right\|_1\right).$$

In order to prove the sharpness of the time decay in Theorem 1.1, we describe the asymptotic expansion of oscillating integrals.

Lemma 3.2 (see [17]) *Suppose ϕ is a smooth function on $\mathbb{R}^p$ and has a non-degenerate critical point at $\bar{\lambda}$. If ψ is supported in a sufficiently small neighborhood of $\bar{\lambda}$, then*

$$\left|\int_{\mathbb{R}^p} e^{it\phi(\lambda)} \psi(\lambda)\, d\lambda\right| \sim |t|^{-p/2}, \quad \text{as } t \to \infty.$$

Besides, it will involve the Laguerre functions when we estimate the oscillatory integrals. We need the following estimates.

Lemma 3.3 (see [4])

$$\left|\left(\tau\frac{d}{d\tau}\right)^{\gamma}\mathfrak{L}_m^{(d-1)}(\tau)\right|\le C_{\gamma,d}(2m+d)^{d-1/4}$$

for all $0\le\gamma\le d$.

Finally, we introduce the following properties of the Fourier transform of surface-carried measures.

Theorem 3.1 (see [18]) *Let S be a smooth hypersurface in $\mathbb{R}^p$ with non-vanishing Gaussian curvature and $d\mu$ a C_0^∞ measure on S. Suppose that $\Gamma\subset\mathbb{R}^p\setminus\{0\}$ is the cone consisting of all ξ which are normal of some point $x\in S$ belonging to a fixed relatively compact neighborhood $\mathcal{N}$ of* $\operatorname{supp} d\mu$. *Then*

$$\left(\frac{\partial}{\partial\xi}\right)^{\nu}\widehat{d\mu}(\xi)=O\bigl(\bigl(1+|\xi|\bigr)^{-M}\bigr),\quad \forall M\in\mathbb{N},\ \textit{if } \xi\notin\Gamma,$$
$$\widehat{d\mu}(\xi)=\sum e^{-i(x_j,\xi)}a_j(\xi),\quad \textit{if } \xi\in\Gamma,$$

where the (*finite*) *sum is taken over all points $x\in\mathcal{N}$ having ξ as a normal and*

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\nu}a_j(\xi)\right|\le C_\nu\bigl(1+|\xi|\bigr)^{-(p-1)/2-|\nu|}.$$

Here, we need the following properties of the Fourier transform of the measure $d\sigma$ on the sphere $\mathbb{S}^{p-1}$. Obviously, $\widehat{d\sigma}$ is radial. By Theorem 3.1, we have the radical decay properties of the Fourier transform of the spherical measure.

Lemma 3.4 *For any $\xi\in\mathbb{R}^p$, the estimate holds*

$$\widehat{d\sigma}(\xi)=e^{i|\xi|}\phi_+\bigl(|\xi|\bigr)+e^{-i|\xi|}\phi_-\bigl(|\xi|\bigr),$$

where

$$\bigl|\phi_\pm^{(k)}(r)\bigr|\le c_k(1+r)^{-(p-1)/2-k},\quad \textit{for all } r>0, k\in\mathbb{N}.$$

4 Dispersive estimates

Lemma 4.1 *Let $0<\alpha<1$. The kernel of φ of $R(\Delta)$ introduced in Section 2 satisfies the estimate*

$$\sup_z\bigl|e^{it\Delta^\alpha}\varphi(z,s)\bigr|\le C_\alpha|t|^{-1/2}|s|^{(1-p)/2}.$$

Proof By the inversion Fourier formula (4) and polar coordinate changes, we have

$$\begin{aligned}e^{it\Delta^\alpha}\varphi(z,s)={}&\left(\frac{1}{2\pi}\right)^{d+p}\sum_{m\in\mathbb{N}}\int_{\mathbb{R}^p}e^{-i\lambda\cdot s+it(2m+d)^\alpha|\lambda|^\alpha}\\&\times R\bigl((2m+d)|\lambda|\bigr)\mathfrak{L}_m^{(d-1)}\left(\frac{|\lambda|}{2}|z|^2\right)|\lambda|^d\,d\lambda\end{aligned}$$

$$= \left(\frac{1}{2\pi}\right)^{d+p} \sum_{m\in\mathbb{N}} \int_{\mathbb{S}^{p-1}} \int_0^{+\infty} e^{-i\lambda\varepsilon\cdot s+it(2m+d)^\alpha\lambda^\alpha} \times R\big((2m+d)\lambda\big)\mathfrak{L}_m^{(d-1)}\left(\frac{\lambda}{2}|z|^2\right)\lambda^{d+p-1}\,d\lambda\,d\sigma(\varepsilon). \tag{6}$$

The expression after the $\mathbb{S}^{p-1}$ integral sign in (6) is very similar to an integral computed in [1] or [4] (see the proof of Lemma 4.1). Integrating the result over $\mathbb{S}^{p-1}$ gives us

$$\sup_z \left|e^{it\Delta^\alpha}\varphi(z,s)\right| \le C_\alpha \min\left\{1, |t|^{-1/2}\right\}, \tag{7}$$

and Lemma 4.1 will come out only if we prove the case for $p \ge 2$ and $|s| > 1$. By switching the order of the integration in (6), it follows from Lemma 3.4 that

$$\begin{aligned} e^{it\Delta^\alpha}\varphi(z,s) &= \left(\frac{1}{2\pi}\right)^{d+p} \sum_{m\in\mathbb{N}} \int_0^{+\infty} \widehat{d\sigma}(\lambda s)e^{it(2m+d)^\alpha\lambda^\alpha} R\big((2m+d)\lambda\big) \\ &\quad \times \mathfrak{L}_m^{(d-1)}\left(\frac{\lambda}{2}|z|^2\right)\lambda^{d+p-1}\,d\lambda \\ &= \left(\frac{1}{2\pi}\right)^{d+p} \sum_{m\in\mathbb{N}} \int_0^{+\infty} \left(e^{i\lambda|s|}\phi_+\big(\lambda|s|\big) + e^{-i\lambda|s|}\phi_-\big(\lambda|s|\big)\right)e^{it(2m+d)^\alpha\lambda^\alpha} \\ &\quad \times R\big((2m+d)\lambda\big)\mathfrak{L}_m^{(d-1)}\left(\frac{\lambda}{2}|z|^2\right)\lambda^{d+p-1}\,d\lambda \\ &:= \left(\frac{1}{2\pi}\right)^{d+p} \sum_{m\in\mathbb{N}} \left(I_m^+ + I_m^-\right). \end{aligned}$$

Then it suffices to study

$$I_m^\pm = \int_0^{+\infty} e^{i(\pm\lambda|s|+t(2m+d)^\alpha\lambda^\alpha)}\phi_\pm\big(\lambda|s|\big)R\big((2m+d)\lambda\big)\mathfrak{L}_m^{(d-1)}\left(\frac{\lambda}{2}|z|^2\right)\lambda^{d+p-1}\,d\lambda.$$

Performing the change of variables, $\mu = (2m+d)\lambda$, recall that $R \in C_c^\infty(\mathbb{R})$,

$$I_m^\pm = \int_{1/2}^4 e^{itg_{m,s,t}^\pm(\mu)}h_{m,s,z}(\mu)\,d\lambda,$$

where

$$\begin{aligned} g_{m,s,t}^\pm(\mu) &= \pm\frac{\mu|s|}{(2m+d)t} + \mu^\alpha, \\ h_{m,s,z}(\mu) &= \phi_\pm\left(\frac{\mu|s|}{2m+d}\right)R(\mu)\mathfrak{L}_m^{(d-1)}\left(\frac{\mu|z|^2}{2(2m+d)}\right)\frac{\mu^{d+p-1}}{(2m+d)^{d+p}}. \end{aligned}$$

By Lemma 3.3 and Lemma 3.4, we get

$$\|h_{m,s,z}\|_\infty + \left\|h'_{m,s,z}\right\|_1 \le C(2m+d)^{-(2p+3)/4}|s|^{-(p-1)/2}.$$

Since $|(g_{m,s,t}^\pm)''| \ge \alpha|\alpha-1|2^{-\alpha-4}$, applying Lemma 3.1 on $I_m^\pm$ gives us

$$\left|I_m^\pm\right| \le C_\alpha(2m+d)^{-(2p+3)/4}|t|^{-1/2}|s|^{-(p-1)/2}. \tag{8}$$

To conclude it suffices to sum these estimates since

$$\sum_{m\in\mathbb{N}}(2m+d)^{-(2p+3)/4}<+\infty.$$

The decay estimate of time is sharp in the joint space-time cone

$$\{(s,t)\in\mathbb{R}^p\times\mathbb{R}:s=Ct\}.\qquad\square$$

We will prove the sharp dispersive estimate.

Lemma 4.2 *Let* $0<\alpha<1$. *The kernel of* φ *of* $R(\Delta)$ *introduced in Section* 2 *satisfies the estimate*

$$\sup_{z,s}\left|e^{it\Delta^\alpha}\varphi(z,s)\right|\leq C_\alpha|t|^{-p/2}.$$

Proof From (7), it suffices to show the inequality $|t|>1$. Recall from (6) that

$$e^{it\Delta^\alpha}\varphi(z,s)=\left(\frac{1}{2\pi}\right)^{d+p}\sum_{m\in\mathbb{N}}\int_{\mathbb{S}^{p-1}}I_{m,\varepsilon}\,d\sigma(\varepsilon),$$

where

$$\begin{aligned}I_{m,\varepsilon}&=\int_0^{+\infty}e^{-i\lambda\varepsilon\cdot s+it(2m+d)^\alpha\lambda^\alpha}R\bigl((2m+d)\lambda\bigr)\mathfrak{L}_m^{(d-1)}\left(\frac{\lambda}{2}|z|^2\right)\lambda^{d+p-1}\,d\lambda\\&=\int_{1/2}^{4}e^{itG_{m,\varepsilon,s,t}(\mu)}H_{m,z}(\mu)\,d\mu\end{aligned}$$

with

$$\begin{aligned}G_{m,\varepsilon,s,t}(\mu)&=\mu^\alpha-\frac{\mu}{(2m+d)t}\varepsilon\cdot s,\\H_{m,z}(\mu)&=R(\mu)\mathfrak{L}_m^{(d-1)}\left(\frac{\mu|z|^2}{2(2m+d)}\right)\frac{\mu^{d+p-1}}{(2m+d)^{d+p}}.\end{aligned}$$

We will try to apply Q times a non-critical phase estimate to the oscillatory integral $I_{m,\varepsilon}$.

Case 1: $|s|\geq\alpha2^{-\alpha-3}(2m+d)|t|$. By (8),

$$\left|\int_{\mathbb{S}^{p-1}}I_{m,\varepsilon}\,d\sigma(\varepsilon)\right|=\left|I_m^++I_m^-\right|\leq C_\alpha(2m+d)^{-p-1/4}|t|^{-p/2}.$$

Case 2: $|s|\leq\alpha2^{-\alpha-3}(2m+d)|t|$. We get

$$G'_{m,\varepsilon,s,t}(\mu)=\alpha\mu^{\alpha-1}-\frac{\varepsilon\cdot s}{(2m+d)t}\geq\alpha2^{-\alpha-2}-\frac{|s|}{(2m+d)|t|}\geq\alpha2^{-\alpha-3}.$$

Here the phase function $G_{m,\varepsilon,s,t}$ has no critical point on $[1/2,4]$. By Q-fold ($1\leq Q\leq d$) integration by parts, we have

$$I_{m,\varepsilon}=(it)^{-Q}\int_{1/2}^{4}e^{itG_{m,\varepsilon,s,t}(\mu)}D^Q\bigl(H_{m,z}(\mu)\bigr)\,d\mu,$$

where the differential operator D is defined by

$$DH_{m,z} = \frac{d}{d\mu}\left(\frac{H_{m,z}(\mu)}{G'_{m,\varepsilon,s,t}(\mu)}\right).$$

By a direct induction,

$$D^Q H_{m,z} = \sum_{k=Q}^{2Q}\sum_{\wr\beta\wr=k} C(\beta,k,Q)\frac{H_{m,z}^{(\beta_1)}(G''_{m,\varepsilon,s,t})^{\beta_2}\cdots(G_{m,\varepsilon,s,t}^{(Q+1)})^{\beta_{Q+1}}}{(G'_{m,\varepsilon,s,t})^k},$$

where $\beta = (\beta_1,\dots,\beta_{Q+1}) \in \{0,\dots,Q\}\times\mathbb{N}^Q$ and $\wr\beta\wr = \sum_{j=1}^{Q+1} j\beta_j$.

A direct calculation shows that

$$\left|G_{m,\varepsilon,s,t}^{(l)}(\mu)\right| = \alpha\prod_{j=1}^{l-1}(j-\alpha)\mu^{-l+\alpha} \le 2^{l+2\alpha}\alpha\prod_{j=1}^{l-1}(j-\alpha) \le C(\alpha,Q), \quad l\ge 2.$$

Using Lemma 3.3,

$$\left|H_{m,z}^{(\beta_1)}(\mu)\right| \le C(\beta_1)(2m+d)^{-p-1/4}.$$

Hence, we have

$$|I_{m,\varepsilon}| \le C(\alpha,Q)|t|^{-Q}\sup_{1\le\beta_1\le Q}\left\|H_{m,z}^{(\beta_1)}\right\|_\infty \le C(\alpha,Q)|t|^{-Q}(2m+d)^{-p-1/4}.$$

Taking $Q=d$, since $|t|>1$ and $p\le 2d-1$, which implies $p/2<d$, it follows that

$$|I_{m,\varepsilon}| \le C_\alpha|t|^{-p/2}(2m+d)^{-p-1/4}.$$

It immediately leads to

$$\left|\int_{\mathbb{S}^{p-1}} I_{m,\varepsilon}\,d\sigma(\varepsilon)\right| \le C_\alpha|t|^{-p/2}(2m+d)^{-p-1/4}.$$

Combining the two cases, by a straightforward summation

$$\left|e^{it\Delta^\alpha}\varphi(z,s)\right| \le C_\alpha|t|^{-p/2}\sum_{m\in\mathbb{N}}(2m+d)^{-p-1/4} \le C_\alpha|t|^{-p/2}.$$

The lemma is proved. □

Proof of Theorem 1.1 The dispersive inequality in Theorem 1.1 is a direct consequence of Lemma 4.2 (see [1]). It suffices to show the sharpness of the estimate. Let $Q\in C_c^\infty([1/2,2])$ with $Q(1)=1$. Choose u_0 such that

$$\hat{u}_0(\lambda,m) = \begin{cases} Q(|\lambda|), & m=0,\\ 0, & m\ge 1.\end{cases}$$

By the inversion Fourier formula (4), then we have

$$e^{it\Delta^\alpha} u_0(z,s) = \left(\frac{1}{2\pi}\right)^{d+p} \int_{\mathbb{R}^p} e^{-i\lambda\cdot s+itd^\alpha|\lambda|^\alpha} Q(|\lambda|) e^{-|\lambda||z|^2/4} |\lambda|^d \, d\lambda.$$

Consider $e^{it\Delta^\alpha} u_0(0,t\bar{s})$ for a fixed $\bar{s} = \alpha d^\alpha(0,\ldots,0,1)$. The above oscillatory integral has a phase

$$\Phi(\lambda) = -\lambda\cdot\bar{s} + d^\alpha|\lambda|^\alpha$$

with a unique non-degenerate critical point $\bar{\lambda} = \alpha^{-1}d^{-\alpha}\bar{s} = (0,\ldots,0,1)$. Indeed, the Hessian is equal to

$$H(\bar{\lambda}) = \alpha d^\alpha |\bar{\lambda}|^{\alpha-4}\big((\alpha-2)\bar{\lambda}_k\bar{\lambda}_l + |\bar{\lambda}|^2\delta_{k,l}\big)_{1\le k,l\le p} = \alpha d^\alpha \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \alpha-1 \end{pmatrix}.$$

So by Lemma 3.2, it yields

$$e^{it\Delta^\alpha} u_0(0,t\bar{s}) \sim C|t|^{-p/2}. \qquad \square$$

5 Strichartz inequalities

In this section, we shall prove the Strichartz inequalities by the decay estimate in Lemma 4.2. We obtain the intermediate results as follows. We omit the proof and refer to [2, 3].

Theorem 5.1 *Let* $0<\alpha<1$. *For* $i=1,2$, *let* $q_i, r_i \in [2,\infty]$ *and* $\rho_i\in\mathbb{R}$ *such that*

(1) $2/q_i = p(1/2-1/r_i)$;

(2) $\rho_i = -(N-p/2)(1/2-1/r_i)$,

except for $(q_i,r_i,p)=(2,\infty,2)$. *Let* q_i', r_i' *denote the conjugate exponent of* q_i, r_i *for* $i=1,2$. *Then the following estimates are satisfied*:

$$\big\|e^{it\Delta^\alpha}u_0\big\|_{L^{q_1}(\mathbb{R},\dot{B}^{\rho_1}_{r_1,2})} \le C\|u_0\|_{L^2},$$

$$\left\|\int_0^t e^{i(t-\tau)\Delta^\alpha} f(\tau)\,d\tau\right\|_{L^{q_1}((0,T),\dot{B}^{\rho_1}_{r_1,2})} \le C\|f\|_{L^{q_2'}((0,T),\dot{B}^{-\rho_2}_{r_2',2})},$$

where the constant $C>0$ *does not depend on* u_0, f *or* T.

Consider the non-homogeneous fractional wave equation (1). The general solution is given by

$$u(t) = e^{it\Delta^\alpha}u_0 - i\int_0^t e^{i(t-\tau)\Delta^\alpha} f(\tau)\,d\tau.$$

Theorem 5.2 *Under the same hypotheses as in Theorem* 5.1, *the solution of the fractional wave equation* (1) *satisfies the following estimate*:

$$\|u\|_{L^{q_1}((0,T),\dot{B}^{\rho_1}_{r_1,2})} \le C\big(\|u_0\|_{L^2} + \|f\|_{L^{q_2'}((0,T),\dot{B}^{-\rho_2}_{r_2',2})}\big),$$

where the constant $C > 0$ does not depend on u_0, f or T.

Applying Proposition 2.2, by direct Besov space injections, we immediately obtain the Strichartz inequalities on Lebesgue spaces in Corollary 1.1.

Competing interests
The author declares to have no competing interests.

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