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Fixed point results on subgraphs of directed graphs

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Abstract

In this paper, we obtain some fixed point results on subgraphs of directed graphs. We show that the Caristi fixed point theorem and a version of Knaster-Tarski fixed point theorem are special cases of our results.

Keywords: Directed graph; Fixed point; Self-path map

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Introduction

In 2005, Echenique started combining fixed point theory and graph theory by giving a short constructive proof for the Tarski fixed point theorem using graphs [1]. Afterwards, Espinola and Kirk applied fixed point results in graph theory [2]. A considerable contribution was made by Jachymski [3] and Beg et al. [4]. More recently, the authors, by providing a new notion of (P)-graphs and using arguments similar to those of Reich et al. [5-8], presented some iterative scheme results for G-contractive and G-nonexpansive maps on graphs [9]. In this paper, we obtain some fixed point results on subgraphs of directed graphs. As some consequences of our results, we obtain the Caristi fixed point theorem and Knaster-Tarski fixed point theorem.

Let (X, d) be a metric space and G a directed graph G such that $V(G) = X$ and the set $E(G)$ of its edges contains all loops. We denote the conversion of a graph G by G^{-1}, that is, the graph obtained from G by reversing the direction of the edges. A mapping $f : X \to X$ preserves the edges of G whenever $(x, y) \in E(G)$ implies $(fx, fy) \in E(G)$ for all $x, y \in X$ [3]. Since G is a directed graph, the direction of edge (x, y) is the inverse of the direction of edge (y, x), that is, $(x, y) \neq (y, x)$. Let G be the directed graph. A finite path of length n in G from x to y is a sequence $\{x_i\}_{i=0}^{n}$ of distinct vertices such that $x_0 = x$, $x_n = y$, and $(x_i, x_{i+1}) \in E(G)$ for $i = 0, 1, ..., n-1$ [9]. In fixed point theory, we like to deal with infinite graphs (see [9]). For this reason, we consider infinite paths. In fact, $B \subseteq E(G)$ is an infinite path whenever there is a finite path between any of its two vertices. Throughout this paper, a path could be finite or infinite, and the vertices of the path are pairwise distinct. Also, we consider cycles as finite paths. We denote by $[x]_G$ the set of all vertices in G wherein there is a (finite or infinite) path from those to x.

Let G' be a subgraph of the directed graph G and $x \in G'$. We emphasize that $[x]_G$ denotes the set of all vertices in G wherein there is a path from those to x via the edges in G. Also, we remind here that $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. Let G' be a subgraph of the directed graph G. We say that $b \in G$ is an upper bound for G' whenever $g' \in [b]_G$ for all $g' \in G'$. Also, we say that $c \in G$ is a supremum of G' whenever $c \in [b]_G$ for all upper bounds b. In fact, c is a least upper bound in a sense.

Example 1.1. Let G be the directed graph via the vertices $V(G) = \{a, b, c, d\}$ and the edges $E(G) = \{(a, b), (b, c), (c, d), (d, a)\}$. Suppose that G' is a subgraph of G denied by $V(G') = \{a, b, c\}$ and $E(G') = \{(a, b), (b, c)\}$, then c, d are upper bounds of G'. Thus, an upper bound is not unique in a subgraph necessarily.

Example 1.2. Let G be the directed graph via the vertices $V(G) = \{0, 2, \frac{1}{n} : n \geq 1\}$ and the edges $E(G) = \{(\frac{1}{n}, \frac{1}{n+1}) : n \geq 1\} \bigcup \{(\frac{1}{n}, 0)\} \bigcup (0, 2) \bigcup \{(\frac{1}{n}, 2) : n \geq 1\}$. If $V(G') = \{0, \frac{1}{n} : n \geq 1\}$ and $E(G') = \{(\frac{1}{n}, \frac{1}{n+1})\}$, then 0 and 2 are the supremum of G'. Thus, a supremum is not unique in a subgraph necessarily.

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Let G be a directed graph and $x_0 \in G$. We say that x_0 is an end point whenever there is no $x \in G$ such that $(x_0, x) \in G$ and $x \neq x_0$. There are many directed graphs via end points. In the following result, we give a class of directed graphs which have end points. The proof of this result is straightforward.

Lemma 1.1. *Let G be the directed graph, $X = V(G)$, $\varphi : X \to \mathbb{R}$ a function, $E(G) = \{(x,y) : d(x,y) \leq \varphi(x) - \varphi(y)\}$, and d a metric on X. If there exists $x_0 \in X$ such that $\varphi(x_0) = \inf_{x \in X} \varphi(x)$, then x_0 is an end point of G.*

Main results

Now we are ready to state and prove our main results. Let G be the directed graph and $\mathcal{M}$ the set of all paths in G. Then $\subseteq$ is a partial order on $\mathcal{M}$. By using Hausdorff's maximum principle, $\mathcal{M}$ has a maximal element. This means that G has a maximal path. We use this subject in our results.

Theorem 2.1. *Let G be a directed graph such that every path in G has an upper bound. Then G has an end point or a cycle.*

Proof. Suppose that G has no cycle. Let B be the maximal path in G and u an upper bond of B. If u is not an end point, there exists $x \in G$ such that $x \neq u$ and $(u, x) \in E(G)$. Thus, $B \bigcup \{x\}$ is a path in G and $B \subset B \bigcup \{x\}$. This contradiction shows that u is an end point of G. $\square$

Let G be a directed graph and T a selfmap on G. We say that T is a self-path map whenever $x \in [Tx]_G$ for all $x \in G$.

Theorem 2.2. *Let G be a directed graph. Then G has an end point if and only if each self-path map on G has a fixed point.*

Proof. Suppose that G has an end point x_0 and T is a self-path map. We prove that x_0 is a fixed point of T. Since $x_0 \in [Tx_0]_G$, there is a (finite or infinite) path $\{\lambda_i\}_{i \geq 0}$ between x_0 and Tx_0. Since x_0 is the end point of G and $\lambda_0 = x_0$, we have $x_0 = \lambda_1$. By continuing this process, it is easy to see that $x_0 = \lambda_i$ for all i. Thus, $x_0 = Tx_0$. Now assume that G is a directed graph and each self-path map on G has a fixed point but has no end point. Then for each $x \in G$, there exists $y \in G$ such that $y \neq x$ and $(x, y) \in E(G)$. By using the selection principle, we can define a selfmap T on G by $Tx = y$. Note that T is a self-path map which has no fixed point. $\square$

Example 2.1. Let G be the directed graph via the vertices $V(G) = \{0, \frac{1}{n} : n \geq 1\}$ and the edges $E(G) = \{(\frac{1}{n}, \frac{1}{n+1}) : n \geq 1\} \bigcup \{(\frac{1}{n}, 0)\} \bigcup (0, 1)$. Define the selfmap T on G by $T_0 = 1$ and $T\frac{1}{n} = \frac{1}{n+1}$ for all $n \geq 1$. Then $x \in [Tx]_G$ for all $x \in G$; T has no fixed point and G has no end point.

Theorem 2.3. *Let G be a directed graph such that every path in G has a supremum and T a selfmap on G such that $Tx \in [Ty]_G$ for all $x \in [y]_G$, $G' = \{x \in G : x \in [Tx]_G\} \neq \emptyset$, and G' has no cycle. Then T has a fixed point in G'.*

Proof. Suppose that B is a path in G and b is the supremum of B in G. Since $c \in [b]_G$ for all $c \in B$, $Tc \in [Tb]_G$ and so $c \in [Tb]_G$. It follows that Tb is an upper bound for B. Since b is the supremum, $b \in Tb$. Thus, $b \in G'$. By using Theorem 2.1, G' has an end point. Since $x \in [Tx]$ for all $x \in G'$, $Tx \in [T^2x]_G$ and so T is a self-path map on G'. Now by using Theorem 2.2, T has a fixed point in G'. $\square$

Now we show that a version of Knaster-Tarski fixed point theorem is a consequence of Theorem 2.3.

Theorem 2.4. *Let $(X, \preceq)$ be a partially ordered set such that each chain in X has a supremum and T a monotone selfmap on X. Assume that there exists $a \in X$ such that $a \preceq Ta$. Then T has a fixed point.*

Proof. Define the graph G by $V(G) = X$ and $E(G) = \{(x,y) : x \preceq y \text{ and } x \neq y\}$. Then $Tx \in [Ty]_G$ for all $x \in [y]_G$. Since $G' = \{x \in G : x \in [Tx]_G\} \neq \emptyset$ and G' has no cycle, by using Theorem 2.3, T has a fixed point. $\square$

Let X be a set and $\varphi : X \to (-\infty, \infty)$ a map. Suppose that G is the directed graph defined by $V(G) = X$ and $E(G) = \{(x,y) : d(x,y) \leq \varphi(x) - \varphi(y)\}$. We say that φ is lower semi-continuous whenever $\varphi(x) \leq \varphi(x_n)$ for all sequence $\{x_n\}$ in X with $x_n \to x$.

Lemma 2.5. *Let X be a complete metric space and $\varphi : X \to (-\infty, \infty)$ a map bounded from below. Suppose that G is the directed graph defined by $V(G) = X$ and $E(G) = \{(x,y) : d(x,y) \leq \varphi(x) - \varphi(y)\}$. If φ is lower semi-continuous, then G has an end point.*

Proof. First we prove that G has no cycle. If G has a cycle, then there exists a path $\{\lambda_i\}_{i=1}^n$ in G such that $\lambda_1 = \lambda_n$. It is easy to check that $d(\lambda_1, \lambda_i) \leq \varphi(\lambda_1) - \varphi(\lambda_i)$ and $d(\lambda_i, \lambda_n) \leq \varphi(\lambda_i) - \varphi(\lambda_n)$ for all $i = 2, 3, ..., n-1$, and so $\lambda_i = \lambda_1$ for $i \geq 2$. This contradiction shows that G has no cycle. Now we prove that each path in G has an upper bound. Let $\{x_\alpha\}_{\alpha \in \Omega}$ be a path in G. Then $\{\varphi(x_\alpha)\}_\Omega$ is a decreasing net of real numbers. Since φ is bounded from below, there is an increasing sequence $\{\alpha_n\}_{n \geq 1}$ in Ω such that $\lim_{n \to \infty} \varphi(x_{\alpha_n}) = \inf_{\alpha \in \Omega} \varphi(x_\alpha)$. One can easily show that $\{x_{\alpha_n}\}_{n \geq 1}$ is a Cauchy sequence and so converges to some $x \in X$. Since φ is lower semi-continuous, $x_{\alpha_n} \in [x]_G$ for all $n \geq 1$. Thus, x is an upper bound for $\{x_{\alpha_n}\}_{n \geq 1}$.

Now we show that x is an upper bound for $\{x_\alpha\}_{\alpha\in\Omega}$. If there exists $\beta \in \Omega$ such that $x_{\alpha_n} \in [x_\beta]_G$ for all $n \geq 1$, then $\varphi(x_\beta) \leq \varphi(x_{\alpha_n})$ for all $n \geq 1$ which implies that $\varphi(x_\beta) = \inf_{\alpha\in\Omega}\varphi(x_\alpha)$. Since $d(x_{\alpha_n}, x_\beta) \leq \varphi(x_{\alpha_n}) - \varphi(x_\beta)$, we get $x_{\alpha_n} \to x_\beta$ which implies that $x_\beta = x$. Hence, $\varphi(x) = \inf_{\alpha\in\Omega}\varphi(x_\alpha)$. Now we claim that $x_\alpha \in [x]_G$, and so x is an upper bound for $\{x_{\alpha_n}\}_{n\geq 1}$. In fact if there is $\alpha \in \Omega$ such that $x \in [x_\alpha]_G$, then $d(x, x_\alpha) \leq \varphi(x) - \varphi(x_\alpha) \leq \varphi(x_\alpha) - \varphi(x_\alpha) = 0$, and so $x = x_\alpha$. Since $\{x_\alpha\}_{\alpha\in\Omega}$ is a path in G, if the last case does not hold, then for each $\alpha \in \Omega$ there exists $n \geq 1$ such that $x_\alpha \in [x_{\alpha_n}]_G$. Hence, $x_\alpha \in [x]_G$ for all $\alpha \in \Omega$. Thus, x is an upper bound for $\{x_\alpha\}_{\alpha\in\Omega}$. Now by using Theorem 2.1, G has an end point. □

Now we can consequent the Caristi fixed point theorem.

Theorem 2.5. *Let X be a complete metric space, $\varphi : X \to (-\infty, \infty)$ a map bounded from below and lower semi-continuous, and $T : X \to X$ a selfmap satisfying $d(x, Tx) \leq \varphi(x) - \varphi(Tx)$ for all $x \in X$. Then T has a fixed point.*

Proof. Suppose that G is the directed graph via the vertices $V(G) = X$ and the edges $E(G) = \{(x, y) : d(x, y) \leq \varphi(x) - \varphi(y)\}$. By using Lemma 2.5, G has an end point. It is easy to see that T is a self-path map on G. Now by using Theorem 2.2, T has a fixed point. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
AB carried out in this manuscript. Also, all authors read and approved the final manuscript.

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