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# Optimal bounds for Neuman-Sándor mean in terms of the geometric convex combination of two Seiffert means

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**Abstract**

In this paper, we find the least value $\alpha$ and the greatest value $\beta$ such that the double inequality

$$P^{\alpha}(a,b)T^{1-\alpha}(a,b) < M(a,b) < P^{\beta}(a,b)T^{1-\beta}(a,b)$$

holds for all $a,b>0$ with $a\neq b$, where $M(a,b)$, $P(a,b)$, and $T(a,b)$ are the Neuman-Sándor, the first and second Seiffert means of two positive numbers $a$ and $b$, respectively.

**MSC:** 26E60

**Keywords:** Neuman-Sándor mean; the first Seiffert mean; the second Seiffert mean

## 1 Introduction

For $a,b>0$ with $a\neq b$, the Neuman-Sándor mean $M(a,b)$ [1], the first Seiffert mean $P(a,b)$ [2], and the second Seiffert mean $T(a,b)$ [3] are defined by

$$M(a,b)=\frac{a-b}{2\sinh^{-1}(\frac{a-b}{a+b})}, \tag{1.1}$$

$$P(a,b)=\frac{a-b}{4\tan^{-1}(\sqrt{a/b})-\pi},$$

$$T(a,b)=\frac{a-b}{2\tan^{-1}(\frac{a-b}{a+b})}, \tag{1.2}$$

respectively. It can be observed that the first Seiffert mean $P(a,b)$ can be rewritten as (see [1])

$$P(a,b)=\frac{a-b}{2\sin^{-1}(\frac{a-b}{a+b})}, \tag{1.3}$$

where $\sinh^{-1}(x)=\log(x+\sqrt{x^2+1})$, $\tan^{-1}(x)=\arctan(x)$, and $\sin^{-1}(x)=\arcsin(x)$ are the inverse hyperbolic sine, inverse tangent, inverse sine functions, respectively.

Recently, the means $M$, $P$, and $T$ and other means have been the subject of intensive research. Many remarkable inequalities for means can be found in the literature [4–10].

Let $H(a,b) = 2ab/(a+b)$, $G(a,b) = \sqrt{ab}$, $L(a,b) = (b-a)/(\log b - \log a)$, $I(a,b) = 1/e(b^b/a^a)^{1/(b-a)}$, $A(a,b) = (a+b)/2$, $Q(a,b) = \sqrt{(a^2+b^2)/2}$ and

$$M_p(a,b) = \begin{cases} (\frac{a^p+b^p}{2})^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases}$$

denote the harmonic, geometric, logarithmic, identric, arithmetic, root-square, and the $p$th power means of two positive numbers $a$ and $b$ with $a \neq b$, respectively. Then it is well known that the inequalities

$$H(a,b) < G(a,b) < L(a,b) < P(a,b) < I(a,b) < A(a,b) < M(a,b) < T(a,b) < Q(a,b)$$

hold for $a, b > 0$ with $a \neq b$.

Neuman and Sándor [1] established

$$\frac{\pi}{2}P(a,b) > \sinh^{-1}(1)M(a,b) > \frac{\pi}{4}T(a,b)$$

for all $a, b > 0$ with $a \neq b$.

Gao [11] proved that the optimal double inequalities

$$\frac{e}{\pi}I(a,b) < P(a,b) < I(a,b), \qquad I(a,b) < T(a,b) < \frac{2e}{\pi}I(a,b)$$

hold for all $a, b > 0$ with $a \neq b$.

The following bounds for the Seiffert means $P(a,b)$ and $T(a,b)$ in terms of the power mean were presented by Jagers in [12]:

$$M_{\frac{1}{2}} < P(a,b) < M_{\frac{2}{3}}(a,b)$$

for all $a, b > 0$ with $a \neq b$. Hästö [13] improved the results of [12] and found the sharp lower power mean bound for the Seiffert mean $P(a,b)$ as follows:

$$P(a,b) > M_{\frac{\log 2}{\log \pi}}(a,b)$$

for all $a, b > 0$ with $a \neq b$.

In [14], the authors proved that the sharp double inequality

$$M_{\frac{\log 2}{\log \pi - \log 2}} < T(a,b) < M_{\frac{5}{3}}(a,b)$$

holds for all $a, b > 0$ with $a \neq b$.

Let $\overline{L}_p(a,b) = (a^{p+1}+b^{p+1})/(a^p+b^p)$ be the Lehmer mean of two positive numbers $a$ and $b$ with $a \neq b$. In [15], the authors presented the following best possible Lehmer mean bounds for the Seiffert means $P(a,b)$ and $T(a,b)$:

$$\overline{L}_{-1/6}(a,b) < P(a,b) < \overline{L}_0(a,b) \quad \text{and} \quad \overline{L}_0(a,b) < T(a,b) < \overline{L}_{1/3}(a,b)$$

for all $a, b > 0$ with $a \neq b$.

Let $u$, $v$, and $w$ be bivariate means such that $u(a,b) < v(a,b) < w(a,b)$ for all $a,b > 0$ with $a \neq b$. The problems of finding the best possible parameters $\alpha$ and $\beta$ such that the inequalities $\alpha u(a,b) + (1-\alpha)v(a,b) < w(a,b) < \beta u(a,b) + (1-\beta)v(a,b)$ and $u(a,b)^{\alpha} v^{1-\alpha}(a,b) < w(a,b) < u(a,b)^{\beta} v^{1-\beta}(a,b)$ hold for all $a,b > 0$ with $a \neq b$ have attracted the interest of many mathematicians.

In [16] and [17], the authors proved that the double inequalities

$$\begin{aligned} &\alpha_1 Q(a,b) + (1-\alpha_1)A(a,b) < T(a,b) < \beta_1 Q(a,b) + (1-\beta_1)A(a,b), \\ &Q^{\alpha_2}(a,b)A^{1-\alpha_2}(a,b) < T(a,b) < Q^{\beta_2}(a,b)A^{1-\beta_2}(a,b) \end{aligned}$$

hold for all $a,b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq (4-\pi)/[(\sqrt{2}-1)\pi]$, $\beta_1 \geq 2/3$, $\alpha_2 \leq 2/3$, $\beta_2 \geq 4 - 2\log\pi/\log 2$.

In [1], Neuman and Sándor gave the inequality

$$Q(a,b)^{\frac{1}{3}} A(a,b)^{\frac{2}{3}} < M(a,b) < \frac{1}{3}Q(a,b) + \frac{2}{3}A(a,b).$$

In [8], Sándor proved the inequality

$$G(a,b)^{\frac{1}{3}} A(a,b)^{\frac{2}{3}} < P(a,b) < \frac{1}{3}G(a,b) + \frac{2}{3}A(a,b).$$

In [18] and [19], the authors proved that the double inequalities

$$\begin{aligned} &Q(a,b)^{\alpha_3} A^{1-\alpha_3}(a,b) < M(a,b) < Q(a,b)^{\beta_3} A^{1-\beta_3}(a,b), \\ &\alpha_4 Q(a,b) + (1-\alpha_4)G(a,b) < M(a,b) < \beta_4 Q(a,b) + (1-\beta_4)G(a,b) \end{aligned}$$

hold for all $a,b > 0$ with $a \neq b$ if and only if $\alpha_3 \leq 1/3$, $\beta_3 \geq 2(\log(2+\sqrt{2}) - \log 3)/\log 2$, $\alpha_4 \leq 2/3$, $\beta_4 \geq 1/[\sqrt{2}\log(1+\sqrt{2})]$.

In [20], the authors proved that the double inequality

$$\alpha_5 A(a,b) + (1-\alpha_5)G(a,b) < P(a,b) < \beta_5 A(a,b) + (1-\beta_5)G(a,b)$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $\alpha_5 \leq \pi/2$, $\beta_5 \geq 2/3$.

The main purpose of this paper is to find the least value $\alpha$ and the greatest value $\beta$ such that the double inequality

$$P^{\alpha}(a,b)T^{1-\alpha}(a,b) < M(a,b) < P^{\beta}(a,b)T^{1-\beta}(a,b)$$

holds for all $a,b > 0$ with $a \neq b$.

Theorem 3.1 and Theorem 3.3 in [21] provide the inequality

$$A(a,b)T(a,b) \leq M^2(a,b), P(a,b)M(a,b) \leq A^2(a,b),$$

following which one can get $P^{\frac{1}{3}}(a,b)T^{\frac{2}{3}} \leq M(a,b)$. Then the lower bound of $\alpha$ in Theorem 3.1 of Section 3 is achieved.

## 2 Lemmas

To establish our main result, we need several lemmas, which we present in this section.

For $x \in (0,1)$, the power series expansions of the functions $\tan^{-1}(x)$ and $\sinh^{-1}(x)$ are presented as follows:

$$\sinh^{-1}(x) = x - \frac{x^3}{6} + \frac{3x^5}{40} - \frac{15x^7}{336} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(2n+1)2^{2n}(n!)^2} x^{2n+1}, \tag{2.1}$$

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}. \tag{2.2}$$

**Lemma 2.1** *If* $x \in (0,1)$, *then one has*

$$1 + \frac{6x^2}{15} < \sqrt{1+x^2} < 1 + \frac{x^2}{2}, \tag{2.3}$$

$$\sin^{-1}(x)\sqrt{1-x^2} < x - \frac{x^3}{3} - \frac{2x^5}{15}, \tag{2.4}$$

$$\sinh^{-1}(x) > x - \frac{x^3}{6}, \tag{2.5}$$

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} < \tan^{-1}(x) < x - \frac{x^3}{3} + \frac{x^5}{5}, \tag{2.6}$$

*and*

$$\tan^{-1}(x) > x - \frac{x^3}{3} + \frac{x^5}{9}. \tag{2.7}$$

*Proof* Square every terms of inequality (2.3) at the same time, then it is easy to prove it. Inequality (2.4) was proved in Lemma 2.3 of [22]. Inequalities (2.5) and (2.6) follow immediately from equations (2.1) and (2.2), respectively.

Let $\Phi(x) = \tan^{-1}(x) - (x - \frac{x^3}{3} + \frac{x^5}{9})$. Then $\Phi'(x) = \frac{x^4(4-5x^2)}{9(1+x^2)}$. Thus, $\Phi(x)$ is strictly increasing on $(0, \frac{2\sqrt{5}}{5}]$ and strictly decreasing on $[\frac{2\sqrt{5}}{5}, 1)$. Considering $\Phi(0) = 0$ and $\Phi(1) = 0.0076\ldots > 0$, we can get $\Phi(x) > 0$ for $x \in (0,1)$. Therefore, inequality (2.7) holds. □

**Lemma 2.2** *If* $x \in (0, 0.7)$, *then one has*

$$\sin^{-1}(x)\sqrt{1-x^2} > x - \frac{x^3}{3} - \frac{x^5}{5}, \tag{2.8}$$

$$\tan^{-1}(x) > x - \frac{x^3}{3} + \frac{x^5}{7}, \tag{2.9}$$

*and*

$$\sinh^{-1}(x) < x - \frac{2x^3}{15}. \tag{2.10}$$

*Proof* Let

$$\gamma_1(x) = \sin^{-1}(x)\sqrt{1-x^2} - \left(x - \frac{x^3}{3} - \frac{x^5}{5}\right), \tag{2.11}$$

$$\gamma_2(x) = \tan^{-1}(x) - \left(x - \frac{x^3}{3} + \frac{x^5}{7}\right), \tag{2.12}$$

$$\gamma_3(x) = \left(x - \frac{2x^3}{15}\right) - \sinh^{-1}(x). \tag{2.13}$$

Then

$$\gamma_1'(x) = \frac{x\gamma_1^*(x)}{\sqrt{1-x^2}}, \tag{2.14}$$

$$\gamma_2'(x) = \frac{x^4(2-5x^2)}{7+7x^2}, \tag{2.15}$$

$$\gamma_3'(x) = \frac{\gamma_3^*(x)}{5\sqrt{1+x^2}}, \tag{2.16}$$

where

$$\gamma_1^*(x) = \left(x + x^3\right)\sqrt{1-x^2} - \sin^{-1}(x), \tag{2.17}$$

$$\gamma_3^*(x) = \sqrt{1+x^2}\left(5 - 2x^2\right) - 5. \tag{2.18}$$

Differentiating $\gamma_1^*(x)$ and $\gamma_3^*(x)$, we have

$$\gamma_1^{*\prime}(x) = \frac{x^2(1-4x^2)}{\sqrt{1-x^2}}, \tag{2.19}$$

$$\gamma_3^{*\prime}(x) = \frac{x}{\sqrt{1+x^2}}\left(1 - 6x^2\right). \tag{2.20}$$

Furthermore, direct or numerical computations lead to

$$\gamma_1(0) = 0, \qquad \gamma_1(0.7) = 0.0017\ldots > 0, \tag{2.21}$$

$$\gamma_1^*(0) = 0, \qquad \gamma_1^*(1) = -1.5708\ldots < 0, \tag{2.22}$$

$$\gamma_2(0) = 0, \qquad \gamma_2(0.7) = 0.0010\ldots > 0, \tag{2.23}$$

$$\gamma_3(0) = 0, \qquad \gamma_3(0.7) = 0.0016\ldots > 0, \tag{2.24}$$

and

$$\gamma_3^*(0) = 0, \qquad \gamma_3^*(1) = -0.7574\ldots < 0. \tag{2.25}$$

From (2.19), we can easy to see that $\gamma_1^*(x)$ is strictly increasing on $(0, \frac{1}{2}]$ and strictly decreasing on $[\frac{1}{2}, 1)$. This fact and (2.22) together with (2.14) imply that there exists $x_0 \in (\frac{1}{2}, 1)$, such that $\gamma_1'(x) > 0$ on $(0, x_0)$ and $\gamma_1'(x) < 0$ on $(x_0, 1)$. The monotonicity of $\gamma_1(x)$ and (2.21) lead to

$$\gamma_1(x) > 0$$

for x $\in (0, 0.7)$. Therefore, inequality (2.8) holds.

Equation (2.15) shows that $\gamma_2(x) > 0$ on $(0, \frac{\sqrt{10}}{5})$ and $\gamma_2(x) < 0$ on $(\frac{\sqrt{10}}{5}, 1)$. This fact and (2.23) lead to

$$\gamma_2(x) > 0$$

for $x \in (0, 0.7)$. That is to say inequality (2.9) holds.

By (2.20), we know that $\gamma_3^*(x)$ is strictly increasing on $(0, \frac{\sqrt{6}}{6}]$ and strictly decreasing on $[\frac{\sqrt{6}}{6}, 1)$. This fact and (2.25) together with (2.16) imply that there must exist $x_1 \in (\frac{\sqrt{6}}{6}, 1)$, such that $\gamma_3'(x) > 0$ on $(0, x_1)$ and $\gamma_3'(x) < 0$ on $(x_1, 1)$. It follows from the monotonicity of $\gamma_3(x)$ and (2.24) that

$$\gamma_3(x) > 0$$

for $x \in (0, 0.7)$. This means the inequality (2.10) holds. □

**Lemma 2.3** *If* $x \in (0.7, 1)$, *the double inequality*

$$x - \frac{x^3}{3} + \frac{2x^5}{17} < \tan^{-1}(x) < x - \frac{x^3}{3} + \frac{2x^5}{13} \tag{2.26}$$

*holds.*

*Proof* Let

$$\xi_1(x) = \left(x - \frac{x^3}{3} + \frac{2x^5}{13}\right) - \tan^{-1}(x),$$

$$\xi_2(x) = \left(x - \frac{x^3}{3} + \frac{2x^5}{17}\right) - \tan^{-1}(x).$$

Then

$$\xi_1'(x) = \frac{x^4(10x^2 - 3)}{13(1 + x^2)}, \tag{2.27}$$

$$\xi_2'(x) = \frac{x^4(10x^2 - 7)}{17(1 + x^2)}. \tag{2.28}$$

Equality (2.27) implies that $\xi_1(x)$ is strictly increasing on $[\frac{\sqrt{30}}{10}, 1)$. Additional numerical computations lead to $\frac{\sqrt{30}}{10} < 0.7$ and $\xi_1(0.7) = 0.0007976\ldots > 0$. Therefore, we can get $\xi_1(x) > 0$ for $x \in (0.7, 1)$. This implies the right hand side of the double inequality (2.26) holds.

Equality (2.28) implies $\xi_2(x)$ is strictly decreasing on $(0, \frac{\sqrt{70}}{10}]$ and strictly increasing on $[\frac{\sqrt{70}}{10}, 1)$. Because of $\xi_2(0) = 0$ and $\xi_2(1) = -0.0011\ldots < 0$, it leads to $\xi_2(x) < 0$ for $x \in (0, 1)$. Specially, for $x \in (0.7, 1)$. This means the left hand side of the double inequality (2.26) holds. □

**Lemma 2.4** *Let*

$$\mu_1(x) = \frac{1 + 3x^2}{(x + x^3)^2} - \frac{1}{(1 + x^2)[\sinh^{-1}(x)]^2} - \frac{x}{(1 + x^2)^{\frac{3}{2}} \sinh^{-1}(x)}, \tag{2.29}$$

$$\mu_2(x) = -\frac{1+3x^2}{(x+x^3)^2} + \frac{1}{(1-x^2)[\sin^{-1}(x)]^2} - \frac{x}{(1-x^2)^{\frac{3}{2}}\sin^{-1}(x)}, \tag{2.30}$$

*and*

$$\mu_3(x) = -\frac{2x}{(1+x^2)^2\tan^{-1}(x)} - \frac{1}{[\tan^{-1}(x)]^2(1+x^2)^2} + \frac{1+3x^2}{(x+x^3)^2}. \tag{2.31}$$

*Then, for any* $x \in (0.7, 1)$, *we have*

$$\mu_1(x) < 0.17, \qquad \mu_2(x) < -1.48, \tag{2.32}$$

*and*

$$\mu_3(x) > -0.05. \tag{2.33}$$

*Proof* From Lemmas 2.6 and 2.7 of [22], for any $x \in [0.7, 1)$, we can get $\mu_1'(x) \le 0.167\ldots < 0.17$ and $\mu_2'(x) \le -1.48798\ldots < -1.48$, respectively.

Differentiating $\mu_3(x)$, we have

$$\mu_3'(x) = \frac{2\eta_1(x) + 6x^2\tan^{-1}(x)\eta_2(x)}{[\tan^{-1}(x)(x+x^3)]^3}, \tag{2.34}$$

where

$$\eta_1(x) = x^3 - x^3\left[\tan^{-1}(x)\right]^2 - \left[\tan^{-1}(x)\right]^3$$

and

$$\eta_2(x) = x^2 + x^3\tan^{-1}(x) - \left(2x^2+1\right)\left[\tan^{-1}(x)\right]^2.$$

For any $x \in [0.7, 1)$,

$$\begin{aligned}\eta_1(x) &< x^3 - x^3\left(x - \frac{x^3}{3} + \frac{x^5}{9}\right)^2 - \left(x - \frac{x^3}{3} + \frac{x^5}{9}\right)^3 \\ &= -x^9\left(54 + x^6\right) < 0\end{aligned} \tag{2.35}$$

and

$$\begin{aligned}\eta_2(x) &< x^2 + x^3\left(x - \frac{x^3}{3} + \frac{2x^5}{13}\right) - \left(2x^2+1\right)\left(x - \frac{x^3}{3} + \frac{2x^5}{17}\right)^2 \\ &= \frac{x^4}{1{,}989}\left(-663 + 1{,}300x^2 - 637x^4\right) + \frac{x^8}{33{,}813}\left(-4{,}743 + 4{,}836x^2 - 936x^4\right)\end{aligned}$$

follow from inequalities (2.7) and (2.26), respectively. Because $-663 + 1{,}300x^2 - 637x^4 < 0$ and $-4{,}743 + 4{,}836x^2 - 936x^4 < 0$ for x $\in (0.7, 1)$, we have

$$\eta_2(x) < 0 \tag{2.36}$$

for $x \in (0.7, 1)$. Thus $\mu_3'(x) < 0$ for $x \in (0.7, 1)$ follows from (2.34), (2.35), and (2.36). Therefore, we obtain $\mu_3(x) > \mu_3(1) = -0.0419\ldots > -0.05$ for $x \in (0.7, 1)$. □

**Lemma 2.5** *Let* $f(x) = \frac{1}{\sqrt{1+x^2}\sinh^{-1}(x)} - (1-\lambda_0)\frac{1}{(1+x^2)\tan^{-1}(x)} - \lambda_0\frac{1}{\sqrt{1-x^2}\sin^{-1}(x)}$, *where* $\lambda_0 = \frac{\log(\frac{4\log(1+\sqrt{2})}{\pi})}{\log 2} = 0.1663\ldots$. *Then the function* $f(x)$ *is strictly decreasing on* $(0.7, 1)$.

*Proof* It is obvious that

$$\begin{aligned} f(x) &= \left[\frac{1}{\sqrt{1+x^2}\sinh^{-1}(x)} - \frac{1}{x(1+x^2)}\right] + \lambda_0\left[\frac{1}{x(1+x^2)} - \frac{1}{\sqrt{1-x^2}\sin^{-1}(x)}\right] \\ &\quad + (\lambda_0 - 1)\left[\frac{1}{(1+x^2)\tan^{-1}(x)} - \frac{1}{x(1+x^2)}\right] \\ &:= U_1(x) + \lambda_0 U_2(x) + (\lambda_0 - 1)U_3(x). \end{aligned}$$

Differentiating $f(x)$, we have

$$\begin{aligned} f'(x) &= U_1'(x) + \lambda_0 U_2'(x) + (\lambda_0 - 1)U_3'(x) \\ &= \mu_1(x) + \lambda_0\mu_2(x) + (\lambda_0 - 1)\mu_3(x), \end{aligned} \tag{2.37}$$

where $\mu_1(x)$, $\mu_2(x)$, and $\mu_3(x)$ are defined as in Lemma 2.4. Therefore, Lemma 2.4 and equation (2.37) yield

$$\begin{aligned} f'(x) &< 0.17 + \lambda_0(-1.48) + (\lambda_0 - 1)(-0.05) \\ &= -0.0345\ldots < 0 \end{aligned}$$

for $x \in (0.7, 1)$. The proof is completed. □

## 3 Main results

**Theorem 3.1** *The double inequality*

$$P^{\alpha}(a,b)T^{1-\alpha}(a,b) < M(a,b) < P^{\beta}(a,b)T^{1-\beta}(a,b)$$

*holds for all* $a, b > 0$ *with* $a \neq b$ *if and only if* $a \geq 1/3$ *and* $\beta \leq \frac{\log(\frac{4\log(1+\sqrt{2})}{\pi})}{\log 2} = 0.1663\ldots$.

*Proof* Because $P(a,b)$, $M(a,b)$, and $T(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b$. Let $p \in (0,1)$, $\lambda_0 = \frac{\log(\frac{4\log(1+\sqrt{2})}{\pi})}{\log 2}$, and $x = (a-b)/(a+b)$. Then $x \in (0,1)$ and

$$\begin{aligned} &p\log\bigl[P(a,b)\bigr] + (1-p)\log\bigl[T(a,b)\bigr] - \log\bigl[M(a,b)\bigr] \\ &\quad = p\log\left[\frac{x}{\sin^{-1}(x)}\right] + (1-p)\log\left[\frac{x}{\tan^{-1}(x)}\right] - \log\left[\frac{x}{\sinh^{-1}(x)}\right] \\ &\quad = \log\bigl[\sinh^{-1}(x)\bigr] - (1-p)\log\bigl[\tan^{-1}(x)\bigr] - p\log\bigl[\sin^{-1}(x)\bigr] := D_p(x). \end{aligned} \tag{3.1}$$

It follows that

$$D_p\bigl(0^+\bigr) = 0 \quad \text{and} \quad D_{\lambda_0}\bigl(1^-\bigr) = 0. \tag{3.2}$$

Differentiating $D_p(x)$, we have

$$\begin{aligned} D_p'(x) &= \frac{1}{\sinh^{-1}(x)\sqrt{1+x^2}} - (1-p)\frac{1}{\tan^{-1}(x)(1+x^2)} - p\frac{1}{\sin^{-1}(x)\sqrt{1-x^2}} \\ &= \frac{g_p(x)}{\sinh^{-1}(x)(1+x^2)\tan^{-1}(x)\sin^{-1}(x)\sqrt{1-x^2}}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} g_p(x) &= \left[\sqrt{1+x^2}\tan^{-1}(x) - (1-p)\sinh^{-1}(x)\right]\sin^{-1}(x)\sqrt{1-x^2} \\ &\quad - p\sinh^{-1}(x)\left(1+x^2\right)\tan^{-1}(x). \end{aligned} \tag{3.4}$$

On one hand, when $p = \frac{1}{3}$, Lemma 2.1 and equation (3.4) lead to

$$\begin{aligned} g_{\frac{1}{3}}(x) &= \left[\sqrt{1+x^2}\tan^{-1}(x) - \frac{2}{3}\sinh^{-1}(x)\right]\sin^{-1}(x)\sqrt{1-x^2} \\ &\quad - \frac{1}{3}\sinh^{-1}(x)\left(1+x^2\right)\tan^{-1}(x) \\ &< \left[\left(1+\frac{x^2}{2}\right)\left(x-\frac{x^3}{3}+\frac{x^5}{5}\right) - \frac{2}{3}\left(x-\frac{x^3}{6}\right)\right]\left(x-\frac{x^3}{3}-\frac{2x^5}{15}\right) \\ &\quad - \frac{1}{3}\left(x-\frac{x^3}{6}\right)\left(1+x^2\right)\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}\right) \\ &= \frac{x^6}{3{,}150}\left(-70+80x^2+41x^4-67x^6\right) < 0 \end{aligned} \tag{3.5}$$

for $x \in (0,1)$. According to (3.3) and (3.5), we can see that

$$D'_{\frac{1}{3}}(x) < 0 \tag{3.6}$$

for $x \in (0,1)$.

On the other hand, when $p = \lambda_0$, the inequalities (2.3) and (2.6) and Lemma 2.2 together with equation (3.4) lead to

$$\begin{aligned} g_{\lambda_0}(x) &= \left[\sqrt{1+x^2}\tan^{-1}(x) - (1-\lambda_0)\sinh^{-1}(x)\right]\sin^{-1}(x)\sqrt{1-x^2} \\ &\quad - \lambda_0\sinh^{-1}(x)\left(1+x^2\right)\tan^{-1}(x) \\ &> \left[\left(1+\frac{6x^2}{15}\right)\left(x-\frac{x^3}{3}+\frac{x^5}{7}\right) - (1-\lambda_0)\left(x-\frac{2x^3}{15}\right)\right]\left(x-\frac{x^3}{3}-\frac{x^5}{5}\right) \\ &\quad - \lambda_0\left(x-\frac{2x^3}{15}\right)\left(1+x^2\right)\left(x-\frac{x^3}{3}+\frac{x^5}{5}\right) \\ &= \frac{x^4}{1{,}575}F_{\lambda_0}(x) \end{aligned} \tag{3.7}$$

for $x \in (0, 0.7)$, where

$$F_{\lambda_0}(x) = (315 - 1{,}575\lambda_0) + (105\lambda_0 - 90)x^2 + (22 - 301\lambda_0)x^4 + (42\lambda_0 - 33)x^6 - 18x^8.$$

Because of $42\lambda_0 - 33 = -26.0142\ldots < 0$ and $105\lambda_0 - 90 = -72.5354\ldots < 0$, it follows that

$$\begin{aligned} F_{\lambda_0}(x) &> (315 - 1{,}575\lambda_0) + (105\lambda_0 - 90)x^2 + (22 - 301\lambda_0)x^4 + (42\lambda_0 - 33)x^4 - 18x^4 \\ &= (315 - 1{,}575\lambda_0) + (105\lambda_0 - 90)x^2 + (-29 - 259\lambda_0)x^4 \\ &:= F^*(x) \end{aligned} \tag{3.8}$$

and

$$F^{*\prime}(x) = 2(105\lambda_0 - 90)x + 4(-29 - 259\lambda_0)x^3 < 0$$

for $x \in (0, 0.7)$. Thus, we can get

$$F_{\lambda_0}(x) > F^*(x) > F^*(0.7) = 0.1825\ldots > 0 \tag{3.9}$$

for $x \in (0, 0.7)$. Therefore, equation (3.3) and inequalities (3.7)-(3.9) imply

$$D'_{\lambda_0}(x) > 0 \tag{3.10}$$

for $x \in (0, 0.7)$.

It follows from equation (3.3) and Lemma 2.5 that $D'_{\lambda_0}(x)$ is strictly decreasing on $(0.7, 1)$. Then from equation (3.10) and $D'_{\lambda_0}(1^-) = -\infty$, we know that there exists $x_* \in (0.7, 1)$ such that $D_{\lambda_0}(x)$ is strictly increasing on $(0, x_*]$ and strictly decreasing on $[x_*, 1)$. This in conjunction with (3.2) means that

$$D_{\lambda_0}(x) > 0 \tag{3.11}$$

for x $\in (0, 1)$.

Therefore, for all $a, b > 0$ with $a \neq b$,

$$M(a,b) > P^{\frac{1}{3}}(a,b)T^{\frac{2}{3}}(a,b), \tag{3.12}$$

follows from equations (3.1), (3.2), and (3.6) as well as

$$M(a,b) < P^{\lambda_0}(a,b)T^{1-\lambda_0}(a,b) \tag{3.13}$$

follows from equations (3.1), (3.2), and (3.11).

Finally, by easy computations, equations (1.1), (1.2), and (1.3) lead to

$$\frac{\log[T(a,b)] - \log[M(a,b)]}{\log[T(a,b)] - \log[P(a,b)]} = \frac{\log[\sinh^{-1}(x)] - \log[\tan^{-1}(x)]}{\log[\sin^{-1}(x)] - \log[\tan^{-1}(x)]}, \tag{3.14}$$

$$\lim_{x\to 0^+} \frac{\log[\sinh^{-1}(x)] - \log[\tan^{-1}(x)]}{\log[\sin^{-1}(x)] - \log[\tan^{-1}(x)]} = \frac{1}{3} \tag{3.15}$$

and

$$\lim_{x\to 1^-} \frac{\log[\sinh^{-1}(x)] - \log[\tan^{-1}(x)]}{\log[\sin^{-1}(x)] - \log[\tan^{-1}(x)]} = \lambda_0. \tag{3.16}$$

Thus, we have the following two claims.

**Claim 1** *If* $\alpha < \frac{1}{3}$, *then from* (3.14) *and* (3.15), *there must exist* $\delta_1 \in (0,1)$ *such that* $M(a,b) < P^{\alpha}(a,b)T^{1-\alpha}(a,b)$ *for all* $a,b > 0$ *with* $(a-b)/(a+b) \in (0,\delta_1)$.

**Claim 2** *If* $\beta > \lambda_0$, *then from* (3.14) *and* (3.16), *there must exist* $\delta_2 \in (0,1)$ *such that* $M(a,b) > P^{\beta}(a,b)T^{1-\beta}(a,b)$ *for all* $a,b > 0$ *with* $(a-b)/(a+b) \in (1-\delta_2,1)$.

Inequalities (3.12) and (3.13) in conjunction with the above two claims mean the proof is completed. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
H-YH carried out the proof of Theorem 3.1. NW carried out the proof of Lemmas 2.1-2.3. B-YL provided the main idea and carried out the proof of Lemmas 2.4 and 2.5. All authors read and approved the final manuscript.

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