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Some equalities and inequalities for probabilistic frames

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Abstract

Probabilistic frames have some properties which are similar to those of frames in Hilbert space. Some equalities and inequalities have been established for traditional frames. In this paper, we give some equalities and inequalities for probabilistic frames. Our results generalize and improve the remarkable results which have been obtained.

Keywords: frame; probabilistic frame; equality

1 Introduction

Frames are redundant systems of vectors for a Hilbert space, which can yield many different and stable representations for a given vector [1]. The frame was first introduced by Duffin and Schaeffer in the context of nonharmonic Fourier series [2]. To date, frame theory has broad applications in pure mathematics, for instance, the Kadison-Singer problem [3] and statistics [4], as well as in applied mathematics, computer science, and emerging applications.

Due to the redundancy of frames, the frame has become an essential tool in signal processing such as wireless communication [5, 6], image processing [7], coding theory [8], and sampling theory [9]. These applications led to resilience to additive noise and quantization [10, 11], resilience to erasures [12–15], and numerical stability of reconstructions [16, 17].

By viewing the frame vectors as discrete mass distributions on $\mathbb{R}^N$, being the generation of frames, probabilistic frames were developed by Ehler [18] and further expanded in [19]. Due to the connections between probability measures and frame theory, probabilistic frames are tightly related to various notions that appeared in areas such as the theory of t-designs [20], positive operator valued measures encountered in quantum computing [21, 22], and isometric measures used in the study of convex bodies [23]. Now, some excellent results of class frames have been obtained and applied successfully. It is necessary to extend some important results of conventional frames to the probabilistic frames.

In this paper, we mainly research the equalities and inequalities of probabilistic frames. Balan *et al.* obtained an identity when studying the optimal decomposition of Parseval frames [24], and they discovered a surprising identity for Parseval frames when working on reconstructing signal without noisy phase or its estimation in [25]. Subsequently, some authors found and improved some equalities or inequalities of the traditional frames based on the work of Balan *et al.*

First we will recall the definition and some properties of probabilistic frames in Hilbert spaces.

Throughout this paper $\mathcal{H}$ will always denote a Hilbert space, I denotes a countable indexing set and $I_{\mathcal{H}}$ denotes the identity operator on $\mathcal{H}$. A system $\{f_i\}_{i\in I}$ is called a frame for $\mathcal{H}$ if there exist the constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2$$

for all $f \in \mathcal{H}$. The constants A and B are called lower and upper frame bounds, respectively. If $A = B$, then the frame is called an A-tight frame, and if $A = B = 1$, then it is called a Parseval frame. A Bessel sequence $\{f_i\}_{i\in I}$ is only required to fulfill the upper frame bound estimate but not necessarily the lower estimate.

For more details on conventional frames we refer to [1, 26].

Let I be a nonempty subset of $\mathbb{R}^N$ and let $\mathcal{M}(\mathcal{B}, I)$ denote the collection of probability measures on I with respect to the Borel σ-algebra $\mathcal{B}$.

Definition 1 A probability measure $\mu \in \mathcal{M}(\mathcal{B}, I)$ is called a probabilistic frame for $\mathcal{H}$ if there are constants $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \int_I |\langle x, y\rangle|^2 \, d\mu(y) \leq B\|x\|^2 \quad \text{for all } f \in \mathcal{H}.$$

The constants A and B are called lower and upper probabilistic frame bounds, respectively. If $A = B$, then the frame is called a probabilistic A-tight frame for $\mathcal{H}$, and if $A = B = 1$, then it is called a probabilistic Parseval frame. If only the upper inequality holds, then we call μ a Bessel measure.

Let $\mu \in \mathcal{M}(\mathcal{B}, I)$ be a probabilistic frame. The probabilistic analysis operator is given by

$$T : \mathcal{H} \to L^2(I, \mu), \qquad x \mapsto \langle x, \cdot\rangle.$$

The adjoint operator T^* of T is called the probabilistic synthesis operator which is given by

$$T^* : L^2(I, \mu) \to \mathcal{H}, \qquad f \mapsto \int_I f(x)x \, d\mu(x).$$

The probabilistic frame operator of μ is $S = T^*T$, and one easily verifies that

$$S : \mathcal{H} \to \mathcal{H}, \qquad S(x) = \int_I \langle x, y\rangle y \, d\mu(y)$$

is positive, self-adjoint, and invertible.

For any $J \subset I$, we define a bounded linear operator S_J as

$$S_J(x) = \int_J \langle x, y\rangle y \, d\mu(y) \quad \text{for all } x \in \mathcal{H},$$

and denote $J^c = I \setminus J$.

Moreover, for $\tilde{\mu} = \mu \circ S$, we have

$$\int_I f\bigl(S^{-1}(y)\bigr)\,d\mu(y) = \int_I f(y)\,d\tilde{\mu}.$$

Using the fact that $S^{-1}S = SS^{-1} = I_{\mathcal{H}}$, the reconstruction formula is given by

$$x = \int_I \langle x, y\rangle Sy\,d\tilde{\mu}(y) = \int_I \bigl\langle x, S^{-1}y\bigr\rangle y\,d\mu(y)$$

for all $x \in \mathcal{H}$.

Definition 2 If $\mu \in \mathcal{M}(\mathcal{B}, I)$ is a probabilistic frame, then $\tilde{\mu} = \mu \circ S$ is called the probabilistic canonical dual frame of μ.

Proposition 1 *If $\mu \in \mathcal{M}(\mathcal{B}, I)$ is a probabilistic frame, then $\tilde{\mu} = \mu \circ S^{1/2}$ is a probabilistic Parseval frame for $\mathcal{H}$.*

We refer to [18, 19, 27] for more details on probabilistic frames.

In order to compare with our result, we list some important equalities as follows.

Theorem 1 [28] *Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ with canonical dual frame $\{g_i\}_{g_i}$. Then for all $J \subset I$ and all $f \in \mathcal{H}$ we have*

$$\sum_{i\in J}\bigl|\langle f, f_i\rangle\bigr|^2 - \sum_{i\in I}\bigl|\langle S_J f, g_i\rangle\bigr|^2 = \sum_{i\in J^c}\bigl|\langle f, f_i\rangle\bigr|^2 - \sum_{i\in I}\bigl|\langle S_{J^c} f, g_i\rangle\bigr|^2.$$

In the situation of Parseval frames, the authors of [28] gave the new identity which is given by

$$\sum_{i\in J}\bigl|\langle f, f_i\rangle\bigr|^2 - \biggl\|\sum_{i\in J}\langle f, f_i\rangle f_i\biggr\|^2 = \sum_{i\in J^c}\bigl|\langle f, f_i\rangle\bigr|^2 - \biggl\|\sum_{i\in J^c}\langle f, f_i\rangle f_i\biggr\|^2. \tag{1}$$

Then the general result for (1) was established in [29] as follows.

Theorem 2 *Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ with canonical dual frame $\{g_i\}_{g_i}$. Then for all $J \subset I$ and all $f \in \mathcal{H}$, we have*

$$\operatorname{Re}\biggl(\sum_{i\in J}\langle f, g_i\rangle\overline{\langle f, f_i\rangle}\biggr) - \sum_{i\in I}\bigl|\langle S_J f, g_i\rangle\bigr|^2 = \operatorname{Re}\biggl(\sum_{i\in J^c}\langle f, g_i\rangle\overline{\langle f, f_i\rangle}\biggr) - \sum_{i\in I}\bigl|\langle S_{J^c} f, g_i\rangle\bigr|^2.$$

Note that the above result involves the real parts of some complex number. Zhu and Wu [30] generalized the above equality to a more general form which does not involve the real parts of the complex numbers.

Theorem 3 *Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ with canonical dual frame $\{g_i\}_{g_i}$. Then for all $J \subset I$ and all $f \in \mathcal{H}$, we have*

$$\sum_{i\in J}\langle f, g_i\rangle\overline{\langle f, f_i\rangle} - \sum_{i\in I}\bigl|\langle S_J f, g_i\rangle\bigr|^2 = \overline{\sum_{i\in J^c}\langle f, g_i\rangle\overline{\langle f, f_i\rangle}} - \sum_{i\in I}\bigl|\langle S_{J^c} f, g_i\rangle\bigr|^2.$$

Next, we extend these equalities to probabilistic frames.

2 The main result for probabilistic Parseval frames

In this section, we continue the work [28, 29] about probabilistic Parseval frames and obtain some important equalities and inequalities of these frames.

Lemma 1 [30] *Let P and Q be two linear bounded operators on $\mathcal{H}$ such that $P + Q = I_{\mathcal{H}}$. Then*

$$P - P^*P = Q^* - Q^*Q.$$

Then we have the following result.

Theorem 4 *If $\mu \in \mathcal{M}(\mathcal{B}, I)$ is a probabilistic Parseval frame for $\mathcal{H}$, then for all $J \subset I$ and all $x \in \mathcal{H}$, we have*

$$\begin{aligned}&\int_J \bigl|\langle x,y\rangle\bigr|^2\,d\mu(y) - \biggl\|\int_J \langle x,y\rangle y\,d\mu(y)\biggr\|^2\\&\quad= \int_{J^c} \bigl|\langle x,y\rangle\bigr|^2\,d\mu(y) - \biggl\|\int_{J^c} \langle x,y\rangle y\,d\mu(y)\biggr\|^2.\end{aligned} \tag{2}$$

Proof Since μ is a Parseval frame, we have $S = I_{\mathcal{H}}$, clearly, $S_J + S_{J^c} = I_{\mathcal{H}}$. Thus, by Lemma 1, we have

$$\begin{aligned}\int_J \bigl|\langle x,y\rangle\bigr|^2\,d\mu(y) - \biggl\|\int_J \langle x,y\rangle y\,d\mu(y)\biggr\|^2 &= \int_J \bigl|\langle x,y\rangle\bigr|^2\,d\mu(y) - \langle S_J x, S_J x\rangle\\&= \langle S_J x, x\rangle - \bigl\langle S_J^* S_J x, x\bigr\rangle\\&= \bigl\langle \bigl(S_J - S_J^* S_J\bigr)x, x\bigr\rangle\\&= \bigl\langle \bigl(S_{J^c}^* - S_{J^c}^* S_{J^c}\bigr)x, x\bigr\rangle\\&= \bigl\langle S_{J^c}^* x, x\bigr\rangle - \bigl\langle S_{J^c}^* S_{J^c} x, x\bigr\rangle\\&= \langle x, S_{J^c} x\rangle - \langle S_{J^c} x, S_{J^c} x\rangle\\&= \int_{J^c} \bigl|\langle x,y\rangle\bigr|^2\,d\mu(y) - \langle S_{J^c} x, S_{J^c} x\rangle\\&= \int_{J^c} \bigl|\langle x,y\rangle\bigr|^2\,d\mu(y) - \biggl\|\int_{J^c} \langle x,y\rangle y\,d\mu(y)\biggr\|^2.\end{aligned}$$ □

Note that each side of (2) is non-negative. An overlapping division of (2) is given as follows.

Proposition 2 *Let $\mu \in \mathcal{M}(\mathcal{B}, I)$ be a probabilistic Parseval frame for $\mathcal{H}$. For every $J \subset I$, every $E \subset J^c$, and all $x \in \mathcal{H}$, we have*

$$\begin{aligned}&\biggl\|\int_{J\cup E} \langle x,y\rangle y\,d\mu(y)\biggr\|^2 - \biggl\|\int_{J^c\setminus E} \langle x,y\rangle y\,d\mu(y)\biggr\|^2\\&\quad= \biggl\|\int_J \langle x,y\rangle y\,d\mu(y)\biggr\|^2 - \biggl\|\int_{J^c} \langle x,y\rangle y\,d\mu(y)\biggr\|^2 + 2\int_E \bigl|\langle x,y\rangle\bigr|^2\,d\mu(y).\end{aligned}$$

Proof Applying Theorem 4 twice, then we have

$$\begin{aligned}
&\left\|\int_{J\cup E}\langle x,y\rangle y\,d\mu(y)\right\|^2-\left\|\int_{J^c\setminus E}\langle x,y\rangle y\,d\mu(y)\right\|^2\\
&\quad=\int_{J\cup E}\left|\langle x,y\rangle\right|^2d\mu(y)-\int_{J^c\setminus E}\left|\langle x,y\rangle\right|^2d\mu(y)\\
&\quad=\int_J\left|\langle x,y\rangle\right|^2d\mu(y)-\int_{J^c}\left|\langle x,y\rangle\right|^2d\mu(y)+2\int_E\left|\langle x,y\rangle\right|^2d\mu(y)\\
&\quad=\left\|\int_J\langle x,y\rangle y\,d\mu(y)\right\|^2-\left\|\int_{J^c}\langle x,y\rangle y\,d\mu(y)\right\|^2+2\int_E\left|\langle x,y\rangle\right|^2d\mu(y).
\end{aligned}$$

□

Corollary 1 *Let $\mu\in\mathcal{M}(\mathcal{B},I)$ be a probabilistic Parseval frame for $\mathcal{H}$. For every $J\subset I$, every $F\subset J$, every $E\subset J^c$ and all $x\in\mathcal{H}$, we have*

$$\begin{aligned}
&\left\|\int_{(J\cup E)\setminus F}\langle x,y\rangle y\,d\mu(y)\right\|^2-\left\|\int_{(J^c\cup F)\setminus E}\langle x,y\rangle y\,d\mu(y)\right\|^2\\
&\quad=\left\|\int_J\langle x,y\rangle y\,d\mu(y)\right\|^2-\left\|\int_{J^c}\langle x,y\rangle y\,d\mu(y)\right\|^2+2\int_{E\cup F}\left|\langle x,y\rangle\right|^2d\mu(y).
\end{aligned}$$

The proof of Corollary 1 is immediate.

By Proposition 1, each probabilistic A-tight frame can be turned into a probabilistic Parseval frame.

Corollary 2 *Let $\mu\in\mathcal{M}(\mathcal{B},I)$ be a probabilistic tight Parseval frame with bound A for $\mathcal{H}$. For every $J\subset I$ and every $x\in\mathcal{H}$, we have*

$$A\int_J\left|\langle x,y\rangle\right|^2d\mu(y)-\left\|\int_J\langle x,y\rangle y\,d\mu(y)\right\|^2=A\int_{J^c}\left|\langle x,y\rangle\right|^2d\mu(y)-\left\|\int_{J^c}\langle x,y\rangle y\,d\mu(y)\right\|^2.$$

The proof of Corollary 2 is straightforward.

Next, we give a discussion of Theorem 4. From Theorem 4, for every $J\subset I$ and every $f\in\mathcal{H}$, we have

$$\int_J\left|\langle x,y\rangle\right|^2d\mu(y)+\left\|\int_{J^c}\langle x,y\rangle y\,d\mu(y)\right\|^2=\int_{J^c}\left|\langle x,y\rangle\right|^2d\mu(y)+\left\|\int_J\langle x,y\rangle y\,d\mu(y)\right\|^2.$$

The above equality leads us to introduce some notation: $\nu_-(\mathcal{U};J)$ and $\nu_+(\mathcal{U};J)$. Let $\mu\in\mathcal{M}(\mathcal{B},I)$ be a probabilistic Parseval frame for $\mathcal{H}$. For every $J\subset I$, we define

$$\begin{aligned}
\nu_-(\mathcal{U};J)&=\inf_{x\neq0}\frac{\int_{J^c}|\langle x,y\rangle|^2\,d\mu(y)+\|\int_J\langle x,y\rangle y\,d\mu(y)\|^2}{\|x\|^2},\\
\nu_+(\mathcal{U};J)&=\sup_{x\neq0}\frac{\int_{J^c}|\langle x,y\rangle|^2\,d\mu(y)+\|\int_J\langle x,y\rangle y\,d\mu(y)\|^2}{\|x\|^2}.
\end{aligned}$$

Some propositions of these notations are given in the following results.

Theorem 5 *The notations $\nu_-(\mathcal{U};J)$ and $\nu_+(\mathcal{U};J)$ have the following properties:*

(i) $\frac{3}{4} \leq \nu_-(\mathcal{U};J) \leq \nu_+(\mathcal{U};J) \leq 1$;
(ii) $\nu_-(\mathcal{U};J) = \nu_-(\mathcal{U};J^c)$ *and* $\nu_+(\mathcal{U};J) = \nu_+(\mathcal{U};J^c)$;
(iii) $\nu_-(\mathcal{U};I) = \nu_+(\mathcal{U};I)$ *and* $\nu_-(\mathcal{U};\emptyset) = \nu_+(\mathcal{U};\emptyset)$.

Proof (i) We first proof the first inequality. Since $S_J + S_{J^c} = I_{\mathcal{H}}$, then we have

$$S_J - S_{J^c} = 2S_J - I_{\mathcal{H}} = S_J^2 - \left(I_{\mathcal{H}} - 2S_J + S_J^2\right) = S_J^2 - (I_{\mathcal{H}} - S_J)^2 = S_J^2 - S_{J^c}^2.$$

Hence,

$$\begin{aligned} S_J + S_{J^c}^2 &= S_{J^c} + S_J^2 \\ &= \frac{1}{2}\left(S_J + S_{J^c} + S_J^2 + S_{J^c}^2\right) \\ &= \frac{1}{2}\left(I_{\mathcal{H}} + S_J^2 + S_{J^c}^2\right) \\ &= \frac{1}{2}\left(I_{\mathcal{H}} + \left(S_J^2 - \frac{1}{2}I_{\mathcal{H}}\right)^2 + \frac{1}{2}I_{\mathcal{H}}\right) \\ &\geq \frac{3}{4}I_{\mathcal{H}}, \end{aligned}$$

with equality if and only if $S_J^2 = \frac{1}{2}I_{\mathcal{H}}$. Therefore, for every $x \in \mathcal{H}$ and $x \neq 0$, we have

$$\begin{aligned} \nu_+(\mathcal{U};J) \geq \nu_-(\mathcal{U};J) &= \inf_{x\neq 0} \frac{\int_{J^c} |\langle x,y\rangle|^2 \, d\mu(y) + \|\int_J \langle x,y\rangle y \, d\mu(y)\|^2}{\|x\|^2} \\ &= \inf_{x\neq 0} \frac{\langle S_{J^c}x, x\rangle + \langle S_J x, S_J x\rangle}{\|x\|^2} \\ &= \frac{1}{2}\left(I_{\mathcal{H}} + \left(S_J^2 - \frac{1}{2}I_{\mathcal{H}}\right)^2 + \frac{1}{2}I_{\mathcal{H}}\right) \\ &\geq \frac{3}{4}, \end{aligned}$$

with equality if and only if $S_J^2 = \frac{1}{2}I_{\mathcal{H}}$.

Next, we prove the second inequality. Since

$$\begin{aligned} \left\| \int_J \langle x,y\rangle y \, d\mu(y) \right\|^2 &= \sup_{\|z\|=1} \left\| \left\langle \int_J \langle x,y\rangle y \, d\mu(y), z \right\rangle \right\|^2 \\ &= \sup_{\|z\|=1} \left\| \int_J \langle x,y\rangle \langle y,z\rangle \, d\mu(y) \right\|^2 \\ &\leq \sup_{\|z\|=1} \int_I \left|\langle z,y\rangle\right|^2 d\mu(y) \int_J \left|\langle x,y\rangle\right|^2 d\mu(y) \\ &= \sup_{\|z\|=1} \|z\|^2 \int_J \left|\langle x,y\rangle\right|^2 d\mu(y) \\ &= \int_J \left|\langle x,y\rangle\right|^2 d\mu(y), \end{aligned}$$

we have

$$\begin{aligned}\nu_-(\mathcal{U};J) \le \nu_+(\mathcal{U};J) &= \sup_{x\neq 0} \frac{\int_{J^c} |\langle x,y\rangle|^2\, d\mu(y) + \| \int_J \langle x,y\rangle y\, d\mu(y)\|^2}{\|x\|^2} \\ &\le \sup_{x\neq 0} \frac{\int_{J^c} |\langle x,y\rangle|^2\, d\mu(y) + \int_J |\langle x,y\rangle|^2\, d\mu(y)}{\|x\|^2} \\ &= \sup_{x\neq 0} \frac{\|x\|^2}{\|x\|^2} = 1;\end{aligned}$$

(ii) and (iii) follow from Theorem 4. □

Corollary 3 *Let $\mu \in \mathcal{M}(\mathcal{B}, I)$ be a probabilistic Parseval frame for $\mathcal{H}$. For every $J \subset I$ and every $x \in \mathcal{H}$, $\nu_-(\mathcal{U};J) = \nu_+(\mathcal{U};J) = 1$ if and only if $S_J x = S_J^2 x$.*

Proof From the definition of ν, $\nu_-(\mathcal{U};J) = \nu_+(\mathcal{U};J) = 1$ if and only if

$$\left\| \int_J \langle x,y\rangle y\, d\mu(y) \right\|^2 = \int_J \left|\langle x,y\rangle\right|^2 d\mu(y).$$

And

$$\left\| \int_J \langle x,y\rangle y\, d\mu(y) \right\|^2 - \int_J \left|\langle x,y\rangle\right|^2 d\mu(y) = \left\langle \left(S_J - S_J^2\right)x, x\right\rangle,$$

which proves the results. □

3 The main result for probabilistic frames

In this section, we extend some results of conventional frames to general probabilistic frames.

Theorem 6 *Let $\mu \in \mathcal{M}(\mathcal{B}, I)$ be a probabilistic frame for $\mathcal{H}$ with probabilistic canonical dual frame $\tilde{\mu}$. For every $J \subset I$ and every $x \in \mathcal{H}$, we have*

$$\int_J \left|\langle x,y\rangle\right|^2 d\mu(y) + \int_I \left|\langle S_{J^c}x,y\rangle\right|^2 d\tilde{\mu}(y) = \int_{J^c} \left|\langle x,y\rangle\right|^2 d\mu(y) + \int_I \left|\langle S_J x,y\rangle\right|^2 d\tilde{\mu}(y).$$

Proof The equality in Theorem 6 can be written as

$$\int_J \left|\langle x,y\rangle\right|^2 d\mu(y) - \int_I \left|\langle S_J x,y\rangle\right|^2 d\tilde{\mu}(y) = \int_{J^c} \left|\langle x,y\rangle\right|^2 d\mu(y) - \int_I \left|\langle S_{J^c}x,y\rangle\right|^2 d\tilde{\mu}(y).$$

Also

$$\begin{aligned}\int_J \left|\langle x,y\rangle\right|^2 d\mu(y) - \int_I \left|\langle S_J x,y\rangle\right|^2 d\tilde{\mu}(y) &= \langle S_J x, x\rangle - \int_I \left|\langle S_J x,y\rangle\right|^2 d\tilde{\mu}(y) \\ &= \langle S_J x, x\rangle - \int_I \left|\left\langle S_J x, S^{-1}y\right\rangle\right|^2 d\mu(y) \\ &= \langle S_J x, x\rangle - \left\langle S^{-1}S_J x, S_J x\right\rangle. \qquad (3)\end{aligned}$$

Simultaneously,

$$\int_{J^c} \left|\langle x,y\rangle\right|^2 \, d\mu(y) - \int_I \left|\langle S_{J^c}x,y\rangle\right|^2 \, d\tilde{\mu}(y) = \langle S_{J^c}x,x\rangle - \left\langle S^{-1}S_{J^c}x, S_{J^c}x\right\rangle. \tag{4}$$

Since $S = S_J + S_{J^c}$, it follows that $I_{\mathcal{H}} = S^{-1}S_J + S^{-1}S_{J^c}$. From the proof of Theorem 5, we have

$$S^{-1}S_J - S^{-1}S_J S^{-1}S_J = S^{-1}S_{J^c} - S^{-1}S_{J^c}S^{-1}S_{J^c}.$$

Moreover, for every $x, z \in \mathcal{H}$, we have

$$\left\langle S^{-1}S_J x, z\right\rangle - \left\langle S^{-1}S_J S^{-1}S_J x, z\right\rangle = \left\langle S^{-1}S_{J^c}x, z\right\rangle - \left\langle S^{-1}S_{J^c}S^{-1}S_{J^c}x, z\right\rangle. \tag{5}$$

If we choose z to be $z = Sx$, by the equalities (3) and (4), (5) is equal to

$$\begin{aligned}
&\left\langle S^{-1}S_J x, z\right\rangle - \left\langle S^{-1}S_J S^{-1}S_J x, z\right\rangle = \left\langle S^{-1}S_{J^c}x, z\right\rangle - \left\langle S^{-1}S_{J^c}S^{-1}S_{J^c}x, z\right\rangle, \\
&\langle S_J x, x\rangle - \left\langle S^{-1}S_J x, S_J x\right\rangle = \langle S_{J^c}x, x\rangle - \left\langle S^{-1}S_{J^c}x, S_{J^c}x\right\rangle, \\
&\int_J \left|\langle x,y\rangle\right|^2 \, d\mu(y) - \int_I \left|\langle S_J x,y\rangle\right|^2 \, d\tilde{\mu}(y) = \int_{J^c} \left|\langle x,y\rangle\right|^2 \, d\mu(y) - \int_I \left|\langle S_{J^c}x,y\rangle\right|^2 \, d\tilde{\mu}(y).
\end{aligned}$$

Hence, the proof is completed. □

In the case of general probabilistic frames, we define notations as follows:

$$\begin{aligned}
\nu'_-(\mathcal{U};J) &= \inf_{x\neq 0} \frac{\int_J |\langle x,y\rangle|^2 \, d\mu(y) + \int_I |\langle S_{J^c}x,y\rangle|^2 \, d\tilde{\mu}(y)}{\int_I |\langle x,y\rangle|^2 \, d\mu(y)}, \\
\nu'_+(\mathcal{U};J) &= \sup_{x\neq 0} \frac{\int_J |\langle x,y\rangle|^2 \, d\mu(y) + \int_I |\langle S_{J^c}x,y\rangle|^2 \, d\tilde{\mu}(y)}{\int_I |\langle x,y\rangle|^2 \, d\mu(y)}.
\end{aligned}$$

These notations of general probabilistic frames also satisfy the properties (i)-(iii) in Theorem 5. We give a detailed proof for the property (i).

Proposition 3 *The notations* $\nu'_-(\mathcal{U};J)$ *and* $\nu'_+(\mathcal{U};J)$ *satisfy*

$$\frac{3}{4} \leq \nu'_-(\mathcal{U};J) \leq \nu'_+(\mathcal{U};J) \leq 1.$$

Before the proof of Proposition 3, we need the following lemma.

Lemma 2 [29] *If* P, Q *are self-adjoint operators on* $\mathcal{H}$ *satisfying* $P + Q = I_{\mathcal{H}}$, *then*

$$\langle Px, x\rangle + \|Qx\|^2 = \langle Qx, x\rangle + \|Px\|^2 \geq \frac{3}{4}\|x\|^2,$$

for all $x \in \mathcal{H}$.

Proof of Proposition 3 First, we prove the left inequality. Since $S = S_J + S_{J^c}$, it follows that

$$I_{\mathcal{H}} = S^{-1/2}S_J S^{-1/2} + S^{-1/2}S_{J^c}S^{-1/2}.$$

By Lemma 2, we get

$$\langle S^{-1/2}S_JS^{-1/2}x,x\rangle + \|S^{-1/2}S_{J^c}S^{-1/2}x\|^2 = \langle S^{-1/2}S_{J^c}S^{-1/2}x,x\rangle + \|S^{-1/2}S_JS^{-1/2}x\|^2.$$

Replacing x by $S^{1/2}x$ for the above equality, then we have

$$\begin{aligned}\langle S^{-1/2}S_Jx,S^{1/2}x\rangle + \|S^{-1/2}S_{J^c}x\|^2 &= \langle S^{-1/2}S_{J^c}x,S^{1/2}x\rangle + \|S^{-1/2}S_Jx\|^2\\ &= \langle S^{-1}S_{J^c}x,x\rangle + \|S^{-1/2}S_Jx\|^2\\ &= \langle S_{J^c}x,x\rangle + \langle S^{-1}S_Jx,S_Jx\rangle\\ &\geq \frac{3}{4}\|S^{1/2}x\|^2\\ &= \frac{3}{4}\langle Sx,x\rangle = \frac{3}{4}\int_I\langle x,y\rangle y\,d\mu(y).\end{aligned}$$

Since

$$\langle S_{J^c}x,x\rangle + \langle S^{-1}S_Jx,S_Jx\rangle = \int_{J^c}|\langle x,y\rangle|^2\,d\mu(y) + \int_I|\langle S_Jx,y\rangle|^2\,d\tilde{\mu},$$

we have $\nu'_+(\mathcal{U};J)\geq\nu'_-(\mathcal{U};J)\geq\frac{3}{4}$.

The right inequality is also true. In fact,

$$\begin{aligned}\langle Px,x\rangle + \|Qx\|^2 &= \langle Qx,x\rangle + \|Px\|^2\\ &= \langle (P^2-P+I_{\mathcal{H}})x,x\rangle\\ &= \left\langle\left(\left(P-\frac{1}{2}I_{\mathcal{H}}\right)^2+\frac{3}{4}I_{\mathcal{H}}\right)x,x\right\rangle\leq\|x\|^2.\end{aligned}$$

It follows $\nu'_+(\mathcal{U};J)\leq 1$. The proof is completed. □

Next, we give a generalization of the equality from Theorem 4 for general probabilistic frames with probabilistic canonical dual frames.

Theorem 7 *Let $\mu\in\mathcal{M}(\mathcal{B},I)$ be a probabilistic frame for $\mathcal{H}$ with the probabilistic canonical dual frame $\tilde{\mu}$. Let $z=Sy$, for every $J\subset I$ and every $x\in\mathcal{H}$, we have*

$$\int_J\langle x,y\rangle\langle x,z\rangle\,d\tilde{\mu}(y) - \left\|\int_J\langle x,y\rangle z\,d\tilde{\mu}(y)\right\|^2 = \int_{J^c}\langle x,y\rangle\langle x,z\rangle\,d\tilde{\mu}(y) - \left\|\int_{J^c}\langle x,y\rangle z\,d\tilde{\mu}(y)\right\|^2.$$

Proof Let $z=Sy$, for every $x\in\mathcal{H}$, we have

$$x=\int_I\langle x,y\rangle Sy\,d\tilde{\mu}(y)=\int_I\langle x,y\rangle z\,d\tilde{\mu}(y). \tag{6}$$

For every $J\subset I$, we define the operator V_J as follows:

$$V_Jx=\int_J\langle x,y\rangle z\,d\tilde{\mu}(y).$$

It follows that $V_J + V_{J^c} = I_{\mathcal{H}}$ by (6). Thus, by Lemma 1, we have

$$\begin{aligned}\int_J \langle x,y\rangle\langle x,z\rangle\,d\tilde{\mu}(y) - \left\|\int_J \langle x,y\rangle z\,d\tilde{\mu}(y)\right\|^2 &= \langle V_J x,x\rangle - \left\langle V_J^* V_J x,x\right\rangle \\ &= \left\langle V_{J^c}^* x,x\right\rangle - \left\langle V_{J^c}^* V_{J^c} x,x\right\rangle \\ &= \langle x, V_{J^c}x\rangle - \|V_{J^c}x\|^2 \\ &= \left\langle x, \int_{J^c}\langle x,y\rangle z\,d\tilde{\mu}(y)\right\rangle - \|V_{J^c}x\|^2 \\ &= \int_{J^c}\langle x,y\rangle\langle x,z\rangle\,d\tilde{\mu}(y) - \left\|\int_{J^c}\langle x,y\rangle z\,d\tilde{\mu}(y)\right\|^2. \quad \square\end{aligned}$$

If we take the real part on both sides of equality in Theorem 7, we can get a more general result.

Theorem 8 *Let $\mu \in \mathcal{M}(\mathcal{B},I)$ be a probabilistic frame for $\mathcal{H}$ with probabilistic canonical dual frame $\tilde{\mu}$. Let $z = Sy$, for every $J \subset I$, every continue bounded sequence $\{b_i\}nL^2(I)$ and every $x \in \mathcal{H}$, we have*

$$\begin{aligned}&\int_J b_i\langle x,y\rangle\langle x,z\rangle\,d\tilde{\mu}(y) - \left\|\int_J b_i\langle x,y\rangle z\,d\tilde{\mu}(y)\right\|^2 \\ &\quad= \int_{J^c}(1-b_i)\langle x,y\rangle\langle x,z\rangle\,d\tilde{\mu}(y) - \left\|\int_{J^c}(1-b_i)\langle x,y\rangle z\,d\tilde{\mu}(y)\right\|^2.\end{aligned}$$

The proof of Theorem 8 is immediate.

For example, we can take $b_i = 1$ if $i \in J$ and $b_i = 0$ if $i \in J^c$. As a special case we have the following result.

Corollary 4 *If $\mu \in \mathcal{M}(\mathcal{B},I)$ is a probabilistic A-tight frame for $\mathcal{H}$ with probabilistic canonical dual frame $\tilde{\mu}$. Let $z = Sy$, for every $J \subset I$, every continue bounded sequence $\{b_i\}nL^2(I)$ and every $x \in \mathcal{H}$, we have*

$$\begin{aligned}&A\int_J b_i\left|\langle x,y\rangle\right|^2\,d\mu(y) - \left\|\int_J \langle x,y\rangle y\,d\mu(y)\right\|^2 \\ &\quad= A\int_{J^c}(1-b_i)\left|\langle x,y\rangle\right|^2\,d\mu(y) - \left\|\int_{J^c}(1-b_i)\langle x,y\rangle y\,d\mu(y)\right\|^2.\end{aligned}$$

Applying Corollary 1 and Theorem 8 proves the result.

4 Conclusions

In this paper, we mainly study some equalities and inequalities for probabilistic frames. We extend some good results of frames to probabilistic frames, and we obtain some new results because not all of properties of probabilistic frames are similar to those of traditional frames. Our results generalize and improve the remarkable results which have been established.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
This work was carried out in collaboration among the authors. All authors made a good contribution to design the research. All authors read and approved the final manuscript.

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