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Smoothing of the lower-order exact penalty function for inequality constrained optimization

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Abstract
In this paper, we propose a method to smooth the general lower-order exact penalty function for inequality constrained optimization. We prove that an approximation global solution of the original problem can be obtained by searching a global solution of the smoothed penalty problem. We develop an algorithm based on the smoothed penalty function. It is shown that the algorithm is convergent under some mild conditions. The efficiency of the algorithm is illustrated with some numerical examples.

Keywords: inequality constrained optimization; exact penalty function; lower-order penalty function; smoothing method

1 Introduction

We consider the following nonlinear constrained optimization problem:

$$[P] \quad \begin{array}{l} \min f(x) \\ \text{s.t. } g_i(x) \le 0, \quad i = 1, 2, \ldots, m, \end{array}$$

where $f : R^n \to R$ and $g_i : R^n \to R$, $i \in I = \{1, 2, \ldots, m\}$ are twice continuously differentiable functions. Let

$$G_0 = \{x \in R^n | g_i(x) \le 0, i = 1, 2, \ldots, m\}.$$

The penalty function methods have been proposed to solve problem [P] in much of the literature. In Zangwill [1], the classical l_1 exact penalty function is defined as follows:

$$p_1(x, q) = f(x) + q \sum_{i=1}^{m} \max\{g_i(x), 0\}, \tag{1.1}$$

where $q > 0$ is a penalty parameter, but it is not a smooth function. Differentiable approximations to the exact penalty function have been obtained in various places in the literature, such as [2–10].

Recently lower-order penalty functions have been proposed in some literature. In [11], Luo gave a global exact penalty result for a lower-order penalty function of the form

$$f(x) + \alpha\left(\sum_{i=1}^{m}\max\{g_i(x),0\}\right)^{1/\gamma}, \tag{1.2}$$

where $\alpha > 0, \gamma \geq 1$ are the penalty parameters. Obviously, it is the l_1 penalty function when $\gamma = 1$.

The nonlinear penalty function has been investigated in [12] and [13] as follows:

$$L^k(x,d) = \left[f(x)^k + \sum_{i=1}^{m} d_i\big(\max\{g_i(x),0\}\big)^k\right]^{1/k}, \tag{1.3}$$

where $f(x)$ is assumed to be positive, $k > 0$ is a given number, and $d = (d_1, d_2, \ldots, d_m) \in R_+^m$ is the penalty parameter. It was shown in [12] that the exact penalty parameter corresponding to $k \in (0,1]$ is substantially smaller than that of the classical l_1 exact penalty function.

In [14], the lower-order penalty functions

$$\varphi_{q,k}(x) = f(x) + q\sum_{i=1}^{m}\big(\max\{g_i(x),0\}\big)^k, \quad k \in (0,1), \tag{1.4}$$

have been introduced and shown to be exact under some conditions, but its smoothing does not discussed for $k \in (0,1)$. When $k = \frac{1}{2}$, we have the following function:

$$\varphi_q(x) = f(x) + q\sum_{i=1}^{m}\sqrt{\max\{g_i(x),0\}}. \tag{1.5}$$

Its smoothing has been investigated in [14, 15] and [16]. The smoothing of the lower-order exact penalty function (1.4) has been investigated in [17] and [18].

In this paper, we aim to smooth the lower-order penalty function (1.4). The rest of this paper is organized as follows. In Section 2, a new smoothing function to the lower-order penalty function (1.4) is introduced. The error estimates are obtained among the optimal objective function values of the smoothed penalty problem, the nonsmooth penalty problem and the original problem. In Section 3, we present an algorithm to compute an approximate solution to [P] based on the smooth penalty function and show that it is globally convergent. In Section 4, three numerical examples are given to show the efficiency of the algorithm. In Section 5, we conclude the paper.

2 Smoothing exact lower-order penalty function

We consider the following lower-order penalty problem:

$$[LOP]_k \quad \min_{x\in R^n}\varphi_{q,k}(x).$$

In order to establish the exact penalization property, we need the following assumptions as given in [14].

Assumption 1 $f(x)$ satisfies the following coercive condition:

$$\lim_{\|x\|\to+\infty} f(x) = +\infty.$$

Under Assumption 1, there exists a box X such that $G([P]) \subset \mathrm{int}(X)$, where $G([P])$ is the set of global minima of problem $[P]$, $\mathrm{int}(X)$ denotes the interior of the set X. Consider the following problem:

$$[P'] \quad \begin{aligned} &\min f(x) \\ &\text{s.t. } g_i(x) \le 0, \quad i = 1,2,\ldots,m, \\ &\quad\ x \in X. \end{aligned}$$

Let $G([P'])$ denote the set of global minima of problem $[P']$. Then $G([P']) = G([P])$.

Assumption 2 The set $G([P])$ is a finite set.

Then for any $k \in (0,1)$, we consider the penalty problem of the form

$$[LOP']_k \quad \min_{x\in X} \varphi_{q,k}(x).$$

We know that the lower-order penalty function $\varphi_{q,k}(x)(k \in (0,1))$ is an exact penalty function in [14] under Assumption 1 and Assumption 2. But the lower-order exact penalty function $\varphi_{q,k}(x)$ ($k \in (0,1)$) is a nondifferentiable function. Now we consider its smoothing.

Let $p_k(u) = (\max\{u,0\})^k$, that is,

$$p_k(u) = \begin{cases} u^k & \text{if } u > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

then

$$\varphi_{q,k}(x) = f(x) + q\sum_{i=1}^{m} p_k\bigl(g_i(x)\bigr). \tag{2.2}$$

For any $\epsilon > 0$, let

$$p_{\epsilon,k}(u) = \begin{cases} 0 & \text{if } u \le 0, \\ \frac{1}{2}\epsilon^{-k}u^{2k} & \text{if } 0 < u \le \epsilon, \\ u^k - \frac{\epsilon^k}{2} & \text{if } u > \epsilon. \end{cases} \tag{2.3}$$

It is easy to see that $p_{\epsilon,k}(u)$ is continuously differentiable on R. Furthermore, we see that $p_{\epsilon,k}(u) \to p_k(u)$ as $\epsilon \to 0$.

Figure 1 shows the behavior of $p_{2/3}(u)$ (represented by the solid line), $p_{0.1,2/3}(u)$ (represented by the dot line), $p_{0.01,2/3}(u)$ (represented by the broken line) and $p_{0.001,2/3}(u)$ (represented by the dash and dot line).

Let

$$\varphi_{q,\epsilon,k}(x) = f(x) + q\sum_{i=1}^{m} p_{\epsilon,k}\bigl(g_i(x)\bigr). \tag{2.4}$$

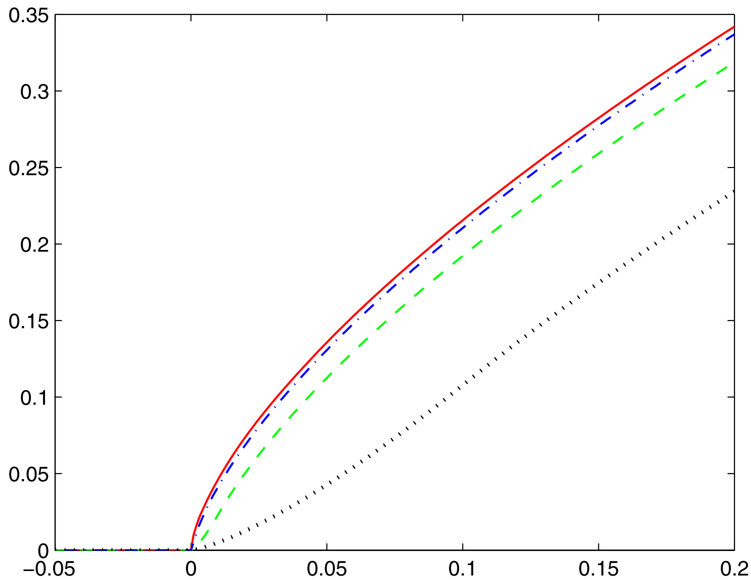

Figure 1 The behavior of $p_{\epsilon,2/3}(u)$ and $p_{2/3}(u)$.

Then $\varphi_{q,\epsilon,k}(x)$ is continuously differentiable on R^n. Consider the following smoothed optimization problem:

$$[SP] \quad \min_{x\in X} \varphi_{q,\epsilon,k}(x).$$

Lemma 2.1 *For any $x \in X, \epsilon > 0$, we see that*

$$0 \le \varphi_{q,k}(x) - \varphi_{q,\epsilon,k}(x) \le \frac{1}{2} mq\epsilon^k.$$

Proof Note that

$$p_k\big(g_i(x)\big) - p_{\epsilon,k}\big(g_i(x)\big) = \begin{cases} 0 & \text{if } g_i(x) \le 0, \\ (g_i(x))^k - \frac{1}{2}\epsilon^{-k}(g_i(x))^{2k} & \text{if } 0 < g_i(x) \le \epsilon, \\ \frac{\epsilon^k}{2} & \text{if } g_i(x) > \epsilon. \end{cases}$$

Let

$$F(u) = u^k - \frac{1}{2}\epsilon^{-k}u^{2k}.$$

We get

$$F'(u) = ku^{k-1} - k\epsilon^{-k}u^{2k-1} = k\epsilon^{-k}u^{k-1}\big(\epsilon^k - u^k\big).$$

When $u \in (0, \epsilon)$, $F'(u) \ge 0$. It is easy to see that $F(u)$ is monotone increasing in $[0, \epsilon]$. When $g_i(x) \in [0, \epsilon]$, we can get

$$0 \le p_k\big(g_i(x)\big) - p_{\epsilon,k}\big(g_i(x)\big) < \frac{1}{2}\epsilon^k.$$

Thus we see that

$$0 \le \varphi_{q,k}(x) - \varphi_{q,\epsilon,k}(x) \le \frac{1}{2} mq\epsilon^k.$$

This completes the proof. □

Theorem 2.1 *Let $\{\epsilon_j\} \to 0^+$ be a sequence of positive numbers and assume that x_j is a solution to $\min_{x\in X} \varphi_{q,\epsilon_j,k}(x)$ for some $q > 0$, $k \in (0,1)$. Let $\bar{x}$ be an accumulation point of the sequence $\{x_j\}$. Then $\bar{x}$ is an optimal solution to $\min_{x\in X} \varphi_{q,k}(x)$.*

Proof Because x_j is a solution to $\min_{x\in X} \varphi_{q,\epsilon_j,k}(x)$, we see that

$$\varphi_{q,\epsilon_j,k}(x_j) \le \varphi_{q,\epsilon_j,k}(x), \quad \forall x \in X.$$

By Lemma 2.1, we see that

$$\varphi_{q,\epsilon_j,k}(x) \le \varphi_{q,k}(x)$$

and

$$\varphi_{q,k}(x) \le \varphi_{q,\epsilon_j,k}(x) + \frac{1}{2} mq\epsilon_j^k.$$

It follows that

$$\varphi_{q,k}(x_j) \le \varphi_{q,\epsilon_j,k}(x_j) + \frac{1}{2} mq\epsilon_j^k \le \varphi_{q,\epsilon_j,k}(x) + \frac{1}{2} mq\epsilon_j^k \le \varphi_{q,k}(x) + \frac{1}{2} mq\epsilon_j^k.$$

Let $j \to \infty$, we see that

$$\varphi_{q,k}(\bar{x}) \le \varphi_{q,k}(x).$$

This completes the proof. □

Theorem 2.2 *Let $x^*_{q,k} \in X$ be an optimal solution of problem $[LOP']_k$ and $\bar{x}_{q,\epsilon,k} \in X$ be an optimal solution of problem $[SP]$ for some $q > 0, k \in (0,1)$ and $\epsilon > 0$. Then we see that*

$$0 \le \varphi_{q,k}\bigl(x^*_{q,k}\bigr) - \varphi_{q,\epsilon,k}(\bar{x}_{q,\epsilon,k}) \le \frac{1}{2} mq\epsilon^k.$$

Proof By Lemma 2.1, we see that

$$\begin{aligned} 0 &\le \varphi_{q,k}\bigl(x^*_{q,k}\bigr) - \varphi_{q,\epsilon,k}\bigl(x^*_{q,k}\bigr) \\ &\le \varphi_{q,k}\bigl(x^*_{q,k}\bigr) - \varphi_{q,\epsilon,k}(\bar{x}_{q,\epsilon,k}) \\ &\le \varphi_{q,k}(\bar{x}_{q,\epsilon,k}) - \varphi_{q,\epsilon,k}(\bar{x}_{q,\epsilon,k}) \\ &\le \frac{1}{2} mq\epsilon^k. \end{aligned}$$

This completes the proof. □

Theorem 2.1 and Theorem 2.2 mean that an approximately optimal solution to [SP] is also an approximately optimal solution to $[LOP']_k$ when the error ϵ is sufficiently small.

Definition 2.1 For $\epsilon > 0$, a point $x \in X$ is an ϵ-feasible solution or an ϵ-solution of problem [P], if

$$g_i(x) \le \epsilon, \quad i = 1, 2, \dots, m.$$

We say that the pair (x^*, λ^*) satisfies the second-order sufficiency condition in [19] if

$$\begin{aligned}
&\nabla_x L(x^*, \lambda^*) = 0, \\
&g_i(x^*) \le 0, \quad i = 1, 2, \dots, m, \\
&\lambda_i^* \ge 0, \quad i = 1, 2, \dots, m, \\
&\lambda_i^* g_i(x^*) = 0, \quad i = 1, 2, \dots, m, \\
&y^T \nabla^2 L(x^*, \lambda^*) y > 0, \quad \text{for any } y \in V(x^*),
\end{aligned} \tag{2.5}$$

where $L(x, \lambda) = f(x) + \sum_{i=1}^m \lambda_i g_i(x)$, and

$$\begin{aligned}
&V(x^*) = \{y \in R^n | \nabla^T g_i(x^*) y = 0, i \in A(x^*); \nabla^T g_i(x^*) y \le 0, i \in B(x^*)\}, \\
&A(x^*) = \{i \in \{1, 2, \dots, m\} | g_i(x^*) = 0, \lambda^* > 0\}, \\
&B(x^*) = \{i \in \{1, 2, \dots, m\} | g_i(x^*) = 0, \lambda^* = 0\}.
\end{aligned}$$

Theorem 2.3 *Suppose that Assumptions* 1 *and* 2 *hold, and that for any* $x^* \in G([P])$, *there exists a* $\lambda^* \in R_+^m$ *such that the pair* (x^*, λ^*) *satisfies the second-order sufficiency condition* (2.5). *Then for* $\forall k \in (0, 1)$, *let* $x^* \in X$ *be a global solution of problem* $[P]$ *and* $\bar{x}_{q,\epsilon,k} \in X$ *be a global solution of problem* $[SP]$ *for* $k \in (0, 1), \epsilon > 0$. *Then there exists* $q^* > 0$ *such that for any* $q > q^*$,

$$0 \le f(x^*) - \varphi_{q,\epsilon,k}(\bar{x}_{q,\epsilon,k}) \le \frac{1}{2} mq\epsilon^k. \tag{2.6}$$

Furthermore, if $\bar{x}_{q,\epsilon,k}$ *be an* ϵ-*feasible solution of problem* $[P]$, *then we see that*

$$0 \le f(x^*_{q,k}) - f(\bar{x}_{q,\epsilon,k}) \le mq\epsilon^k,$$

where $q^* > 0$ *is defined in Corollary* 2.3 *in* [14].

Proof By Corollary 2.3 in [14], we see that $x^* \in X$ is a global solution of problem $[LOP']_k$. Then by Theorem 2.2, we see that

$$0 \le \varphi_{q,k}(x^*) - \varphi_{q,\epsilon,k}(\bar{x}_{q,\epsilon,k}) \le \frac{1}{2} mq\epsilon^k. \tag{2.7}$$

Since $\sum_{i=1}^m p_k(g_i(x^*)) = 0$, we have

$$\varphi_{q,k}(x^*) = f(x^*) + q \sum_{i=1}^m p_k(g_i(x^*)) = f(x^*). \tag{2.8}$$

By (2.7) and (2.8), we see that (2.6) holds.

Furthermore, it follows from (2.4) and (2.6) that

$$0 \le f(x^*) - \left(f(\bar{x}_{q,\epsilon,k}) + q \sum_{i=1}^{m} p_{\epsilon,k}\big(g_i(\bar{x}_{q,\epsilon,k})\big) \right) \le \frac{1}{2} m q \epsilon^k.$$

It follows that

$$q \sum_{i=1}^{m} p_{\epsilon,k}\big(g_i(\bar{x}_{q,\epsilon,k})\big) \le f(x^*) - f(\bar{x}_{q,\epsilon,k}) \le \frac{1}{2} m q \epsilon^k + q \sum_{i=1}^{m} p_{\epsilon,k}\big(g_i(\bar{x}_{q,\epsilon,k})\big). \tag{2.9}$$

From (2.3) and the fact that $\bar{x}_{q,\epsilon,k}$ is an ϵ-feasible solution of problem [P], we see that

$$0 \le \sum_{i=1}^{m} p_{\epsilon,k}\big(g_i(\bar{x}_{q,\epsilon,k})\big) \le \frac{1}{2} m \epsilon^k. \tag{2.10}$$

Then it follows from (2.9) and (2.10) that

$$0 \le f(x^*) - f(\bar{x}_{q,\epsilon,k}) \le m q \epsilon^k.$$

This completes the proof. □

Theorem 2.3 means that an approximately optimal solution to [SP] is an approximately optimal solution to [P] if the solution to [SP] is ϵ-feasible.

3 A smoothing method

We propose the following algorithm to solve [P].

Algorithm 3.1

Step 1 Choose an initial point x_0, and a stoping tolerance $\epsilon > 0$. Given $\epsilon_0 > 0, q_0 > 0, 0 < \eta < 1$, and $\sigma > 1$, let $j = 0$ and go to Step 2.

Step 2 Use x_j as the starting point to solve $\min_{x \in R^n} \varphi_{q_j,\epsilon_j,k}(x)$. Let x_j^* be the optimal solution obtained (x_j^* is obtained by a quasi-Newton method and a finite difference gradient).

Step 3 If x_j^* is ϵ-feasible to [P], then stop and we have obtained an approximately optimal solution x_j^* of the original problem [P]. Otherwise, let $q_{j+1} = \sigma q_j, \epsilon_{j+1} = \eta \epsilon_j, x_{j+1} = x_j^*$, and $j = j+1$, then go to Step 2.

From $0 < \eta < 1$ and $\sigma > 1$, we can easily see that the sequence $\{\epsilon_j\}$ is decreasing to 0 and the sequence $\{q_j\}$ is increasing to $+\infty$ as $j \longrightarrow +\infty$.

Now we prove the convergence of the algorithm under some mild conditions.

Theorem 3.1 *Suppose that Assumption* 1 *holds and for any* $q \in [q_0, +\infty), \epsilon \in (0, \epsilon_0]$, *the set*

$$\arg\min_{x \in R^n} \varphi_{q,\epsilon,k}(x) \neq \emptyset.$$

Let $\{x_j^*\}$ *be the sequence generated by Algorithm* 3.1. *If the sequence* $\{\varphi_{q_j,\epsilon_j,k}(x_j^*)\}$ *is bounded, then* $\{x_j^*\}$ *is bounded and any limit point of* $\{x_j^*\}$ *is the solution of* [*P*].

Proof First we show that $\{x_j^*\}$ is bounded. Note that

$$\varphi_{q_j,\epsilon_j,k}\left(x_j^*\right) = f\left(x_j^*\right) + q_j \sum_{i=1}^{m} p_{\epsilon_j,k}\left(g_i\left(x_j^*\right)\right), \quad j = 0, 1, 2, \ldots.$$

By the assumptions, there is some number L such that

$$L > \varphi_{q_j,\epsilon_j,k}\left(x_j^*\right), \quad j = 0, 1, 2, \ldots.$$

Suppose to the contrary that $\{x_j^*\}$ is unbounded. Without loss of generality, we assume that $\|x_j^*\| \to \infty$ as $j \to \infty$. Then we get

$$L > f\left(x_j^*\right), \quad j = 0, 1, 2, \ldots,$$

which results in a contradiction since f is coercive.

We show next that any limit point of $\{x_j^*\}$ is the optimal solution of [P]. Let $\bar{x}$ be any limit point of $\{x_j^*\}$. Then there exists a natural number set $J \subseteq N$, such that $x_j^* \to \bar{x}, j \in J$. If we can prove that (i) $\bar{x} \in G_0$ and (ii) $f(\bar{x}) \leq \inf_{x \in G_0} f(x)$ hold, then $\bar{x}$ is the optimal solution of [P].

(i) Suppose to the contrary that $\bar{x} \notin G_0$, then there exist $\delta_0 > 0, i_0 \in I$ and the subset $J_1 \subset J$ such that

$$g_{i_0}\left(x_j^*\right) \geq \delta_0 > \epsilon_j$$

for any $j \in J_1$.

And by step 2 in Algorithm 3.1 and (2.3), we see that

$$f\left(x_j^*\right) + \frac{1}{2} q_j \delta_0^k \leq f\left(x_j^*\right) + q_j \left(\left(g_{i_0}\left(x_j^*\right)\right)^k - \frac{1}{2}\epsilon_j^k\right) \leq \varphi_{q_j,\epsilon_j,k}\left(x_j^*\right) \leq \varphi_{q_j,\epsilon_j,k}(x) = f(x)$$

for any $x \in G_0$, which contradicts with $q_j \to +\infty$. Then we see that $\bar{x} \in G_0$.

(ii) For any $x \in G_0$, we have

$$f\left(x_j^*\right) \leq \varphi_{q_j,\epsilon_j,k}\left(x_j^*\right) \leq \varphi_{q_j,\epsilon_j,k}(x) = f(x),$$

then $f(\bar{x}) \leq \inf_{x \in G_0} f(x)$ holds.

This completes the proof. □

4 Numerical examples

In this section, we solve three numerical examples to show the applicability of Algorithm 3.1.

Example 4.1 (Example 4.2 in [17], Example 3.3 in [18] and Example 4.1 in [20])

$$\min f(x) = x_1^2 + x_2^2 - \cos(17x_1) - \cos(17x_2) + 3,$$

$$\text{s.t. } g_1(x) = (x_1 - 2)^2 + x_2^2 - 1.6^2 \leq 0,$$

Table 1 Numerical results for Example 4.1 with $k = 1/3$

j	x_j^*	q_j	ϵ_j	$g_1(x_j^*)$	$g_2(x_j^*)$	$f(x_j^*)$
0	$\begin{pmatrix} -0.362270 \\ 0.366667 \end{pmatrix}$	1	0.1	3.154764	–0.224317	1.277367
1	$\begin{pmatrix} 0.724975 \\ 0.399152 \end{pmatrix}$	2	0.01	–0.774989	–0.000000	1.837569

Table 2 Numerical results for Example 4.1 with $k = 2/3$

j	x_j^*	q_j	ϵ_j	$g_1(x_j^*)$	$g_2(x_j^*)$	$f(x_j^*)$
0	$\begin{pmatrix} 0.725362 \\ 0.399226 \end{pmatrix}$	10	0.1	0.775917	0.000175	1.715609
1	$\begin{pmatrix} 0.725353 \\ 0.399257 \end{pmatrix}$	20	0.01	–0.775869	0.000000	1.837547

$$g_2(x) = x_1^2 + (x_2 - 3)^2 - 2.7^2 \le 0,$$

$$0 \le x_1 \le 2,$$

$$0 \le x_2 \le 2.$$

Let $k = 1/3, x_0 = (0.5, 1.5), q_0 = 1.0, \epsilon_0 = 0.1, \eta = 0.1, \sigma = 2, \epsilon = 10^{-15}$, we obtain the results by Algorithm 3.1 shown in Table 1.

Let $k = 2/3, x_0 = (0.5, 1.5), q_0 = 10, \epsilon_0 = 0.1, \eta = 0.1, \sigma = 2, \epsilon = 10^{-15}$, we obtain the results by Algorithm 3.1 shown in Table 2.

When $k = 1/3$ and $k = 2/3$, numerical results are given in Table 1 and Table 2, respectively. It is clear from Table 1 and Table 2 that the obtained approximate solutions are similar. In [17], the given solution for Example 4.2 is $(0.7254, 0.3993)$ with objective function value 1.8375 when $k = 1/3$. In [18], the given solution for Example 3.3 is $(0.7239410, 0.3988712)$ with objective function value 1.837919 when $k = 1/3$. The given solution for Example 3.3 is $(0.7276356, 0.3998984)$ with objective function value 1.838380 when $k = 2/3$. In [20], the given solution for Example 4.1 is $(0.7255, 0.3993)$ with objective function value 1.8376. Numerical results are similar to the results of [17] and [20], and they are better than the results of [20] in this example.

Example 4.2 (Test Problem 6 in Section 4.6 in [21])

$$\min f(x) = -x - y,$$

$$\text{s.t. } g_1(x, y) = y - 2x^4 + 8x^3 - 8x^2 - 2 \le 0,$$

$$g_2(x, y) = y - 4x^4 + 32x^3 - 88x^2 + 96x - 36 \le 0,$$

$$0 \le x \le 3,$$

$$0 \le y \le 4.$$

Let $k = 2/3, x_0 = (2.5, 0), q_0 = 5, \epsilon_0 = 0.1, \eta = 0.1, \sigma = 2, \epsilon = 10^{-15}$, the results by Algorithm 3.1 are shown in Table 3.

Let $k = 2/3, x_0 = (0, 4), q_0 = 5, \epsilon_0 = 0.1, \eta = 0.1, \sigma = 2, \epsilon = 10^{-15}$, the results by Algorithm 3.1 are shown in Table 4.

Table 3 Numerical results for Example 4.2 with $x_0 = (2.5, 0)$

k	x_k^*	q_k	ϵ_k	$g_1(x_k^*)$	$g_2(x_k^*)$	$f(x_k^*)$
0	(2.329720, 3.177613)	5	0.1	–0.002508	0.000057	–5.507333
1	(2.329648, 3.177624)	10	0.01	–0.001917	–0.000266	–5.507273
2	(2.329674, 3.177610)	20	0.001	–0.002136	–0.000163	–5.507283

Table 4 Numerical results for Example 4.2 with $x_0 = (0, 4)$

k	x_k^*	q_k	ϵ_k	$g_1(x_k^*)$	$g_2(x_k^*)$	$f(x_k^*)$
0	(2.329741, 3.177865)	5	0.1	–0.002428	0.000408	–5.507606
1	(2.329649, 3.177847)	10	0.01	–0.001698	–0.000041	–5.507496

Table 5 Numerical results for Example 4.2 with $x_0 = (1.0, 1.5)$

k	x_k^*	q_k	ϵ_k	$g_1(x_k^*)$	$g_2(x_k^*)$	$f(x_k^*)$
0	(2.329625, 3.178285)	5	0.1	–0.001067	0.000286	–5.507911
1	(2.329538, 3.178429)	10	0.01	–0.000208	0.000019	–5.507967
2	(2.329517, 3.178421)	20	0.001	–0.000049	–0.000085	–5.507938

Let $k = 2/3, x_0 = (1.0, 1.5), q_0 = 5, \epsilon_0 = 0.1, \eta = 0.1, \sigma = 2, \epsilon = 10^{-15}$, the results by Algorithm 3.1 are shown in Table 5.

With different starting points $x_0 = (2.5, 0), x_0 = (0, 4)$, and $x_0 = (1.0, 1.5)$, numerical results are given in Table 3, Table 4 and Table 5, respectively. One can see that the numerical results in Tables 3-5 are similar. This means that Algorithm 3.1 does not completely depend on how to choose a starting point in this example. In [21], the given solution for Example 4.2 is (2.3295, 3.1783) with objective function value –5.5079. Numerical results are similar to the result of [21] in this example.

For the jth iteration of the algorithm, we define a constraint error e_j by

$$e_j = \sum_{i=1}^{m} \max\{g_i(x_j^*), 0\}.$$

It is clear that x_j^* is ϵ-feasible to (P) when $e_j < \epsilon$.

Example 4.3 (Example 4.5 in [15] and Example 3.4 in [18])

$$\min f(x) = 10x_2 + 2x_3 + x_4 + 3x_5 + 4x_6,$$

$$\text{s.t. } g_1(x) = x_1 + x_2 - 10 = 0,$$

$$g_2(x) = -x_1 + x_3 + x_4 + x_5 = 0,$$

$$g_3(x) = -x_2 - x_3 + x_5 + x_6 = 0,$$

$$g_4(x) = 10x_1 - 2x_3 + 3x_4 - 2x_5 - 16 \leq 0,$$

Table 6 Numerical results for Example 4.3

j	x_j^*	q_j	ϵ_j	e_j	$f(x_j^*)$
0	(1.620510, 8.377264, 0.013437, 0.606651 1.000285, 7.390398)	100	0.5	–0.000017	116.968612
1	(1.620468, 8.379530, 0.013229, 0.607246 0.999994, 7.392764)	200	0.005	0.000000	117.000044

$$g_5(x) = x_1 + 4x_3 + x_5 - 10 \le 0,$$

$$0 \le x_1 \le 12,$$

$$0 \le x_2 \le 18,$$

$$0 \le x_3 \le 5,$$

$$0 \le x_4 \le 12,$$

$$0 \le x_5 \le 1,$$

$$0 \le x_6 \le 16.$$

Let $k = 2/3, x_0 = (2, 2, 1, 2, 1, 2), q_0 = 100, \epsilon_0 = 0.5, \eta = 0.01, \sigma = 2, \epsilon = 10^{-15}$, the results by Algorithm 3.1 are shown in Table 6.

It is clear from Table 6 that the obtained approximately optimal solution is $x^* = (1.620468, 8.379530, 0.013229, 0.607246, 0.999994, 7.392764)$ with corresponding objective function value 117.000044. In [15], the obtained approximately optimal solution is $x^* = (1.805996, 8.194004, 0.497669, 0.308327, 1.000000, 7.691673)$ with corresponding objective function value 117.010399. In [18], the given solution for Example 3.4 is $(1.635022, 8.364978, 0.060010, 0.5812775, 0.9937346, 7.431253)$ with objective function value 117.0573 when $k = 2/3$. Numerical results are better than the results of [15] and [18] in this example.

5 Concluding remarks

In this paper, we propose a method for smoothing the nonsmooth lower-order exact penalty function for inequality constrained optimization. We prove that the algorithm based on the smoothed penalty functions is convergent under mild conditions.

According to the numerical results given in Section 4, we can obtain an approximately optimal solution of the original problem [P] by Algorithm 3.1.

Finally, we give some advices on how to choose parameter in the algorithm. Usually, the initial value of q_0 may be 1, 5, 10, 100, 1000 or 10000, and $\sigma = 2, 5, 10$ or 100. The initial value of ϵ_0 may be 1, 0.5, 0.1 or 0.01, and $\eta = 0.5, 0.1, 0.05$ or 0.01.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YD drafted the manuscript. SL helped to draft the manuscript and revised it. All authors read and approved the final manuscript.

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