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# Convergence on successive over-relaxed iterative methods for non-Hermitian positive definite linear systems

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**Abstract**
Some convergence conditions on successive over-relaxed (SOR) iterative method and symmetric SOR (SSOR) iterative method are proposed for non-Hermitian positive definite linear systems. Some examples are given to demonstrate the results obtained.

**MSC:** 15A06; 15A15; 15A48

**Keywords:** SOR iterative method; SSOR iterative method; non-Hermitian positive definite linear system; convergence

## 1 Introduction

Many problems in scientific computing give rise to a system of $n$ linear equations

$$Ax = b, \quad A = (a_{ij}) \in \mathbb{C}^{n\times n} \text{ is nonsingular, and } x, b \in \mathbb{C}^n, \tag{1}$$

where $A$ is a large sparse non-Hermitian positive definite matrix, that is, its Hermitian part $H = (A + A^*)/2$ is Hermitian positive definite, where $A^*$ denotes the conjugate transpose of a matrix $A$. In order to solve system (1) by iterative methods, usually, efficient splittings of the coefficient matrix $A$ are required. For example, the classic Jacobi and Gauss-Seidel iterations [1–3] split the matrix $A$ into its diagonal and off-diagonal parts. Recently, considerable interest appears in the work on the Hermitian and skew-Hermitian splitting (HSS) method for this system introduced by Bai et al. [4] and some generalized HSS methods such as the NSS method [5], PSS method [6], PHSS method [7, 8], and LHSS method [9], and several significant theoretical results are proposed. Meanwhile, these methods and theoretical results are applied to this linear system directly or indirectly (as a preconditioner); see [4–14]. It is shown in [3, 15, 16] that the successive over-relaxed (SOR) iterative method and symmetric SOR (SSOR) iterative method for Hermitian positive definite linear systems are convergent. But, is the same true for these iterative methods for non-Hermitian positive definite linear systems? In this paper, we mainly study the convergence of the SOR iterative and SSOR iterative method for non-Hermitian positive definite linear systems and propose some convergence conditions.

## 2 Main results

The main theoretical results in this paper are given to propose some convergence conditions on the SOR and SSOR iterative methods. For convenience of presentation, in Section 2.1, we give some classic iterative methods for linear systems.

### 2.1 SOR iterative methods

In order to solve system (1) by iterative methods, we split the coefficient matrix $A$ in (1) into

$$A = D - L - U, \tag{2}$$

where $D = \operatorname{diag}(\operatorname{Re}(a_{11}), \operatorname{Re}(a_{22}), \ldots, \operatorname{Re}(a_{nn}))$, $L$ is a lower triangular matrix, and $U$ is a strictly upper triangular matrix. Then,

$$D^{-1/2}AD^{-1/2} = I - D^{-1/2}LD^{-1/2} - D^{-1/2}UD^{-1/2}, \tag{3}$$

where $I$ is the identity matrix. Without loss of generality, in (2), we can assume that $D = I$. Decomposition (2) becomes

$$A = I - L - U. \tag{4}$$

The forward, backward, and symmetric Gauss-Seidel (FGS, BGS, and SGS) iterative methods are defined by the iteration matrices

$$T = (I - L)^{-1}U, \qquad F = (I - U)^{-1}L, \quad \text{and} \quad S = FT = (I - U)^{-1}L(I - L)^{-1}U, \tag{5}$$

respectively. The forward, backward, and symmetric successive over-relaxation (FSOR, BSOR, and SSOR) iterative methods are defined by the iteration matrices

$$L_\omega = (I - \omega L)^{-1}\bigl[\omega U + (1 - \omega)I\bigr], \tag{6}$$

$$F_\omega = (I - \omega U)^{-1}\bigl[\omega L + (1 - \omega)I\bigr], \tag{7}$$

and

$$S_\omega = F_\omega L_\omega = (I - \omega U)^{-1}\bigl[\omega L + (1 - \omega)I\bigr](I - \omega L)^{-1}\bigl[\omega U + (1 - \omega)I\bigr], \tag{8}$$

respectively.

### 2.2 Convergence results for SOR iterative method

Throughout the paper, we denote the conjugate transpose of a vector $x \in \mathbb{C}$ and a matrix $A \in \mathbb{C}^{n\times n}$, the spectrum of $A$, and the set of all eigenvectors of $A$ by $x^*$ and $A^*$, $\sigma(A) = \{\lambda \in \mathbb{C} : \det(\lambda I - A) = 0\}$, and $\vartheta(A) = \{x \in \mathbb{C}^n : Ax = \lambda x, \lambda \in \sigma(A)\}$, respectively. Let $\rho(A) = \max_{\lambda\in\sigma(A)|\lambda|}$ be the spectral radius of $A$, and $\vartheta_{\max}(A) = \{x \in \vartheta(A) : Ax = \lambda x, |\lambda| = \rho(A)\}$. The following lemmas will be used in this paper.

**Lemma 1** *Let $A = M - N \in \mathbb{C}^{n\times n}$ with $M$ nonsingular and $F = M^{-1}N$. Then $\rho(F) < 1$ if and only if $H - F^*HF$ is Hermitian positive definite on $\vartheta(F)$ for any $n \times n$ Hermitian positive definite matrix $H$.*

*Proof* Let $\lambda$ be any eigenvalue of the matrix $F$, and $x \in \vartheta(F)$ be a corresponding eigenvector with $x \neq 0$, that is, $Fx = \lambda x$. Then, for all $x \in \vartheta(F)$, we have

$$x^*\big(H - F^*HF\big)x = x^*Hx - x^*F^*HFx = \big(1 - |\lambda|^2\big)x^*Hx,$$

which indicates that this lemma is true. □

From Lemma 1 we have the following conclusion.

**Lemma 2** *Let $A = M - N \in \mathbb{C}^{n\times n}$ with $M$ nonsingular and $F = M^{-1}N$. Then $\rho(F) < 1$ if and only if $M^*M - N^*N$ is Hermitian positive definite on $\vartheta(F)$.*

In what follows, we propose some convergence conditions on SOR and SSOR iterative methods for non-Hermitian positive definite linear systems.

**Theorem 1** *Let $A = I - L - U \in \mathbb{C}^{n\times n}$ be positive definite with $H = (A^* + A)/2$, and $L_\omega$ be defined in (6). Then the SOR iterative method converges to the unique solution of (1) for any choice of the initial guess $x_0$, that is, $\rho(L_\omega) < 1$ if and only if $0 < \omega < 1/\eta$ if $\eta > 0$ or $1/\eta < \omega$ if $\eta < 0$ or $0 < \omega$ if $\eta = 0$, where*

$$\eta = \frac{x^*[(I-U)^*(I-U) - L^*L]x}{2x^*Hx}$$

*for any $x \in \vartheta_{\max}(L_\omega)$.*

*Proof* Let $\alpha = 1/\omega - 1$. Then (6) is changed as

$$L_\alpha = [(\alpha+1)I - L]^{-1}(U + \alpha I). \tag{9}$$

Let $M = (\alpha+1)I - L$ and $N = U + \alpha I$. Then (7) is changed into $L_\alpha = M^{-1}N$. Assume that $\lambda$ is an eigenvalue of $L_\alpha$ with $|\lambda| = \rho(L_\alpha)$ and $x \in \vartheta_{\max}(L_\omega)$ is its corresponding eigenvector. Then, $M^{-1}Nx = \lambda x$, $Nx = \lambda Mx$, and consequently $x^*N^*Nx = |\lambda|^2 x^*M^*Mx$. Thus,

$$\big[\rho(L_\alpha)\big]^2 = x^*N^*Nx/x^*M^*Mx. \tag{10}$$

For any $x \in \vartheta_{\max}(L_\omega)$, we have

$$\begin{aligned} x^*\big(M^*M - N^*N\big)x &= x^*\big\{[(\alpha+1)I - L]^*[(\alpha+1)I - L] - (U+\alpha I)^*(U+\alpha I)\big\}x \\ &= 2\alpha x^*Hx + x^*\big[(I-L)^*(I-L) - U^*U\big]x \\ &= 2(\alpha+1)x^*Hx - x^*\big[(I-U)^*(I-U) - L^*L\big]x \\ &= 2x^*Hx\left(\frac{1}{\omega} - \frac{x^*[(I-U)^*(I-U) - L^*L]x}{x^*Hx}\right) \\ &= 2x^*Hx\left(\frac{1}{\omega} - \eta\right), \end{aligned} \tag{11}$$

where

$$\eta = \frac{x^*[(I-U)^*(I-U) - L^*L]x}{2x^*Hx}.$$

It follows from Lemma 2, (8), and (9) that $\rho(L_\omega) < 1$ if and only if $2x^*Hx(1/\omega - \eta) > 0$ for $x \in \vartheta_{\max}(L_\omega)$. Since $A$ is positive definite, $H = (A + A^*)/2$ is Hermitian positive definite, that is, $x^*Hx > 0$ for any $x \neq 0$, $x \in \mathbb{C}^n$. As a consequence, $x^*Hx > 0$ for any $x \in \vartheta_{\max}(L_\omega)$. Thus, $\rho(L_\omega) < 1$ if and only if $1/\omega > \eta$. Again, since $1/\omega > \eta$ holds if and only if $0 < \omega < 1/\eta$ if $\eta > 0$ or $1/\eta < \omega$ if $\eta < 0$ or $0 < \omega$ if $\eta = 0$, this completes the proof. □

**Theorem 2** *Let $A = I - L - U \in \mathbb{C}^{n\times n}$ be positive definite with $H = (A^* + A)/2$, and $L_\omega$ be defined in* (6). *Then the SOR iterative method converges to the unique solution of* (1) *for any choice of the initial guess $x_0$, that is, $\rho(L_\omega) < 1$ if and only if $0 < \omega < 1/(1-\tau)$ if $\tau < 1$ or $\omega > 1/(1-\eta)$ if $\tau > 1$ or $\omega > 0$ if $\tau = 1$, where*

$$\tau = \frac{x^*[(I-L)^*(I-L) - U^*U]x}{2x^*Hx}$$

*for any $x \in \vartheta_{\max}(L_\omega)$.*

*Proof* Since

$$(I-L)^*(I-L) - U^*U = 2H - \bigl[(I-U)^*(I-U) - L^*L\bigr]$$

yields $\tau = 1 - \eta$, the conclusion of the theorem follows from Theorem 1. □

**Theorem 3** *Let $A - L - U \in \mathbb{C}^{n\times n}$ be positive definite with $H = (A^* + A)/2$, and $L_\omega$ be defined in* (6). *Suppose that A satisfies one of the two conditions*: (i) $\|T\|_2 \leq 1$ *and* $0 < \omega < 1$; (ii) $\|T\|_2 > 1$ *and* $0 < \omega < \omega_0$. *Then the SOR iterative method converges to the unique solution of* (1) *for any choice of the initial guess $x_0$, that is, $\rho(L_\omega) < 1$, where*

$$\omega_0 = \frac{2\lambda_{\min}([(I-L)(I-L)^*]^{-1}H)}{\|T\|_2^2 + 2\lambda_{\min}([(I-L)(I-L)^*]^{-1}H) - 1}.$$

*Proof* According to the proof of Theorem 1,

$$\begin{aligned} x^*\bigl(M^*M - N^*N\bigr)x &= x^*\bigl[(\alpha I + I - L)^*(\alpha I + I - L) - (\alpha I - U)^*(\alpha I - U)\bigr]x \\ &= 2\alpha x^*Hx + x^*\bigl[(I-L)^*(I-L) - U^*U\bigr]x \\ &= x^*(I-L)\bigl[2\alpha(I-L)^{-1}H(I-L)^{-*} + I - TT^*\bigr](I-L)^*x \\ &\geq \bigl[2\lambda_{\min}\bigl(\bigl[(I-L)(I-L)^*\bigr]^{-1}H\bigr) + 1 - \|T\|_2^2\bigr]x^*(I-L)(I-L)^*x. \end{aligned}$$

(i) If $\|T\|_2 \leq 1$ and $0 < \omega < 1$, then it is obvious that $x^*(M^*M - N^*N)x > 0$ for all $x \neq 0$, $x \in \mathbb{C}^n$. Hence, $M^*M - N^*N \succ 0$. As a result, $I - (M^{-1}N)(M^{-1}N)^* \succ 0$ and $\rho(L_\omega) \leq \|M^{-1}N\|_2 < 1$, that is, the SOR iterative method converges to the unique solution of (1) for any choice of the initial guess $x_0$.

(ii) If $\|T\|_2 > 1$ and $0 < \omega < \omega_0$, then by the same method we can prove that $\rho(L_\omega) \leq \|M^{-1}N\|_2 < 1$, that is, the SOR iterative method converges to the unique solution of (1) for any choice of the initial guess $x_0$. This completes the proof. □

**Theorem 4** *Let $A = I - L - U \in \mathbb{C}^{n\times n}$ be positive definite with $H = (A^* + A)/2$, and $L_\omega$ be defined in* (6). *Suppose that $A$ satisfies one of the two conditions*: (i) $\sigma_{\min}(F) \leq 1$ *and* $0 < \omega < 1$; (ii) $\sigma_{\min}(F) > 1$ *and* $0 < \omega < \omega_1$. *Then the SOR iterative method converges to the unique solution of* (1) *for any choice of the initial guess $x_0$, that is, $\rho(L_\omega) < 1$, where $\sigma_{\min}(F)$ denotes the minimal singular value of the matrix $F$, and*

$$\omega_1 = \frac{2\lambda_{\min}([(I-U)(I-U)^*]^{-1}H)}{1-\sigma_{\min}^2(F)}.$$

*Proof* According to the proofs of Theorems 1 and 3,

$$\begin{aligned} x^*\bigl(M^*M - N^*N\bigr)x &= x^*\bigl\{[(\alpha+1)I - L]^*[(\alpha+1)I - L] - (U+\alpha I)^*(U+\alpha I)\bigr\}x \\ &= 2(\alpha+1)x^*Hx + x^*\bigl[L^*L - (I-U)^*(I-U)\bigr]x \\ &= x^*(I-U)\bigl[2\alpha(I-U)^{-1}H(I-U)^{-*} + FF^* - I\bigr](I-U)^*x \\ &\geq \bigl[2\lambda_{\min}\bigl(\bigl[(I-U)(I-U)^*\bigr]^{-1}H\bigr) + \sigma_{\min}^2(F) - 1\bigr]x^*(I-U)(I-U)^*x. \end{aligned}$$

(i) If $\sigma_{\min}(F) \leq 1$ and $0 < \omega < 1$, then it is obvious that $x^*(M^*M - N^*N)x > 0$ for all $x \neq 0$, $x \in \mathbb{C}^n$. Hence, $M^*M - N^*N \succ 0$. As a result, $I - (M^{-1}N)(M^{-1}N)^* \succ 0$ and $\rho(L_\omega) \leq \|M^{-1}N\|_2 < 1$, that is, the SOR iterative method converges to the unique solution of (1) for any choice of the initial guess $x_0$.

(ii) If $\sigma_{\min}(F) > 1$ and $0 < \omega < \omega_0$, then by the same method we can prove that $\rho(L_\omega) \leq \|M^{-1}N\|_2 < 1$, that is, the SOR iterative method converges to the unique solution of (1) for any choice of the initial guess $x_0$. This proof is completed. □

**Remark 1**

(1) It follows from Theorem 3 that whether the forward Gauss-Seidel method converges or not, there always exists a positive constant $\omega_0$ such that, for $0 < \omega < \omega_0$, the SOR iterative method converges to the unique solution of (1) for any choice of the initial guess $x_0$.

(2) Theorem 4 shows that whether the backward Gauss-Seidel method converges or not, there always exists a positive constant $\omega_1$ such that, for $0 < \omega < \omega_1$ the SOR iterative method converges to the unique solution of (1) for any choice of the initial guess $x_0$.

### 2.3 Convergence results for SSOR iterative method

In what follows, convergence results for the SSOR iterative method are established for non-Hermitian positive definite linear systems.

**Theorem 5** *Let $A = I - L - U \in \mathbb{C}^{n\times n}$ with $A_L = I - L - L^*$ and $A_U = I - U - U^*$ both Hermitian positive semidefinite. Then for $0 < \omega < 2$, the SSOR iterative method converges to the unique solution of* (1) *for any choice of the initial guess $x_0$, that is, $\rho(S_\omega) < 1$ for $0 < \omega < 2$, where $S_\omega$ is defined in* (8).

*Proof* If $0 < \omega < 2$, then

$$\rho(S_\omega) = \rho(F_\omega F_\omega) = \rho\bigl\{(I-\omega U)^{-1}\bigl[\omega L + (1-\omega)I\bigr](I-\omega L)^{-1}\bigl[\omega U + (1-\omega)I\bigr]\bigr\} < 1.$$

Let $y$ be an eigenvector corresponding to the eigenvalue $\mu$ of $S_\omega$ with $|\mu| = \rho(S_\omega)$. Then

$$S_\omega y = (I-\omega U)^{-1}\bigl[\omega L + (1-\omega)I\bigr](I-\omega L)^{-1}\bigl[\omega U + (1-\omega)I\bigr]y = \mu y$$

and

$$\bigl[\omega L + (1-\omega)I\bigr](I-\omega L)^{-1}\bigl[\omega U + (1-\omega)I\bigr]y = \mu(I-\omega U)y.$$

Hence,

$$\bigl\{\mu(I-\omega U) - \bigl[\omega L + (1-\omega)I\bigr](I-\omega L)^{-1}\bigl[\omega U + (1-\omega)I\bigr]\bigr\}y = 0,$$

which shows that

$$Q_\mu = \mu(I-\omega U) - \bigl[\omega L + (1-\omega)I\bigr](I-\omega L)^{-1}\bigl[\omega U + (1-\omega)I\bigr] = R_\mu/(I-\omega L)$$

is singular, where $R_\mu/(I-\omega L)$ denotes the Schur complement of

$$R_\mu = \begin{bmatrix} I-\omega L & \omega U + (1-\omega)I \\ \omega L + (1-\omega)I & \mu(I-\omega U) \end{bmatrix}$$

with respect to $I-\omega L$. It then follows from Lemma 3.13 in [17] that $R_\mu$ is singular. Assume that $\rho(S_\omega) = |\mu| \geq 1$. Since $A_U = I-U-U^*$ is Hermitian positive semidefinite and $0 < \omega < 2$,

$$\begin{aligned} &\bigl[\mu(I-\omega U)\bigr]^*\bigl[\mu(I-\omega U)\bigr] - \bigl[\omega U + (1-\omega)I\bigr]^*\bigl[\omega U + (1-\omega)I\bigr] \\ &\quad\succeq (I-\omega U)^*(I-\omega U) - \bigl[\omega U + (1-\omega)I\bigr]^*\bigl[\omega U + (1-\omega)I\bigr] \\ &\quad= \omega(2-\omega)\bigl(I-U-U^*\bigr) \end{aligned}$$

is Hermitian positive semidefinite. Thus,

$$\begin{aligned} &\rho\bigl[\bigl(\bigl[\omega U + (1-\omega)I\bigr]\bigl[\mu(I-\omega U)\bigr]^{-1}\bigr)^*\bigl(\bigl[\omega U + (1-\omega)I\bigr]\bigl[\mu(I-\omega U)\bigr]^{-1}\bigr)\bigr] \\ &\quad= \bigl\|\bigl[\omega U + (1-\omega)I\bigr]\bigl[\mu(I-\omega U)\bigr]^{-1}\bigr\|_2 \\ &\quad\leq 1, \end{aligned}$$

which implies

$$\bigl\|\bigl[\omega U + (1-\omega)I\bigr]\bigl[\mu(I-\omega U)\bigr]^{-1}\bigr\|_2 \leq 1. \tag{12}$$

In the same way, the Hermitian positive semidefiniteness of $A_L = I-L-L^*$ and $0 < \omega < 2$ also yield the inequality

$$\bigl\|\bigl[\omega L + (1-\omega)I\bigr]\bigl[\mu(I-\omega L)\bigr]^{-1}\bigr\|_2 \leq 1. \tag{13}$$

Inequalities (12) and (13) show that $R_\mu$ is block diagonally dominant. By Theorem 2.1 in [18], $R_\mu$ is nonsingular, which contradicts the fact that $R_\mu$ is singular. This indicates that the assumption $\rho(S_\omega) = |\mu| \geq 1$ is not correct. Thus, $\rho(S_\omega) = |\mu| < 1$, i.e., the SSOR iterative method converges to the unique solution of (1) for any choice of the initial guess $x_0$. The proof is completed. □

We know from (5) and (8) that the SSOR method changes into SGS method when $\omega = 1$. Thus, it follows from Theorem 5 that the following conclusion is true.

**Theorem 6** *Let $A = I - L - U \in \mathbb{C}^{n\times n}$ with $A_L = I - L - L^*$ and $A_U = I - U - U^*$ both Hermitian positive semidefinite. Then the SGS iterative method converges to the unique solution of* (1) *for any choice of the initial guess $x_0$.*

## 3 Numerical examples

Two numerical examples are given in this section to demonstrate that the convergence results are true and the SOR iterative and SSOR iterative methods are very effective.

**Example 1** The coefficient matrix $A$ of the linear system (1) is given as

$$A = \begin{bmatrix} 1.0000 & 1.1000 & 1.2000 & 1.0000 \\ -0.5000 & 1.0000 & 1.4000 & 0.6500 \\ -0.2000 & -0.6000 & 1.0000 & 1.5500 \\ 0.2000 & 0.2500 & -0.4500 & 1.0000 \end{bmatrix}.$$

It is verified that $A$ is positive definite. By Matlab 7.0 computations we get Table 1.

Table 1 shows that for the matrix $A$, the SOR iterative method is convergent for $0 < \omega < 0.87$. By direct computations, $\|T\|_2 = 4.0258 > 1$ when $\omega_0 = 0.0281$. Thus, Theorem 3 shows that the SOR iterative method converges for the matrix $A$ when $0 < \omega < 0.0281$. The same is shown in Theorem 4 for $0 < \omega < 0.2292$. It follows from Table 1 that Theorems 3 and 4 are both true. However, both intervals given by the theorems are very narrow. Sometimes these intervals fail to include the optimal value such that $\rho(L_\omega)$ reaches minimization.

**Example 2** The coefficient matrix $A$ of the linear system (1) is given as

$$A = \begin{bmatrix} 1 & 1 & & & & & \\ -1 & 3 & 2 & & & & \\ & -1 & 5 & 3 & & & \\ & & -1 & 7 & 4 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -1 & 2n-5 & n-2 & \\ & & & & & -1 & 2n-3 & n-1 \\ & & & & & & -1 & 2n-1 \end{bmatrix}.$$

**Table 1 The comparison results on the spectral radius of the SOR iterative matrix of *A* for different *ω***

| $\omega$ | 0.01 | 0.02 | 0.0281 | 0.05 | 0.1 | 0.15 | 0.20 | 0.2292 |
|---|---|---|---|---|---|---|---|---|
| $\rho(L_\omega)$ | 0.9929 | 0.9858 | 0.9801 | 0.9645 | 0.9290 | 0.8936 | 0.8585 | 0.8382 |
| $\omega$ | 0.25 | 0.30 | 0.42 | 0.52 | 0.6 | 0.72 | 0.82 | 0.87 |
| $\rho(L_\omega)$ | 0.8239 | 0.7901 | 0.7155 | 0.6679 | 0.6512 | 0.7078 | 0.8769 | 1.0007 |

**Table 2 The comparison results on the spectral radius of the SSOR iterative matrix of *B* for different $\omega$**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\omega$ | 0.125 | 0.250 | 0.375 | 0.500 | 0.625 | 0.750 | 0.875 | 0.1 |
| $\rho(S_\omega)$ | 0.76945 | 0.56558 | 0.39654 | 0.26311 | 0.16501 | 0.10674 | 0.27623 | 0.26139 |
| $\omega$ | 1.125 | 1.250 | 1.375 | 1.5000 | 1.625 | 1.750 | 1.875 | 1.925 |
| $\rho(S_\omega)$ | 0.15931 | 0.18860 | 0.26281 | 0.36777 | 0.49742 | 0.64802 | 0.81648 | 0.88823 |

It is verified that $A_L$ and $A_U$ are both Hermitian positive definite for $n = 100$. Matlab computations yield some comparison results on the spectral radius of SSOR iterative matrices; see Table 2.

Table 2 shows that the spectral radius $\rho(S_\omega)$ gradually decreases to 0.10674 with $\omega$ increasing from 0.125 to 0.750, whereas $\rho(S_\omega)$ gradually increases from 0.15931 to 0.88823 with $\omega$ increasing from 1.125 to 1.925. However, when $\omega$ increases from 0.750 to 1.125, $\rho(S_\omega)$ gradually increases from 0.10674 to 0.27623 and gradually decreases from 0.27623 to 0.15931. Therefore, the optimal value of $\omega$ should be $\omega_{opt} \in (0.625, 1.250)$ such that the SSOR iterative method converges faster to the unique solution of (1) for any choice of the initial guess $x_0$.

## 4 Conclusions

In this paper, we study the convergence of the SOR and SSOR iterative methods for non-Hermitian positive definite linear systems. we propose some necessary and sufficient conditions for the convergence of the SOR iterative method. But, these conditions are only theoretically significant and are difficult to apply to practical computations. In what follows, two conditions are presented such that there always exists a positive constant $\omega_0$ ($\omega_1$) such that, for $0 < \omega < \omega_0$ ($0 < \omega < \omega_1$), the SOR iterative method converges for linear system (1) whether the forward or backward Gauss-Seidel method converges or not.

It is more important that a practical condition for both $A_L = I - L - L^*$ and $A_U = I - U - U^*$ to be Hermitian positive semidefinite is proposed such that both the SSOR iterative method for any $\omega \in (0, 2)$ and the SGS iterative method converge to the unique solution of (1) for any choice of the initial guess $x_0$.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to this work. All authors read and approved the final manuscript.

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