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Notions of generalized s-convex functions on fractal sets

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Abstract

The purpose of this article is to present some new inequalities for products of generalized convex and generalized s-convex functions on fractal sets. Furthermore, some applications are given.

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Keywords: s-convex functions; fractal space; local fractional derivative

1 Introduction

Let $f: I \subset \mathbb{R} \longrightarrow \mathbb{R}^\alpha$. For any $x_1, x_2 \in I$ and $\gamma \in [0,1]$ if the inequality

$$f\big(\gamma x_1 + (1-\gamma)x_2\big) \leq \gamma^\alpha f(x_1) + (1-\gamma)^\alpha f(x_2)$$

holds, then f is called a generalized convex function on I [1]. In $\alpha = 1$, we have convex function, convexity is defined only in geometrical terms as being the property of a function whose graph bears tangents only under it [2].

The convexity of functions plays a significant role in many fields, for example, in biological system, economy, optimization, and so on [3–5].

In recent years, the fractal theory has received significantly remarkable attention from scientists and engineers. In the sense of Mandelbrot, a fractal set is one whose Hausdorff dimension strictly exceeds the topological dimension [6, 7]. Many researchers studied the properties of functions on fractal space and constructed many kinds of fractional calculus by using different approaches [8–12]. Particularly, in [13], Yang gave the analysis of local fractional functions on fractal space systematically, which includes local fractional calculus and the monotonicity of function.

Let $\mathbb{R}^\alpha$ be the real line numbers on fractal space. Then by using Gao-Yang-Kang's concept one can explain the definitions of the local fractional derivative and local fractional integral as in [12–16]. Now if r_1^α, r_2^α and $r_3^\alpha \in \mathbb{R}^\alpha$ $(0 < \alpha \leq 1)$, then

(1) $r_1^\alpha + r_2^\alpha \in \mathbb{R}^\alpha$, $r_1^\alpha r_2^\alpha \in \mathbb{R}^\alpha$,
(2) $r_1^\alpha + r_2^\alpha = r_2^\alpha + r_1^\alpha = (r_1 + r_2)^\alpha = (r_2 + r_1)^\alpha$,
(3) $r_1^\alpha + (r_2^\alpha + r_3^\alpha) = (r_1^\alpha + r_2^\alpha) + r_3^\alpha$,
(4) $r_1^\alpha r_2^\alpha = r_2^\alpha r_1^\alpha = (r_1 r_2)^\alpha = (r_2 r_1)^\alpha$,
(5) $r_1^\alpha (r_2^\alpha r_3^\alpha) = (r_1^\alpha r_2^\alpha) r_3^\alpha$,
(6) $r_1^\alpha (r_2^\alpha + r_3^\alpha) = (r_1^\alpha r_2^\alpha) + (r_1^\alpha r_3^\alpha)$,

(7) $r_1^\alpha + 0^\alpha = 0^\alpha + r_1^\alpha = r_1^\alpha$ and $r_1^\alpha \cdot 1^\alpha = 1^\alpha \cdot r_1^\alpha = r_1^\alpha$.

Let us start with some definitions as regards the local fractional calculus on $\mathbb{R}^\alpha$.

Definition 1.1 [13] Let y be a local fractional continuous function on the interval $[a_1, a_2]$. The local fractional integral of the function $y(m)$ of order α is defined by

$$\begin{aligned} {}_{a_1}I_{a_2}^{(\alpha)} y(m) &= \bigl(\Gamma(1+\alpha)\bigr)^{-1} \int_{a_1}^{a_2} y(\mu)(d\mu)^\alpha \\ &= \bigl(\Gamma(1+\alpha)\bigr)^{-1} \lim_{\Delta\mu \longrightarrow 0} \sum_{i=1}^{n} y(\mu_i)(\Delta\mu_i)^\alpha \end{aligned}$$

with $\Delta\mu_i = \mu_{i+1} - \mu_i$ and $\Delta\mu = \max\{\Delta\mu_i : i = 1, 2, \ldots, n-1\}$ where $[\mu_i, \mu_{i+1}]$, $i = 0, 1, \ldots, n-1$, $\mu_0 = a_1 < \mu_1 < \cdots < \mu_{n-1} < \mu_n = a_2$ is a partition of the interval $[a_1, a_2]$ and Γ is the well-known Gamma function $\Gamma(m) = \int_0^\infty t^{m-1} e^{-t}\, dt$.

In [17], Mo and Sui introduced the definitions of two kinds of generalized s-convex functions on fractal sets as follows.

Definition 1.2

(i) A function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}^\alpha$, is called a generalized s-convex $(0 < s < 1)$ in the first sense if

$$f(\gamma_1 a_1 + \gamma_2 a_2) \le \gamma_1^{s\alpha} f(a_1) + \gamma_2^{s\alpha} f(a_2) \tag{1}$$

for all $a_1, a_2 \in \mathbb{R}_+$ and all $\gamma_1, \gamma_2 \ge 0$ with $\gamma_1^s + \gamma_2^s = 1$. This class of functions is denoted by GK_s^1.

(ii) A function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}^\alpha$, is called a generalized s-convex $(0 < s < 1)$ in the second sense if (1) holds for all $a_1, a_2 \in \mathbb{R}_+$ and all $\gamma_1, \gamma_2 \ge 0$ with $\gamma_1 + \gamma_2 = 1$. This class of functions is denoted by GK_s^2.

In the same paper, [17], Mo and Sui proved that all functions from GK_s^2, for $s \in (0,1)$, are nonnegative.

In this article, recall that gs_1 and gs_2 are the generic classes consistent for those functions that are generalized s-convex in the first sense, and in the second sense, respectively. It is well known that there are many important established inequalities for the class of generalized convex functions, however, one of the most famous is known as the generalized Hermit-Hadamard inequality, or the 'generalized Hadamard inequality' and stated as follows (see [1]): let f be a generalized convex function on $[a_1, a_2] \subseteq \mathbb{R}$, $a_1 < a_2$, then

$$f\left(\frac{a_1 + a_2}{2}\right) \le \frac{\Gamma(1+\alpha)}{(a_2 - a_1)^\alpha}\, {}_{a_1}I_{a_2}^{(\alpha)} f(x) \le \frac{f(a_1) + f(a_2)}{2^\alpha}.$$

2 Basic results for generalized s-convex functions

Lemma 2.1 *If $f \in GK_s^1$ or $f \in GK_s^2$, then*

$$f(\gamma_1 x_1 + \gamma_2 x_2) \le \gamma_1^{\alpha s} f(x_1) + \gamma_2^{\alpha s} f(x_2)$$

with $\gamma_1, \gamma_2 \in [0,1]$, exclusively.

So, we might re-write the definitions of generalized s-convexity in the first sense and the second sense as follows.

Definition 2.1 A function $f: I \subset \mathbb{R}_+ \longrightarrow \mathbb{R}^\alpha$ is called a generalized s-convex in the first sense if

$$f\bigl(\gamma x_1 + \bigl(1-\gamma^s\bigr)^{\frac{1}{s}} x_2\bigr) \leq \gamma^{\alpha s} f(x_1) + \bigl(1-\gamma^s\bigr)^\alpha f(x_2)$$

for all $x_1, x_2 \in I$ and for all $0 \leq \gamma \leq 1$.

Definition 2.2 A function $f: I \subset \mathbb{R}_+ \longrightarrow \mathbb{R}^\alpha$ is called a generalized s-convex in the second sense if

$$f\bigl(\gamma x_1 + (1-\gamma)x_2\bigr) \leq \gamma^{\alpha s} f(x_1) + (1-\gamma)^{\alpha s} f(x_2)$$

for all $x_1, x_2 \in I$ and for all $0 \leq \gamma \leq 1$.

Theorem 2.1 *The classes GK_1^1, GK_1^2 and the class of generalized convex functions are equivalent when the domain is restricted to $\mathbb{R}_+$.*

Proof Simply an issue of applying the definitions. □

Now, some results as regards generalized s-convex functions are given.

Remark 2.1
(i) If $f \in GK_s^1$, then $f(\frac{x_1+x_2}{2^{\frac{1}{s}}}) \leq \frac{f(x_1)+f(x_2)}{2^\alpha}$.
(ii) If $f \in GK_s^2$, then $f(\frac{x_1+x_2}{2}) \leq \frac{f(x_1)+f(x_2)}{2^{\alpha s}}$.
(iii) For a function that is both generalized gs_1 and gs_2, there is a one-to-one correspondence between the set of all pairs of the sort (γ, η) with respect to gs_1 and the set of all pairs of the sort (γ, η) with respect to gs_2. (So we may write each γ as a γ_1^s and each η as a η_1^s and *vice versa*. This happens in the light of the fact that $0 \leq \gamma, \eta \leq 1$, $0 \leq s \leq 1$.)

Theorem 2.2 *If a function $f \in GK_s^1$ and $f \in GK_s^2$, then*

$$f(\gamma_1 x_1 + \eta_1 x_2) \leq \gamma_1^{\alpha s} f(x_1) + \eta_1^{\alpha s} f(x_2) \leq \gamma_2^{\alpha s} f(x_1) + \eta_2^{\alpha s} f(x_2)$$

for some $\{\gamma_i, \eta_i, i = 1, 2\} \subset [0, 1]$.

Proof It follows from the one-to-one correspondence proved before. To each γ_2, η_2 such that $\gamma_2 + \eta_2 = 1$, there corresponds γ_1, η_1 where $\gamma_1^s + \eta_1^s = 1$ and $\gamma_2 \geq \gamma_1$, $\eta_2 \geq \eta_1$ since $\{\gamma_i, \eta_i, i = 1, 2\} \subset [0, 1]$. □

Theorem 2.3 *If a function $f \in GK_s^1$ and $f \in GK_s^2$ and its domain concurs with its counter-domain, then $fof \in GK_s^2$.*

Proof $f(\gamma_1 x_1 + (1-\gamma_1^s)^{\frac{1}{s}} x_2) \le \gamma_1^{\alpha s} f(x_1) + (1-\gamma_1^s)^{\alpha} f(x_2)$, then

$$\begin{aligned} f\big(\gamma_1^{\alpha s} f(x_1) + \big(1-\gamma_1^s\big)^{\alpha} f(x_2)\big) &\le \big(\gamma_1^{\alpha s}\big)^{\alpha s} f\big(f(x_1)\big) + \big(\big(1-\gamma_1^s\big)^{\alpha}\big)^{\alpha s} f\big(f(x_2)\big) \\ &= \gamma_2^{\alpha s} fof(x_1) + \beta_2^{\alpha s} fof(x_2). \end{aligned}$$
□

3 Inequalities for generalized *s*-convex functions

Theorem 3.1 *Let* $g: [0,1] \longrightarrow \mathbb{R}^{\alpha}$ *be a function such that*

$$g(\gamma) = \frac{1}{(a_2-a_1)^{2\alpha}(\Gamma(1+\alpha))^2} \int_{a_1}^{a_2} \int_{a_1}^{a_2} f\big(\gamma x_1 + (1-\gamma) x_2\big)(dx_1)^{\alpha}(dx_2)^{\alpha},$$

where $f: [a_1, a_2] \longrightarrow \mathbb{R}^{\alpha}$ *and* $f \in GK_s^2$. *Then*:
(i) $g \in GK_s^2$ *in* $[0,1]$. *If* $f \in GK_s^1$, *then* $g \in GK_s^1$.
(ii) $2^{\alpha} g(0) = 2^{\alpha} g(1) = \frac{2^{\alpha}}{(a_2-a_1)^{\alpha}\Gamma(1+\alpha)}\, {}_{a_1}I_{a_2}^{(\alpha)} f(x_1)$ *is an upper bound for* $g(\gamma)$.
(iii) $2^{\alpha(1-s)} g(\gamma) \ge g(\frac{1}{2}) = \frac{1}{(a_2-a_1)^{2\alpha}(\Gamma(1+\alpha))^2} \int_{a_1}^{a_2} \int_{a_1}^{a_2} f(\frac{x_1+x_2}{2})(dx_1)^{\alpha}(dx_2)^{\alpha}$, $\gamma \in [0,1]$.

Proof (i) Take $\{\gamma_1, \gamma_2\} \subset [0,1]$, $\gamma_1 + \gamma_2 = 1$, $t_1, t_2 \in D$ and $f \in GK_s^2$, then we have

$$\begin{aligned} &g(\gamma_1 t_1 + \gamma_2 t_2) \\ &\quad= \frac{1}{(a_2-a_1)^{2\alpha}(\Gamma(1+\alpha))^2} \int_{a_1}^{a_2} \int_{a_1}^{a_2} f\big((\gamma_1 t_1 + \gamma_2 t_2) x_1 \\ &\qquad + \big(1 - (\gamma_1 t_1 + \gamma_2 t_2)\big) x_2\big)(dx_1)^{\alpha}(dx_2)^{\alpha} \\ &\quad\le \frac{1}{(a_2-a_1)^{2\alpha}(\Gamma(1+\alpha))^2} \int_{a_1}^{a_2} \int_{a_1}^{a_2} \big[\gamma_1^{\alpha s} f(t_1 x_1 + x_2 - t_1 x_2) \\ &\qquad + \gamma_2^{\alpha s} f(t_2 x_1 + x_2 - t_2 x_2)\big](dx_1)^{\alpha}(dx_2)^{\alpha} \\ &\quad= \gamma_1^{\alpha s} \frac{1}{(a_2-a_1)^{2\alpha}(\Gamma(1+\alpha))^2} \int_{a_1}^{a_2} \int_{a_1}^{a_2} f\big(t_1 x_1 + (1-t_1) x_2\big)(dx_1)^{\alpha}(dx_2)^{\alpha} \\ &\qquad + \gamma_2^{\alpha s} \frac{1}{(a_2-a_1)^{2\alpha}(\Gamma(1+\alpha))^2} \int_{a_1}^{a_2} \int_{a_1}^{a_2} f\big(t_2 x_1 + (1-t_2) x_2\big)(dx_1)^{\alpha}(dx_2)^{\alpha} \\ &\quad= \gamma_1^{\alpha s} g(t_1) + \gamma_2^{\alpha s} g(t_2), \end{aligned}$$

which implies that $g \in GK_s^2$ in $[0,1]$.

(ii) Since $f(\gamma x_1 + (1-\gamma) x_2) \le \gamma^{\alpha s} f(x_1) + (1-\gamma)^{\alpha s} f(x_2)$ we have

$$\begin{aligned} A &= \frac{1}{(\Gamma(1+\alpha))^2} \int_{a_1}^{a_2} \int_{a_1}^{a_2} f\big(\gamma x_1 + (1-\gamma) x_2\big)(dx_1)^{\alpha}(dx_2)^{\alpha} \\ &\le \frac{1}{(\Gamma(1+\alpha))^2} \int_{a_1}^{a_2} \int_{a_1}^{a_2} \gamma^{\alpha s} f(x_1)(dx_1)^{\alpha}(dx_2)^{\alpha} \\ &\quad + \frac{1}{(\Gamma(1+\alpha))^2} \int_{a_1}^{a_2} \int_{a_1}^{a_2} (1-\gamma)^{\alpha s} f(x_2)(dx_1)^{\alpha}(dx_2)^{\alpha} \\ &= \gamma^{\alpha s} \frac{1}{\Gamma(1+\alpha)} (a_2-a_1)^{\alpha}\, {}_{a_1}I_{a_2}^{(\alpha)} f(x_1) + (1-\gamma)^{\alpha s} \frac{1}{\Gamma(1+\alpha)} (a_2-a_1)^{\alpha}\, {}_{a_1}I_{a_2}^{(\alpha)} f(x_2) \\ &= \frac{(\gamma^{\alpha s} + (1-\gamma)^{\alpha s})}{\Gamma(1+\alpha)} (a_2-a_1)^{\alpha}\, {}_{a_1}I_{a_2}^{(\alpha)} f(x_1), \end{aligned}$$

since $\gamma^{\alpha s} \leq 1^{\alpha}$. Because $(1-\gamma)^{\alpha s} \leq 1^{\alpha}$ as well, we have

$$A \leq \frac{2^{\alpha}}{\Gamma(1+\alpha)}(a_2 - a_1)^{\alpha} {}_{a_1}I_{a_2}^{(\alpha)} f(x_1).$$

(iii) Since f is generalized s-convex in the second sense,

$$\frac{f(\gamma x_1 + (1-\gamma)x_2) + f((1-\gamma)x_1 + \gamma x_2)}{2^{\alpha s}} \geq f\left(\frac{x_1 + x_2}{2}\right)$$

for all $\gamma \in [0,1]$ and $x_1, x_2 \in [a,b]$.

If we integrate on $[a_1, a_2] \times [a_1, a_2]$, we obtain

$$\begin{aligned}
&\frac{1}{2^{\alpha s}(\Gamma(1+\alpha))^2} \int_{a_1}^{a_2} \int_{a_1}^{a_2} \big[f\big(\gamma x_1 + (1-\gamma)x_2\big) + f\big((1-\gamma)x_1 + \gamma x_2\big)\big](dx_1)^{\alpha}(dx_2)^{\alpha} \\
&\quad \geq \frac{1}{(\Gamma(1+\alpha))^2} \int_{a_1}^{a_2} \int_{a_1}^{a_2} f\left(\frac{x_1 + x_2}{2}\right)(dx_1)^{\alpha}(dx_2)^{\alpha} \\
&\quad \Rightarrow \\
&\frac{1}{2^{\alpha(s-1)}(a_2 - a_1)^{2\alpha}(\Gamma(1+\alpha))^2} \int_{a_1}^{a_2} \int_{a_1}^{a_2} f\big(\gamma x_1 + (1-\gamma)x_2\big)(dx_1)^{\alpha}(dx_2)^{\alpha} \\
&\quad \geq \frac{1}{(a_2 - a_1)^{2\alpha}(\Gamma(1+\alpha))^2} \int_{a_1}^{a_2} \int_{a_1}^{a_2} f\left(\frac{x_1 + x_2}{2}\right)(dx_1)^{\alpha}(dx_2)^{\alpha},
\end{aligned}$$

which means that

$$2^{\alpha(1-s)} g(\gamma) \geq g\left(\frac{1}{2}\right).$$

The proof is complete. □

Assume the following functions:

(i) $g_{F_1}: [0,1] \longrightarrow \mathbb{R}^{\alpha}$, defined by

$$g_{F_1}(\gamma) = \frac{1}{(\Gamma(1+\alpha))^2} \int_{a_1}^{a_2} \int_{a_1}^{a_2} f\big(\gamma x_1 + (1-\gamma)x_2\big) F(x_1)F(x_2)(dx_1)^{\alpha}(dx_2)^{\alpha},$$

where $f: [a_1, a_2] \subset \mathbb{R}_+ \longrightarrow \mathbb{R}^{\alpha}$ and $f \in GK_s^1$.

(ii) $g_{F_2}: [0,1] \longrightarrow \mathbb{R}^{\alpha}$, defined by

$$g_{F_2}(\gamma) = \frac{1}{(\Gamma(1+\alpha))^2} \int_{a_1}^{a_2} \int_{a_1}^{a_2} f\big(\gamma x_1 + (1-\gamma)x_2\big) F(x_1)F(x_2)(dx_1)^{\alpha}(dx_2)^{\alpha},$$

where $f: [a_1, a_2] \subset \mathbb{R}_+ \longrightarrow \mathbb{R}^{\alpha}$ and $f \in GK_s^2$.

Theorem 3.2 *The following holds*:

(i) g_{F_1} *and* g_{F_2} *are both symmetric about* $\gamma = \frac{1}{2}$.

(ii) $g_{F_2} \in GK_s^2$ *in* $[0,1]$.

(iii) *We have the upper bound*

$$2^{\alpha} g_{F_2}(1) = \frac{2^{\alpha}}{(\Gamma(1+\alpha))^2} \int_{a_1}^{a_2} \int_{a_1}^{a_2} f(x_1)F(x_1)F(x_2)(dx_1)^{\alpha}(dx_2)^{\alpha}$$

for the function $g_{F_2}(\gamma)$.

Proof (i) g_{F_1} and g_{F_2} are both symmetric about $\gamma = \frac{1}{2}$ because $g_{F_1}(\gamma) = g_{F_1}(1-\gamma)$ and $g_{F_2}(\gamma) = g_{F_2}(1-\gamma)$.

(ii) We get

$$\begin{aligned} g_{F_2}\big(\gamma t_1 + (1-\gamma)t_2\big) = {}& \frac{1}{(\Gamma(1+\alpha))^2} \int_{a_1}^{a_2} \int_{a_1}^{a_2} f\big(\big(\gamma t_1 + (1-\gamma)t_2\big)x_1 \\ &+ \big(1 - \big(\gamma t_1 + (1-\gamma)t_2\big)x_2\big)\big)F(x_1)F(x_2)(dx_1)^{\alpha}(dx_2)^{\alpha}. \end{aligned}$$

But

$$\begin{aligned} &\big(\gamma t_1 + (1-\gamma)t_2\big)x_1 + (1 - \big(\gamma t_1 + (1-\gamma)t_2\big)x_2 \\ &\quad = \gamma t_1 x_1 + (1-\gamma)t_2 x_1 + \gamma x_2 + (1-\gamma)x_2 - \gamma t_1 x_2 - (1-\gamma)t_2 x_2 \\ &\quad = \gamma(t_1 x_1 + x_2 - t_1 x_2) + (1-\gamma)(t_2 x_1 + x_2 - t_2 x_2). \end{aligned}$$

Since $f \in GK_s^2$,

$$\begin{aligned} &g_{F_2}\big(\gamma t_1 + (1-\gamma)t_2\big) \\ &\quad = \frac{1}{(\Gamma(1+\alpha))^2} \int_{a_1}^{a_2} \int_{a_1}^{a_2} f\big(\big(\gamma t_1 + (1-\gamma)t_2\big)x_1 \\ &\qquad + \big(1 - \big(\gamma t_1 + (1-\gamma)t_2\big)x_2\big)\big)F(x_1)F(x_2)(dx_1)\alpha(dx_2)^{\alpha} \\ &\quad \leq \gamma^{\alpha s} \frac{1}{(\Gamma(1+\alpha))^2} \int_{a_1}^{a_2} \int_{a_1}^{a_2} f\big(t_1 x_1 + (1-t_1)x_2\big)F(x_1)F(x_2)(dx_1)^{\alpha}(dx_2)^{\alpha} \\ &\qquad + (1-\gamma)^{\alpha s} \frac{1}{(\Gamma(1+\alpha))^2} \int_{a_1}^{a_2} \int_{a_1}^{a_2} f\big(t_2 x_1 + (1-t_2)x_2\big)F(x_1)F(x_2)(dx_1)^{\alpha}(dx_2)^{\alpha} \\ &\quad = \gamma^{\alpha s} g_{F_2}(t_1) + (1-\gamma)^{\alpha s} g_{F_2}(t_2), \end{aligned}$$

which proves that $g_{F_2} \in GK_s^2$.

(iii) From the definition and the assumptions, we get

$$\begin{aligned} g_{F_2}(\gamma) = {}& \frac{1}{(\Gamma(1+\alpha))^2} \int_{a_1}^{a_2} \int_{a_1}^{a_2} f\big(\gamma x_1 + (1-\gamma)x_2\big)F(x_1)F(x_2)(dx_1)^{\alpha}(dx_2)^{\alpha} \\ \leq {}& \gamma^{\alpha s} \frac{1}{(\Gamma(1+\alpha))^2} \int_{a_1}^{a_2} \int_{a_1}^{a_2} f(x_1)F(x_1)F(x_2)(dx_1)^{\alpha}(dx_2)^{\alpha} \\ &+ (1-\gamma)^{\alpha s} \frac{1}{(\Gamma(1+\alpha))^2} \int_{a_1}^{a_2} \int_{a_1}^{a_2} f(x_2)F(x_1)F(x_2)(dx_1)^{\alpha}(dx_2)^{\alpha} \\ = {}& \big(\gamma^{\alpha s} + (1-\gamma)^{\alpha s}\big) \frac{1}{(\Gamma(1+\alpha))^2} \int_{a_1}^{a_2} \int_{a_1}^{a_2} f(x_1)F(x_1)F(x_2)(dx_1)^{\alpha}(dx_2)^{\alpha} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{2^\alpha}{(\Gamma(1+\alpha))^2}\int_{a_1}^{a_2}\int_{a_1}^{a_2} f(x_1)F(x_1)F(x_2)(dx_1)^\alpha(dx_2)^\alpha \\
&= 2^\alpha g_{F_2}(1). \qquad \square
\end{aligned}
$$

Theorem 3.3 *Let $f_1, f_2 : [a_1, a_2] \subset \mathbb{R}_+ \longrightarrow \mathbb{R}^\alpha$, $a_1 < a_2$, be nonnegative, and generalized s-convex functions in the second sense. If $f_1 \in GK^2_{s_1}$ and $f_2 \in GK^2_{s_2}$ on $[a_1, a_2]$ for some $\gamma \in [0,1]$ and $s_1, s_2 \in (0,1]$, then*

$$
\begin{aligned}
&\frac{1}{(\Gamma(1+\alpha))^3}\int_{a_1}^{a_2}\int_{a_1}^{a_2}\int_0^1 f_1\bigl(\gamma x_1+(1-\gamma)x_2\bigr)f_2\bigl(\gamma x_1+(1-\gamma)x_2\bigr)(d\gamma)^\alpha(dx_2)^\alpha(dx_1)^\alpha \\
&\quad\leq \frac{2^\alpha\Gamma(1+(\gamma_1+\gamma_2)\alpha)}{\Gamma(1+(\gamma_1+\gamma_2+1)\alpha)}\frac{1}{(\Gamma(1+\alpha))^2}(a_2-a_1)^\alpha\int_{a_1}^{a_2} f_1(x_1)f_2(x_1)(dx_1)^\alpha \\
&\qquad+\frac{1}{2^\alpha(\Gamma(1+\alpha))^3}\beta_\alpha(a_2-a_1)^{2\alpha}\bigl[T_1(a,b)+T_2(a,b)\bigr],
\end{aligned}
$$

where

$$
\begin{aligned}
&\beta_\alpha = \int_0^1 \gamma^{\alpha s_1}(1-\gamma)^{\alpha s_2}(d\gamma)^\alpha, \\
&T_1(a_1,a_2) = f_1(a_1)f_2(a_1)+f_1(a_2)f_2(a_2),
\end{aligned}
$$

and

$$
T_2(a_1,a_2) = f_1(a_1)f_2(a_2)+f_1(a_2)f_2(a_1).
$$

Proof Since $f_1 \in GK^2_{s_1}$ and $f_2 \in GK^2_{s_2}$ on $[a_1, a_2]$,

$$
\begin{aligned}
&f_1\bigl(\gamma x_1+(1-\gamma)x_2\bigr) \leq \gamma^{\alpha s_1}f_1(x_1)+(1-\gamma)^{\alpha s_1}f_1(x_2), \\
&f_2\bigl(\gamma x_1+(1-\gamma)x_2\bigr) \leq \gamma^{\alpha s_2}f_2(x_1)+(1-\gamma)^{\alpha s_2}f_2(x_2),
\end{aligned}
$$

for all $\gamma \in [0,1]$. Since f_1 and f_1 are nonnegative,

$$
\begin{aligned}
&f_1\bigl(\gamma x_1+(1-\gamma)x_2\bigr)f_2\bigl(\gamma x_1+(1-\gamma)x_2\bigr) \\
&\quad\leq \gamma^{\alpha(s_1+s_2)}f_1(x_1)f_2(x_1)+(1-\gamma)^{\alpha(s_1+s_2)}f_1(x_2)f_2(x_2) \\
&\qquad+\gamma^{\alpha s_1}(1-\gamma)^{\alpha s_2}f_1(x_1)f_2(x_2)+\gamma^{\alpha s_2}(1-\gamma)^{\alpha s_1}f_1(x_2)f_2(x_1).
\end{aligned}
$$

Integrating both sides of the above inequality over $[0,1]$, we obtain

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_0^1 f_1\bigl(\gamma x_1+(1-\gamma)x_2\bigr)f_2\bigl(\gamma x_1+(1-\gamma)x_2\bigr)(d\gamma)^\alpha \\
&\quad\leq \frac{1}{\Gamma(1+\alpha)}f_1(x_1)f_2(x_1)\int_0^1 \gamma^{\alpha(s_1+s_2)}(d\gamma)^\alpha \\
&\qquad+\frac{1}{\Gamma(1+\alpha)}f_1(x_2)f_2(x_2)\int_0^1 (1-\gamma)^{\alpha(s_1+s_2)}(d\gamma)^\alpha \\
&\qquad+\frac{1}{\Gamma(1-\alpha)}f_1(x_1)f_2(x_2)\int_0^1 \gamma^{\alpha s_1}(1-\gamma)^{\alpha s_2}(d\gamma)^\alpha
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma(1+\alpha)} f_1(x_2) f_2(x_1) \int_0^1 \gamma^{\alpha s_2} (1-\gamma)^{\alpha s_1} (d\gamma)^\alpha \\
&= \frac{\Gamma(1+(s_1+s_2)\alpha)}{\Gamma(1+(s_1+s_2+1)\alpha)} f_1(x_1) f_2(x_1) + \frac{\Gamma(1+(s_1+s_2)\alpha)}{\Gamma(1+(s_1+s_2+1)\alpha)} f_1(x_2) f_2(x_2) \\
&+ \frac{1}{\Gamma(1+\alpha)} \beta_\alpha \big[f_1(x_1) f_2(x_2) + f_1(x_2) f_2(x_1) \big].
\end{aligned}
$$

If we integrate both sides of the above inequalities on $[a_1, a_2] \times [a_1, a_2]$, we obtain

$$
\begin{aligned}
&\frac{1}{(\Gamma(1+\alpha))^3} \int_{a_1}^{a_2} \int_{a_1}^{a_2} \int_0^1 f_1\big(\gamma x_1 + (1-\gamma) x_2\big) f_2\big(\gamma x_1 + (1-\gamma) x_2\big) (d\gamma)^\alpha (dx_2)^\alpha (dx_1)^\alpha \\
&\leq \frac{\Gamma(1+(s_1+s_2)\alpha)}{\Gamma(1+(s_1+s_2+1)\alpha)} \frac{1}{(\Gamma(1+\alpha))^2} \bigg[\int_{a_1}^{a_2} \int_{a_1}^{a_2} f_1(x_1) f_2(x_1) (dx_2)^\alpha (dx_1)^\alpha \\
&+ \int_{a_1}^{a_2} \int_{a_1}^{a_2} f_1(x_2) f_2(x_2) (dx_2)^\alpha (dx_1)^\alpha \bigg] \\
&+ \frac{1}{(\Gamma(1+\alpha))^3} \beta_\alpha \bigg[\int_{a_1}^{a_2} \int_{a_1}^{a_2} f_1(x_1) f_2(x_2) (dx_2)^\alpha (dx_1)^\alpha \\
&+ \int_{a_1}^{a_2} \int_{a_1}^{a_2} f_1(x_2) f_2(x_1) (dx_2)^\alpha (dx_1)^\alpha \bigg] \\
&= \frac{\Gamma(1+(s_1+s_2)\alpha)}{\Gamma(1+(s_1+s_2+1)\alpha)} \frac{1}{(\Gamma(1+\alpha))^2} (a_2 - a_1)^\alpha \\
&\times \bigg[\int_{a_1}^{a_2} f_1(x_1) f_2(x_1) (dx_1)^\alpha + \int_{a_1}^{a_2} f_1(x_2) f_2(x_2) (dx_2)^\alpha \bigg] \\
&+ \frac{1}{(\Gamma(1+\alpha))^3} \beta_\alpha \bigg[\int_{a_1}^{a_2} f_1(x_1) (dx_1)^\alpha \int_{a_1}^{a_2} f_2(x_2) (dx_2)^\alpha \\
&+ \int_{a_1}^{a_2} f_1(x_2) (dx_2)^\alpha \int_{a_1}^{a_2} f_2(x_1) (dx_1)^\alpha \bigg].
\end{aligned}
$$

By using the right half of the generalized Hadamard inequality on the right side of the above inequality, we have

$$
\begin{aligned}
&\frac{1}{(\Gamma(1+\alpha))^3} \int_{a_1}^{a_2} \int_{a_1}^{a_2} \int_0^1 f_1\big(\gamma x_1 + (1-\gamma) x_2\big) f_2\big(\gamma x_1 + (1-\gamma) x_2\big) (d\gamma)^\alpha (dx_2)^\alpha (dx_1)^\alpha \\
&\leq \frac{2^\alpha \Gamma(1+(s_1+s_2)\alpha)}{\Gamma(1+(s_1+s_2+1)\alpha)} \frac{1}{(\Gamma(1+\alpha))^2} (a_2 - a_1)^\alpha \int_{a_1}^{a_2} f_1(x_1) f_2(x_1) (dx_1)^\alpha \\
&+ \frac{1}{2^\alpha (\Gamma(1+\alpha))^3} \beta_\alpha (a_2 - a_1)^{2\alpha} \big[T_1(a_1, a_2) + T_2(a_1, a_2) \big].
\end{aligned}
$$

The proof is complete. □

Remark 3.1

(i) If $\alpha = 1$, in Theorem 3.3, then

$$
\beta = \int_0^1 \gamma^{s_1} (1-\gamma)^{s_2} (d\gamma) = \frac{\Gamma(s_1+1)\Gamma(s_2+1)}{\Gamma(s_1+s_2+2)}
$$

and

$$\begin{aligned}&\int_{a_1}^{a_2}\int_{a_1}^{a_2}\int_0^1 f_1\big(\gamma x_1+(1-\gamma)x_2\big)f_2\big(\gamma x_1+(1-\gamma)x_2\big)(d\gamma)(dx_1)(dx_2)\\&\quad\le \frac{2}{s_1+s_2+1}(a_2-a_1)\int_{a_1}^{a_2} f_1(x_1)f_2(x_1)\,dx_1\\&\qquad+\frac{\Gamma(s_1+1)\Gamma(s_2+1)}{2\Gamma(s_1+s_2+2)}(a_2-a_1)^2\big[T_1(a_1,a_2)+T_2(a_1,a_2)\big].\end{aligned}$$

(ii) If $\alpha=1$, and $s_1=s_2=1$, then $\frac{\Gamma(s_1+1)\Gamma(s_2+1)}{\Gamma(s_1+s_2+2)}=\frac{1}{6}$. Also,

$$\beta=\int_0^1\gamma^{s_1}(1-\gamma)^{s_2}(d\gamma)=\frac{\Gamma(s_1+1)\Gamma(s_2+1)}{\Gamma(s_1+s_2+2)},$$

which implies that

$$\begin{aligned}&\int_{a_1}^{a_2}\int_{a_1}^{a_2}\int_0^1 f_1\big(\gamma x_1+(1-\gamma)x_2\big)f_2\big(\gamma x_1+(1-\gamma)x_2\big)(d\gamma)(dx_2)(dx_1)\\&\quad\le\frac{2}{3}(a_2-a_1)\int_{a_1}^{a_2} f_1(x_1)f_2(x_1)\,dx_1\\&\qquad+\frac{1}{12}(a_2-a_1)^2\big[T_1(a_1,a_2)+T_2(a_1,a_2)\big].\end{aligned}$$

Theorem 3.4 *Let $f_1,f_2\colon[a_1,a_2]\subset\mathbb{R}_+\longrightarrow\mathbb{R}^\alpha$, $a_1<a_2$, be nonnegative and generalized s-convex functions in the second sense. If $f_1\in GK^2_{s_1}$ and $f_2\in GK^2_{s_2}$ on $[a_1,a_2]$ for some $\gamma\in[0,1]$ and $s_1,s_2\in(0,1]$, then*

$$\begin{aligned}&\frac{1}{(\Gamma(1+\alpha))^2}\int_{a_1}^{a_2}\int_0^1 f_1\left(\gamma x_1+(1-\gamma)\frac{a+b}{2}\right)f_2\left(\gamma x_1+(1-\gamma)\frac{a+b}{2}\right)(d\gamma)^\alpha(dx_1)^\alpha\\&\quad\le\frac{\Gamma(1+(s_1+s_2)\alpha)}{\Gamma(1+(s_1+s_2+1)\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_{a_1}^{a_2} f_1(x_1)f_2(x_1)(dx_1)^\alpha\\&\qquad+\frac{1}{2^\alpha\Gamma(1+\alpha)}\left[\frac{\Gamma(1+(s_1+s_2)\alpha)}{2^\alpha\Gamma(1+(s_1+s_2+1)\alpha)}+\frac{1}{\Gamma(1+\alpha)}\beta_\alpha\right]\\&\qquad\times(a_2-a_1)^\alpha\big[T_1(a_1,a_2)+T_2(a_1,a_2)\big],\end{aligned}$$

where β_α, $T_1(a_1,a_2)$, and $T_2(a_1,a_2)$ are defined in Theorem 3.3.

Proof Since $f_1\in GK^2_{s_1}$ and $f_2\in GK^2_{s_2}$ on $[a_1,a_2]$,

$$\begin{aligned}f_1\left(\gamma x_1+(1-\gamma)\frac{a_1+a_2}{2}\right)&\le\gamma^{\alpha s_1}f_1(x_1)+(1-\gamma)^{\alpha s_1}f_1\left(\frac{a_1+a_2}{2}\right),\\f_2\left(\gamma x_1+(1-\gamma)\frac{a_1+a_2}{2}\right)&\le\gamma^{\alpha s_2}f_2(x_1)+(1-\gamma)^{\alpha s_2}f_2\left(\frac{a_1+a_2}{2}\right),\end{aligned}$$

for all $x_1\in[a_1,a_2]$ and all $\gamma\in[0,1]$. Because f_1 and f_1 are nonnegative, we have

$$\begin{aligned}
&f_1\left(\gamma x_1+(1-\gamma)\frac{a_1+a_2}{2}\right)f_2\left(\gamma x_1+(1-\gamma)\frac{a_1+a_2}{2}\right)\\
&\quad\le \gamma^{\alpha(s_1+s_2)}f_1(x_1)f_2(x_1)+(1-\gamma)^{\alpha(s_1+s_2)}f_1\left(\frac{a_1+a_2}{2}\right)f_2\left(\frac{a_1+a_2}{2}\right)\\
&\qquad+\gamma^{\alpha s_1}(1-\gamma)^{\alpha s_2}f_1(x_1)f_2\left(\frac{a_1+a_2}{2}\right)+\gamma^{\alpha s_2}(1-\gamma)^{\alpha s_1}f_1\left(\frac{a_1+a_2}{2}\right)f_2(x_1).
\end{aligned}$$

Integrating both sides of the above inequality over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_0^1 f_1\left(\gamma x_1+(1-\gamma)\frac{a_1+a_2}{2}\right)f_2\left(\gamma x_1+(1-\gamma)\frac{a_1+a_2}{2}\right)(d\gamma)^\alpha\\
&\quad\le \frac{1}{\Gamma(1+\alpha)}f_1(x_1)f_2(x_1)\int_0^1\gamma^{\alpha(s_1+s_2)}(d\gamma)^\alpha\\
&\qquad+\frac{1}{\Gamma(1+\alpha)}f_1\left(\frac{a_1+a_2}{2}\right)f_2\left(\frac{a_1+a_2}{2}\right)\int_0^1(1-\gamma)^{\alpha(s_1+s_2)}(d\gamma)^\alpha\\
&\qquad+\frac{1}{\Gamma(1+\alpha)}f_1(x_1)f_2\left(\frac{a_1+a_2}{2}\right)\int_0^1\gamma^{\alpha s_1}(1-\gamma)^{\alpha s_2}(d\gamma)^\alpha\\
&\qquad+\frac{1}{\Gamma(1+\alpha)}f_1\left(\frac{a_1+a_2}{2}\right)f_2(x_1)\int_0^1\gamma^{\alpha s_2}(1-\gamma)^{\alpha s_1}(d\gamma)^\alpha\\
&\quad=\frac{\Gamma(1+(s_1+s_2)\alpha)}{\Gamma(1+(s_1+s_2+1)\alpha)}f_1(x_1)f_2(x_1)\\
&\qquad+\frac{\Gamma(1+(s_1+s_2)\alpha)}{\Gamma(1+(s_1+s_2+1)\alpha)}f_1\left(\frac{a_1+a_2}{2}\right)f_2\left(\frac{a_1+a_2}{2}\right)\\
&\qquad+\frac{1}{\Gamma(1+\alpha)}\beta_\alpha\left[f_1(x_1)f_2\left(\frac{a_1+a_2}{2}\right)+f_1\left(\frac{a_1+a_2}{2}\right)f_2(x_1)\right].
\end{aligned}$$

If we integrate both sides of the above inequalities on $[a_1,a_2]$, we obtain

$$\begin{aligned}
&\frac{1}{(\Gamma(1+\alpha))^2}\int_{a_1}^{a_2}\int_0^1 f_1\left(\gamma x_1+(1-\gamma)\frac{a_1+a_2}{2}\right)f_2\left(\gamma x_1+(1-\gamma)\frac{a_1+a_2}{2}\right)(d\gamma)^\alpha(dx_1)^\alpha\\
&\quad\le\frac{\Gamma(1+(s_1+s_2)\alpha)}{\Gamma(1+(s_1+s_2+1)\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_{a_1}^{a_2}f_1(x_1)f_2(x_1)(dx_1)^\alpha\\
&\qquad+\frac{\Gamma(1+(s_1+s_2)\alpha)}{\Gamma(1+(s_1+s_2+1)\alpha)}\frac{1}{\Gamma(1+\alpha)}(a_2-a_1)^\alpha f_1\left(\frac{a_1+a_2}{2}\right)f_2\left(\frac{a_1+a_2}{2}\right)\\
&\qquad+\frac{1}{(\Gamma(1+\alpha))^2}\beta_\alpha\left[f_2\left(\frac{a_1+a_2}{2}\right)\int_{a_1}^{a_2}f_1(x_1)(dx_1)^\alpha+f_1\left(\frac{a_1+a_2}{2}\right)\int_{a_1}^{a_2}f_2(x_1)(dx_1)^\alpha\right]\\
&\quad\le\frac{\Gamma(1+(s_1+s_2)\alpha)}{\Gamma(1+(s_1+s_2+1)\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_{a_1}^{a_2}f_1(x_1)f_2(x_1)(dx_1)^\alpha\\
&\qquad+\frac{\Gamma(1+(s_1+s_2)\alpha)}{\Gamma(1+(s_1+s_2+1)\alpha)}\frac{1}{\Gamma(1+\alpha)}(a_2-a_1)^\alpha\frac{f_1(a_1)+f_1(a_2)}{2^\alpha}\frac{f_2(a_1)+f_2(a_2)}{2^\alpha}\\
&\qquad+\frac{1}{(\Gamma(1+\alpha))^2}(a_2-a_1)^\alpha\beta_\alpha\left[2^\alpha\frac{f_1(a_1)+f_1(a_2)}{2^\alpha}\frac{f_2(a_1)+f_2(a_2)}{2^\alpha}\right]\\
&\quad=\frac{\Gamma(1+(s_1+s_2)\alpha)}{\Gamma(1+(s_1+s_2+1)\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_{a_1}^{a_2}f_1(x_1)f_2(x_1)(dx_1)^\alpha
\end{aligned}$$

$$+ \frac{\Gamma(1+(s_1+s_2)\alpha)}{4^\alpha\Gamma(1+(s_1+s_2+1)\alpha)}\frac{1}{\Gamma(1+\alpha)}(a_2-a_1)^\alpha\big[T_1(a_1,a_2)+T_2(a_1,a_2)\big]$$
$$+ \frac{1}{2^\alpha(\Gamma(1+\alpha))^2}(a_2-a_1)^\alpha\beta_\alpha\big[T_1(a_1,a_2)+T_2(a_1,a_2)\big].$$

Then

$$\begin{aligned}
&\frac{1}{(\Gamma(1+\alpha))^2}\int_{a_1}^{a_2}\int_0^1 f_1\left(\gamma x_1+(1-\gamma)\frac{a_1+a_2}{2}\right)f_2\left(\gamma x_1+(1-\gamma)\frac{a_1+a_2}{2}\right)(d\gamma)^\alpha(dx_1)^\alpha \\
&\leq \frac{\Gamma(1+(s_1+s_2)\alpha)}{\Gamma(1+(s_1+s_2+1))\alpha}\frac{1}{\Gamma(1+\alpha)}\int_{a_1}^{a_2} f_1(x_1)f_2(x_1)(dx_1)^\alpha \\
&\quad + \frac{1}{2^\alpha\Gamma(1+\alpha)}\left[\frac{\Gamma(1+(s_1+s_2)\alpha)}{2^\alpha\Gamma(1+(s_1+s_2+1))\alpha}+\frac{1}{\Gamma(1+\alpha)}\beta_\alpha\right] \\
&\quad \times (a_2-a_1)^\alpha\big[T_1(a_1,a_2)+T_2(a_1,a_2)\big].
\end{aligned}$$

The proof is complete. □

Remark 3.2 If $\alpha = 1$, then

$$\begin{aligned}
&\int_{a_1}^{a_2}\int_0^1 f_1\left(\gamma x_1+(1-\gamma)\frac{a_1+a_2}{2}\right)f_2\left(\gamma x_1+(1-\gamma)\frac{a_1+a_2}{2}\right)(d\gamma)(dx_1) \\
&\leq \frac{1}{s_1+s_2+1}\int_{a_1}^{a_2} f_1(x_1)f_2(x_1)(dx_1) \\
&\quad + \frac{1}{2}\left[\frac{1}{2(s_1+s_2+1)}+\frac{\Gamma(s_1+1)\Gamma(s_2+1)}{\Gamma(s_1+s_2+1)}\right](a_2-a_1)\big[T_1(a_1,a_2)+T_2(a_1,a_2)\big].
\end{aligned}$$

Theorem 3.5 *Let $f_1,f_2\colon [a_1,a_2]\subset\mathbb{R}_+\longrightarrow\mathbb{R}^\alpha$, $a_1<a_2$, be nonnegative and generalized s-convex functions in the second sense. If $f_1\in GK^2_{s_1}$ and $f_2\in GK^2_{s_2}$ on $[a_1,a_2]$ for some $\gamma\in[0,1]$ and $s_1,s_2\in(0,1]$, then*

$$\begin{aligned}
&\frac{1}{(\Gamma(1+\alpha))^2}\int_{a_1}^{a_2}\int_0^1 f_1\left(\gamma\frac{a_1+a_2}{2}+(1-\gamma)x_2\right)f_2\left(\gamma\frac{a_1+a_2}{2}+(1-\gamma)x_2\right)(d\gamma)^\alpha(dx_2)^\alpha \\
&\leq \frac{\Gamma(1+(s_1+s_2)\alpha)}{\Gamma(1+(s_1+s_2+1)\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_{a_1}^{a_2} f_1(x_2)f_2(x_2)(dx_2)^\alpha \\
&\quad + \frac{1}{2^\alpha\Gamma(1+\alpha)}\left[\frac{\Gamma(1+(s_1+s_2)\alpha)}{2^\alpha\Gamma(1+(s_1+s_2+1)\alpha)}+\frac{1}{\Gamma(1+\alpha)}\beta_\alpha\right] \\
&\quad \times (a_2-a_1)^\alpha\big[T_1(a_1,a_2)+T_2(a_1,a_2)\big],
\end{aligned}$$

where β_α, $T_1(a_1,a_2)$ and $T_2(a_1,a_2)$ are defined in Theorem 3.3.

Proof The proof of this theorem can easily be given like in Theorem 3.3. □

Theorem 3.6 *Let $f_1,f_2\colon [a_1,a_2]\subset\mathbb{R}_+\longrightarrow\mathbb{R}^\alpha$, $a_1<a_2$, be nonnegative, and generalized convex functions. If f_1 and f_2 are generalized convex on $[a_1,a_2]$ for some $\gamma\in[0,1]$, then*

$$\begin{aligned}
&\frac{1}{(\Gamma(1+\alpha))^2}\int_{a_1}^{a_2}\int_0^1 f_1\left(\gamma\frac{a_1+a_2}{2}+(1-\gamma)x_2\right)f_2\left(\gamma\frac{a_1+a_2}{2}+(1-\gamma)x_2\right)(d\gamma)^\alpha(dx_2)^\alpha \\
&\quad\le \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_{a_1}^{a_2} f_1(x_2)f_2(x_2)(dx_2)^\alpha \\
&\qquad + \frac{1}{2^\alpha\Gamma(1+\alpha)}\left[\frac{\Gamma(1+2\alpha)}{2^\alpha\Gamma(1+3\alpha)}+\frac{1}{\Gamma(1+\alpha)}\beta_\alpha\right](a_2-a_1)^\alpha\big[T_1(a_1,a_2)+T_2(a_1,a_2)\big],
\end{aligned}$$

where $T_1(a_1,a_2)$ and $T_2(a_1,a_2)$ are defined in Theorem 3.3, *but*

$$\beta_\alpha=\int_0^1\gamma^\alpha(1-\gamma)^\alpha(d\gamma)^\alpha=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}.$$

Proof Since f_1 and f_2 are generalized convex on $[a_1,a_2]$,

$$\begin{aligned}
&f_1\left(\gamma\frac{a_1+a_2}{2}+(1-\gamma)x_2\right)\le\gamma^\alpha f_1\left(\frac{a_1+a_2}{2}\right)+(1-\gamma)^\alpha f_1(x_2),\\
&f_2\left(\gamma\frac{a_1+a_2}{2}+(1-\gamma)x_2\right)\le\gamma^\alpha f_2\left(\frac{a_1+a_2}{2}\right)+(1-\gamma)^\alpha f_2(x_2),
\end{aligned}$$

for all $x_2\in[a_1,a_2]$, $\gamma\in[0,1]$. Since f_1 and f_1 are nonnegative,

$$\begin{aligned}
&f_1\left(\gamma\frac{a_1+a_2}{2}+(1-\gamma)x_2\right)f_2\left(\gamma\frac{a_1+a_2}{2}+(1-\gamma)x_2\right)\\
&\quad\le\gamma^{2\alpha}f_1\left(\frac{a_1+a_2}{2}\right)f_2\left(\frac{a_1+a_2}{2}\right)+(1-\gamma)^{2\alpha}f_1(x_2)f_2(x_2)\\
&\qquad+\gamma^\alpha(1-\gamma)^\alpha f_1\left(\frac{a_1+a_2}{2}\right)f_2(x_2)+\gamma^\alpha(1-\gamma)^\alpha f_1(x_2)f_2\left(\frac{a_1+a_2}{2}\right).
\end{aligned}$$

Integrating both sides of the above inequality over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_0^1 f_1\left(\gamma\frac{a_1+a_2}{2}+(1-\gamma)x_2\right)f_2\left(\gamma\frac{a_1+a_2}{2}+(1-\gamma)x_2\right)(d\gamma)^\alpha\\
&\quad\le\frac{1}{\Gamma(1+\alpha)}f_1\left(\frac{a_1+a_2}{2}\right)f_2\left(\frac{a_1+a_2}{2}\right)\int_0^1\gamma^{2\alpha}(d\gamma)^\alpha\\
&\qquad+\frac{1}{\Gamma(1+\alpha)}f_1(x_2)f_2(x_2)\int_0^1(1-\gamma)^{2\alpha}(d\gamma)^\alpha\\
&\qquad+\frac{1}{\Gamma(1+\alpha)}f_1\left(\frac{a_1+a_2}{2}\right)f_2(x_2)\int_0^1\gamma^\alpha(1-\gamma)^\alpha(d\gamma)^\alpha\\
&\qquad+\frac{1}{\Gamma(1+\alpha)}f_1(x_2)f_2\left(\frac{a_1+a_2}{2}\right)\int_0^1\gamma^\alpha(1-\gamma)^\alpha(d\gamma)^\alpha\\
&\quad=\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}f_1\left(\frac{a_1+a_2}{2}\right)f_2\left(\frac{a_1+a_2}{2}\right)+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}f_1(x_2)f_2(x_2)\\
&\qquad+\frac{1}{\Gamma(1+\alpha)}\beta_\alpha\left[f_1\left(\frac{a_1+a_2}{2}\right)f_2(x_2)+f_1(x_2)f_2\left(\frac{a_1+a_2}{2}\right)\right].
\end{aligned}$$

If we integrate both sides of the above inequalities on $[a_1, a_2]$, we obtain

$$\begin{aligned}
&\frac{1}{(\Gamma(1+\alpha))^2}\int_{a_1}^{a_2}\int_0^1 f_1\left(\gamma\frac{a_1+a_2}{2}+(1-\gamma)x_2\right)f_2\left(\gamma\frac{a_1+a_2}{2}+(1-\gamma)x_2\right)(d\gamma)^\alpha(dx_2)^\alpha\\
&\quad\le\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\frac{1}{\Gamma(1+\alpha)}(a_2-a_1)^\alpha f_1\left(\frac{a_1+a_2}{2}\right)f_2\left(\frac{a_1+a_2}{2}\right)\\
&\qquad+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_{a_1}^{a_2}f_1(x_2)f_2(x_2)(dx_2)^\alpha\\
&\qquad+\frac{1}{(\Gamma(1+\alpha))^2}\beta_\alpha\left[f_1\left(\frac{a_1+a_2}{2}\right)\int_{a_1}^{a_2}f_2(x_2)(dx_2)^\alpha+f_2\left(\frac{a_1+a_2}{2}\right)\int_{a_1}^{a_2}f_1(x_2)(dx_2)^\alpha\right]\\
&\quad\le\frac{\Gamma(1+2\alpha)}{\Gamma(1+2\alpha)}\frac{1}{\Gamma(1+\alpha)}(a_2-a_1)^\alpha\frac{f_1(a_1)+f_1(a_2)}{2^\alpha}\frac{f_2(a_1)+f_2(a_2)}{2^\alpha}\\
&\qquad+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_{a_1}^{a_2}f_1(x_2)f_2(x_2)(dx_2)^\alpha\\
&\qquad+\frac{1}{(\Gamma(1+\alpha))^2}\beta_\alpha\left[2^\alpha\frac{f_1(a_1)+f_1(a_2)}{2^\alpha}\frac{f_2(a_1)+f_2(a_2)}{2^\alpha}\right](a_2-a_1)^\alpha\\
&\quad=\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_{a_1}^{a_2}f_1(x_2)f_2(x_2)(dx_2)^\alpha+\frac{1}{2^\alpha\Gamma(1+\alpha)}\left[\frac{\Gamma(1+2\alpha)}{2^\alpha\Gamma(1+3\alpha)}\right.\\
&\qquad\left.+\frac{1}{\Gamma(1+\alpha)}\beta_\alpha\right](a_2-a_1)^\alpha\left[T_1(a_1,a_2)+T_2(a_1,a_2)\right].
\end{aligned}$$

The proof is complete. □

Remark 3.3 If $\alpha = 1$, then

$$\begin{aligned}
&\int_{a_1}^{a_2}\int_0^1 f_1\left(\gamma\frac{a_1+a_2}{2}+(1-\gamma)x_2\right)f_2\left(\gamma\frac{a_1+a_2}{2}+(1-\gamma)x_2\right)(d\gamma)(dx_2)\\
&\quad\le\frac{1}{3}\int_{a_1}^{a_2}f_1(x_2)f_2(x_2)(dx_2)+\frac{1}{6}(a_2-a_1)\left[T_1(a_1,a_2)+T_2(a_1,a_2)\right].
\end{aligned}$$

4 Applications

Example 4.1 Let $a_1, a_2 \in \mathbb{R}_+$, $a_1 < a_2$, and $a_2 - a_1 \le 1$, then

$$\begin{aligned}
&\left\{2^\alpha\left[\frac{2^\alpha}{\Gamma(1+\alpha)}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)^2\right.\right.\\
&\qquad\left.\left.+2^\alpha\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^2\right]\right\}K^2(a_1,a_2)\\
&\qquad+\left\{\frac{2^\alpha}{\Gamma(1+\alpha)}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)^2\right.\\
&\qquad\left.+4^\alpha\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^2\right\}G^2(a_1,a_2)\\
&\quad\le 2^\alpha\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)\Gamma(1+3\alpha)}\right)^2\frac{(a_2^{3\alpha}-a_1^{3\alpha})}{(a_2-a_1)^\alpha}+\frac{2^\alpha}{(\Gamma(1+\alpha))^3}\beta_\alpha A^2(a_1,a_2),
\end{aligned}$$

where

$$\beta_\alpha = \int_0^1 \gamma^\alpha (1-\gamma)^\alpha (d\gamma)^\alpha,$$
$$A(a_1,a_2) = \frac{a_1^\alpha + a_2^\alpha}{2^\alpha}, \quad a_1,a_2 \geq 0,$$
$$G(a_1,a_2) = \left(a_1^\alpha a_2^\alpha\right)^{\frac{1}{2}}, \quad a_1,a_2 \geq 0,$$

and

$$K(a_1,a_2) = \left(\frac{a_1^{2\alpha} + a_2^{2\alpha}}{2^\alpha}\right)^{\frac{1}{2}}, \quad a_1,a_2 \geq 0.$$

Proof If $f_1 \in GK_{s_1}^2$ and $f_2 \in GK_{s_2}^2$ on $[a_1,a_2]$ for some $\gamma \in [0,1]$ and $s_1,s_2 \in (0,1]$, then, by Theorem 3.3, if $f_1,f_2\colon [0,1] \longrightarrow [0^\alpha,1^\alpha]$, $f_1(x) = x^{\alpha s_1}$, $f_2(x) = x^{\alpha s_2}$, where $x \in [a_1,a_2]$, $s_1 = s_2 = 1$ and $a_2 - a_1 \leq 1$, then

$$\begin{aligned}
&\frac{1}{(\Gamma(1+\alpha))^3} \int_{a_1}^{a_2} \int_{a_1}^{a_2} \int_0^1 \left(\gamma x_1 + (1-\gamma)x_2\right)^{2\alpha} (d\gamma)^\alpha (dx_2)^\alpha (dx_1)^\alpha \\
&\quad \leq 2^\alpha \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \frac{1}{(\Gamma(1+\alpha))^2} (a_2-a_1)^\alpha \int_{a_1}^{a_2} x_1^{2\alpha} (dx_1)^\alpha \\
&\qquad + \frac{1}{2^\alpha(\Gamma(1+\alpha))^3} \beta_\alpha \left[a_1^{2\alpha} + a_2^{2\alpha} + 2^\alpha a_1^\alpha a_2^\alpha\right] (a_2-a_1)^{2\alpha} \\
&\quad \Rightarrow \\
&(a_2-a_1)^{2\alpha} \Biggl\{ \Biggl[\frac{2^\alpha}{\Gamma(1+\alpha)} \left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)^2 \\
&\qquad + 2^\alpha \left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right] \left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^2 \Biggr] \left(a_1^{2\alpha} + a_2^{2\alpha}\right) \\
&\qquad + \left[\frac{2^\alpha}{\Gamma(1+\alpha)} \left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)^2 + 4^\alpha \left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right] \left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^2 \right] a_1^\alpha a_2^\alpha \Biggr\} \\
&\quad \leq 2^\alpha \left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)\Gamma(1+3\alpha)}\right)^2 (a_2-a_1)^\alpha \left(a_2^{3\alpha} - a_1^{3\alpha}\right) \\
&\qquad + \frac{1}{2^\alpha(\Gamma(1+\alpha))^3} \beta_\alpha (a_2-a_1)^{2\alpha} \left(a_2^\alpha + a_1^\alpha\right)^2 \\
&\quad \Rightarrow \\
&(a_2-a_1)^{2\alpha} \Biggl\{ 2^\alpha \Biggl[\frac{2^\alpha}{\Gamma(1+\alpha)} \left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)^2 \\
&\qquad + 2^\alpha \left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right] \left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^2 \Biggr] K^2(a_1,a_2) \\
&\qquad + \Biggl[\frac{2^\alpha}{\Gamma(1+\alpha)} \left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)^2 \\
&\qquad + 4^\alpha \left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right] \left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^2 \Biggr] G^2(a_1,a_2) \Biggr\}
\end{aligned}$$

$$\begin{aligned}&\leq 2^{\alpha}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)\Gamma(1+3\alpha)}\right)^{2}(a_2-a_1)^{\alpha}\left(a_2^{3\alpha}-a_1^{3\alpha}\right)\\&\quad+\frac{2^{\alpha}}{(\Gamma(1+\alpha))^{3}}\beta_{\alpha}(a_2-a_1)^{2\alpha}A^{2}(a_1,a_2).\end{aligned}$$

The proof is complete. □

Remark 4.1 If $\alpha = 1$, then

$$\begin{aligned}&\frac{22}{36}K^{2}(a_1,a_2)+\frac{14}{36}G^{2}(a_1,a_2)\leq\frac{2}{9}\left(\frac{a_2^{3}-a_1^{3}}{a_2-a_1}\right)+\frac{1}{3}A^{2}(a_1,a_2)\\&\quad\Rightarrow\\&22K^{2}(a_1,a_2)+14G^{2}(a_1,a_2)\leq 8\left(a_2^{2}+a_1a_2+a_1^{2}\right)+12A^{2}(a_1,a_2),\end{aligned}$$

where

$$\begin{aligned}&A(a_1,a_2)=\frac{a_1+a_2}{2},\quad a_1,a_2\geq 0,\text{is the arithmetic mean,}\\&G(a_1,a_2)=(a_1a_2)^{\frac{1}{2}},\quad a_1,a_2\geq 0,\text{is the geometric mean, and}\\&K(a_1,a_2)=\left(\frac{a_1^{2}+a_2^{2}}{2}\right)^{\frac{1}{2}},\quad a_1,a_2\geq 0,\text{is the quadratic mean.}\end{aligned}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally in the design and progress of the study. Both authors read and approved the final manuscript.

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References
1. Mo, H, Sui, X, Yu, D: Generalized convex functions on fractal sets and two related inequalities. Abstr. Appl. Anal. **2014**, Article ID 636751 (2014)
2. Hörmander, L: Notions of Convexity. Springer Science & Business Media, vol. 127 (2007)
3. Kılıçman, A, Saleh, W: Some inequalities for generalized *s*-convex functions. JP J. Geom. Topol. **17**, 63-82 (2015)
4. Grinblatt, M, Linnainmaa, JT: Jensen's inequality, parameter uncertainty, and multiperiod investment. Rev. Asset Pricing Stud. **1**(1), 1-34 (2011)
5. Ruel, JJ, Ayres, MP: Jensen's inequality predicts effects of environmental variation. Trends Ecol. Evol. **14**(9), 361-366 (1999)
6. Edgar, GA: Integral, Probability, and Fractal Measures. Springer, New York (1998)
7. Kolwankar, KM, Gangal, AD: Local fractional calculus: a calculus for fractal space-time. In: Fractals: Theory and Applications in Engineering, pp. 171-181. Springer, London (1999)
8. Baleanu, D, Srivastava, HM, Yang, XJ: Local fractional variational iteration algorithms for the parabolic Fokker-Planck equation defined on Cantor sets. Prog. Fract. Differ. Appl. **1**(1), 1-11 (2015)
9. Carpinteri, A, Chiaia, B, Cornetti, P: Static-kinematic duality and the principle of virtual work in the mechanics of fractal media. Comput. Methods Appl. Mech. Eng. **191**(1-2), 3-19 (2001)
10. Yang, XJ, Baleanu, D, Srivastava, HM: Local fractional similarity solution for the diffusion equation defined on Cantor sets. Appl. Math. Lett. **47**, 54-60 (2015)
11. Yang, XJ, Baleanu, D, Srivastava, HM: Local Fractional Integral Transforms and Their Applications. Academic Press, New York (2015)

12. Zhao, Y, Cheng, DF, Yang, XJ: Approximation solutions for local fractional Schrödinger equation in the on-dimensional Cantorian system. Adv. Math. Phys. **2013**, Article ID 5 (2013)
13. Yang, XJ: Advanced Local Fractional Calculus and Its Applications. World Science Publisher, New York (2012)
14. Yang, XJ, Baleanu, D, Machado, JAT: Mathematical aspects of Heisenberg uncertainty principle within local fractional Fourier analysis. Bound. Value Probl. **2013**, Article ID 131 (2013)
15. Yang, AM, Chen, ZS, Srivastava, HM, Yang, XJ: Application of the local fractional series expansion method and the variational iteration method to the Helmholtz equation involving local fractional derivative operators. Abstr. Appl. Anal. **2013**, Article ID 259125 (2013)
16. Yang, XJ, Baleanu, D, Khan, Y, Mohyud-Din, ST: Local fractional variational iteration method for diffusion and wave equations on Cantor sets. Rom. J. Phys. **59**(1-2), 36-48 (2014)
17. Mo, H, Sui, X: Generalized *s*-convex functions on fractal sets. Abstr. Appl. Anal. **2014**, Article ID 254731 (2014)