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Some expressions for the Riesz angle of weighted Orlicz sequence spaces

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Abstract

We obtain an expression for computation of the Riesz angle in weighted Orlicz sequence spaces. We use this expression to find some estimates of the Riesz angle for large classes of weighted Orlicz sequence spaces.

Introduction

In order to generalize the technique in [1] for c_0 to a larger class of Banach lattices, Borwein and Sims introduced in [2] the notion of a weakly orthogonal Banach lattice and Riesz angle $a(X)$. A Banach lattice is weakly orthogonal if $\lim_{n\to\infty} \| |x_n| \wedge |x| \| = 0$ for all $x \in X$, whenever $\{x_n\}_{n=1}^{\infty}$ is a weakly null sequence, where $|x| \wedge |y| = \min(|x|,|y|)$. The Riesz angle $a(X)$ of a Banach lattice $(X, \| \cdot \|)$ is

$$a(X) = \sup\{\|(|x| \vee |y|)\| : \|x\| \leq 1, \|y\| \leq 1\},$$

where $|x| \vee |y| = \max(|x|,|y|)$. Clearly $1 \leq a(X) \leq 2$. If there exists a weakly orthogonal Banach lattice Y such that $d(X,Y).a(Y) < 2$, where $d(X,Y)$ is the Banach-Mazur distance between the Banach spaces X and Y and $a(Y)$ is the Riesz angle of Y, then X has the weak fixed point property [2]. The coefficient $R(X)$ for a Banach space X is defined, and a connection between $a(X)$ and $R(X)$ is found in [3]. If there exists a Banach space Y with weak Opial condition, such that $d(X,Y).R(Y) < 2$, then X has the fixed point property [3]. An N-dimensional Riesz angle for a Banach lattice is introduced and studied in [4]. A fixed point theorem is proved involving the N-dimensional Riesz angle of the space [4].

The Riesz angle is an important geometric coefficient in Banach lattices [5-7]; therefore, the finding of formulas for its calculation or estimation is an interesting problem. Estimations of the Riesz angles in Orlicz function spaces are found in [8]. Some estimations of the Riesz angles in Orlicz sequence spaces equipped with Luxemburg norm and Orlicz norm were found in [9]. Later, a formula for computing the Riesz angle in Orlicz spaces is obtained

in [10]. We refine the technique in [10] to obtain a formula for the Riesz angle in a wide class of weighted Orlicz sequence spaces. We apply this formula to find the Riesz angle in some classes of weighted Orlicz sequence spaces. An open problem is to find formula for the computation of the N-dimensional Riesz angle, defined recently in [4], in Orlicz spaces.

Methods

We use the standard Banach space terminology from [11]. Let X be a real Banach space, S_X be the unit sphere of X. Let ℓ^0 stand for the space of all real sequences, i.e. $x = \{x_i\}_{i=1}^{\infty} \in \ell^0$, $\mathbb{N}$ is the set of natural numbers and $\mathbb{R}$ is the set of real numbers.

Definition 1. A Banach space $(X, \| \cdot \|)$ is said to be a Köthe sequence space if X is a subspace of ℓ^0 such that
(i) *If $x \in \ell^0$, $y \in X$ and $|x_i| \leq |y_i|$ for all $i \in \mathbb{N}$ then $x \in X$ and $\|x\| \leq \|y\|$*
(ii) *There exists an element $x \in X$ such that $x_i > 0$ for all $i \in \mathbb{N}$.*

We recall that M is an Orlicz function if M is even, convex, $M(0) = 0$, $M(t) > 0$ for $t > 0$. The Orlicz function $M(t)$ is said to have the property Δ_2 if there exists a constant c such that $M(2t) \leq cM(t)$ for every $t \in \mathbb{R}$. A weight sequence $w = \{w_i\}_{i=1}^{\infty}$ is a sequence of positive reals. Following [12], we say that $w = \{w_i\}_{i=1}^{\infty}$ is from the class Λ if there exists a subsequence $w = \{w_{i_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} w_{i_k} = 0$ and $\sum_{k=1}^{\infty} w_{i_k} = \infty$. A weighted Orlicz sequence space $\ell_M(w)$ generated by an Orlicz function M and a weight sequence w is the set of all sequences $x \in \ell^0$, such that the inequality $\widetilde{M}(x/\lambda) = \sum_{i=1}^{\infty} w_i M(x_i/\lambda) < \infty$ holds for some $\lambda < \infty$.

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It is well known that the space $\ell_M(w)$ is a Banach space if endowed with Luxemburg's norm $\|x\|_M = \inf\left\{r > 0 : \sum_{i=1}^{\infty} w_i M(x_i/r) \le 1\right\}$ or with Amemiya's norm $\||x\||_M = \inf\left\{\frac{1}{k}\left(1 + \sum_{i=1}^{\infty} w_i M(kx_i)\right) : k > 0\right\}$.

We will write $\ell_M(w, \|\cdot\|_M)$ and $\ell_M(w, \||\cdot\||_M)$ for the weighted Orlicz sequence spaces equipped with Luxemburg's and Amemiya's norms, respectively. Luxemburg's and Amemiya's norms are connected by the inequalities:

$$\|\cdot\|_M \le \||\cdot\||_M \le 2\|\cdot\|_M.$$

We will write $\ell_M(w)$, when the statement holds for the weighted Orlicz sequence space equipped with both norms - Luxemburg and Amemiya. The space $\ell_M(w)$, endowed with Luxemburg's or Amemiya's norm is a Köthe sequence space. In [13], Ruiz proved that the weighted Orlicz sequence spaces $\ell_M(w)$ are all mutually isomorphic for the weight sequences $w = \{w_n\}_{n=1}^{\infty} \in \Lambda$. Sharp estimates are found in [14] for the cotype of $\ell_M(w)$, which depends only on the generating Orlicz function, when the weight sequence verifies the condition $w = \{w_n\}_{n=1}^{\infty} \in \Lambda$. It is proved in [15] that $\ell_M(w)$, endowed with Luxemburg's or Amemiya's norm, has weak uniform normal structure iff $M \in \Delta_2$ at zero, when weight sequence verifies the condition $w = \{w_n\}_{n=1}^{\infty} \in \Lambda$. Weighted Orlicz sequence spaces were investigated for example in [16-18]. Let us mention that if the weight sequence is from the class Λ, then a lot of the properties of the space $\ell_M(w)$ depend only on the generating Orlicz function M [14,15], which is in contrast with the results when $w \notin \Lambda$ [12,13,15]. All these inspired us to find the Riesz angle in a wide class of weighted Orlicz sequence spaces.

Results and discussion

Theorem 1. *Let M be an Orlicz function with the Δ_2-condition and $w = \{w_i\}_{i=1}^{\infty} \in \Lambda$. Then, the Riesz angle of $X = (\ell_M(w), \|\cdot\|)$ can be expressed as follows:*

$$a(X) = \sup\left\{k_x : \widetilde{M}_w\left(\frac{x}{k_x}\right) = \frac{1}{2}, x \in S_{\ell_M(w)}\right\}.$$

Theorem 2. *Let M be an Orlicz function with the Δ_2-condition and $w = \{w_i\}_{i=1}^{\infty} \in \Lambda$. Then, the Riesz angle of $X = (\ell_M(w), \||\cdot\||)$ can be expressed as follows:*

$$d(X) \le a(X) \le \frac{3}{2} d(X),$$

where

$$d(X) = \sup_{\||x\||=1} \inf_{k>1}\left\{d_{x,k} : \widetilde{M}_w\left(\frac{kx}{d_{x,k}}\right) = \frac{k-1}{2}\right\}.$$

Lemma 1. *Let $w = \{w_i\}_{i=1}^{\infty} \in \Lambda$ and $v = \{v_i\}_{i=1}^{\infty} \in \Lambda$ be an arbitrary subsequence of w. Then, there exist sequences of naturals $\{m_i^{(s)}\}_{i=1}^{\infty}$, $\{k_i^{(s)}\}_{i=1}^{\infty}$, $s \in \mathbb{N}$, such that*

$$1 \le m_1^{(1)} \le k_1^{(1)}$$

$$k_{n-1}^{(1)} < m_1^{(n)}, \quad m_i^{(s)} \le k_i^{(s)}, \quad k_i^{(s)} < m_{i+1}^{(s-1)}$$

for $n, i, s \in \mathbb{N}, n \ge 2, i + s = n + 1$,

and for every $i \in \mathbb{N}$, there holds the equality

$$\sum_{s=1}^{\infty} \sum_{j=m_i^{(s)}}^{k_i^{(s)}} v_j = w_i.$$

Proof. By $v \in \Lambda$, it follows that there is a subsequence $\{v_{i_j}\}_{j=1}^{\infty}$, such that $\lim_{j\to\infty} v_{i_j} = 0$ and $\sum_{j=1}^{\infty} v_{i_j} = \infty$. For the simplicity of the notations, let the subsequence $\{v_{i_j}\}_{j=1}^{\infty}$ denoted by $\{v_j\}_{j=1}^{\infty}$. We will prove the Lemma by induction on n:

(i) Let $n = 1$. We can choose $m_1^{(1)}, k_1^{(1)}$ so that

$$m_1^{(1)} \le k_1^{(1)}$$

and

$$w_1 - \frac{w_1}{2} \le \sum_{j=m_1^{(1)}}^{k_1^{(1)}} v_j < w_1.$$

Let us use the notation

$$f(i, p) = \sum_{s=1}^{p} \sum_{j=m_i^s}^{k_i^s} v_j.$$

(ii) Let $n = 2$. We will show that we can choose $m_i^{(s)}, k_i^{(s)}$, $i + s = 3$, so that

$$k_1^{(1)} < m_1^{(2)} \le k_1^{(2)} < m_2^{(1)} \le k_2^{(1)}$$

and

$$w_i - \frac{w_i}{2^{3-i}} \le \sum_{s=1}^{3-i} \sum_{j=m_i^{(s)}}^{k_i^{(s)}} v_j < w_i, \quad \text{for } i = 1, 2.$$

Indeed, let us choose first $m_1^{(2)}, k_1^{(2)} \in \mathbb{N}$: $k_1^{(1)} < m_1^{(2)} \le k_1^{(2)}$, such that

$$w_1 - \frac{w_1}{2^2} \le f(1, 1) + \sum_{j=m_1^{(2)}}^{k_1^{(2)}} v_j < w_1.$$

Then, we choose $m_2^{(1)}, k_2^{(1)} \in \mathbb{N}$: $k_1^{(2)} < m_2^{(1)} \le k_2^{(1)}$, such that

$$w_2 - \frac{w_2}{2} \le \sum_{j=m_2^{(1)}}^{k_2^{(1)}} v_j < w_2.$$

(iii) Suppose that for $n = p$, we have chosen $\{m_i^{(s)}\}_{i=1}^{\infty}$, $\{k_i^{(s)}\}_{i=1}^{\infty}$, $i + s = p + 1$ with the properties:

$$k_{p-1}^{(1)} < m_1^{(p)}, m_i^{(s)} \le k_i^{(s)}, k_i^{(s)} < m_{i+1}^{(s-1)}$$

for $i + s = p + 1$ and

$$w_i - \frac{w_i}{2^{p+1-i}} \le \sum_{s=1}^{p+1-i} \sum_{j=m_i^{(s)}}^{k_i^{(s)}} v_j < w_i, \text{ for } i \le p.$$

(iv) Let $n = p + 1$. We will show that we can choose $\{m_i^{(s)}\}_{i=1}^{\infty}$, $\{k_i^{(s)}\}_{i=1}^{\infty}$, $i + s = p + 2$, so that

$$k_p^{(1)} < m_1^{(p+1)}, m_i^{(s)} \le k_i^{(s)}, k_i^{(s)} < m_{i+1}^{(s-1)}$$

for $i + s = p + 2$ and

$$w_i - \frac{w_i}{2^{p+1-i}} \le \sum_{s=1}^{p+1-i} \sum_{j=m_i^{(s)}}^{k_i^{(s)}} v_j < w_i, \qquad (1)$$

for $i \le p + 1$.

Indeed, let us choose first $m_1^{(p+1)}, k_1^{(p+1)} \in \mathbb{N}: k_p^{(1)} < m_1^{(p+1)} \le k_1^{(p+1)}$, such that

$$w_1 - \frac{w_1}{2^{p+1}} \le f(1,p) + \sum_{j=m_1^{(p+1)}}^{k_1^{(p+1)}} v_j < w_1.$$

Then, we choose $m_2^{(p)}, k_2^{(p)} \in \mathbb{N}: k_1^{(p+1)} < m_2^{(p)} \le k_2^{(p)}$, such that

$$w_2 - \frac{w_2}{2^p} \le f(2, p-1) + \sum_{j=m_2^{(p)}}^{k_2^{(p)}} v_j < w_2.$$

If for $i_0 \le p - 1$ we have chosen $m_{i_0}^{(p-i_0+2)}, k_{i_0}^{(p-i_0+2)} \in \mathbb{N}$ to satisfy the inequalities $k_{i_0}^{(p-i_0+1)} < m_{i_0}^{(p-i_0+2)} \le k_{i_0}^{(p-i_0+2)}$, such that

$$w_{i_0} - \frac{w_{i_0}}{2^{p-i_0+2}} \le f(i_0, p-i_0+1) + \sum_{j=m_{i_0}^{(p-i_0+2)}}^{k_{i_0}^{(p-i_0+2)}} v_j < w_{i_0},$$

then for $i_0 + 1 \le p$, we choose $m_{i_0+1}^{(p-i_0+1)}, k_{i_0+1}^{(p-i_0+1)} \in \mathbb{N}: k_{i_0+1}^{(p-i_0)} < m_{i_0+1}^{(p-i_0+1)} \le k_{i_0+1}^{(p-i_0+1)}$, such that

$$w_{i_0+1} - \frac{w_{i_0+1}}{2^{p-i_0+1}} \le f(i_0+1, p-i_0)$$

$$+ \sum_{j=m_{i_0+1}^{(p-i_0+1)}}^{k_{i_0+1}^{(p-i_0+1)}} v_j < w_{i_0+1}.$$

On the last step for $i = p + 1$, we choose $m_{p+1}^{(1)}, k_{p+1}^{(1)} \in \mathbb{N}: k_p^{(2)} < m_{p+1}^{(1)} \le k_{p+1}^{(1)}$ such that

$$w_{p+1} - \frac{w_{p+1}}{2} \le \sum_{j=m_{p+1}^{(1)}}^{k_{p+1}^{(1)}} v_j = f(p+1,1) < w_{p+1}.$$

By (1), it follows that $\lim\limits_{n \to \infty} \sum\limits_{s=1}^{n} \sum\limits_{j=m_i^{(s)}}^{k_i^{(s)}} v_j = w_i$ holds for every $i \in \mathbb{N}$. □

Theorem 3. *[19] Let the iterated series $\sum_{n=1}^{\infty} \sum_{s=1}^{\infty} a_n^s$ be given. If the series $\sum_{n=1}^{\infty} \sum_{s=1}^{\infty} |a_n^s|$ is convergent, then for any permutations $\pi, \sigma : \mathbb{N} \to \mathbb{N}$ the series $\sum_{n,s} a_{\pi(n)}^{\sigma(s)}$ is convergent and $\sum_{n,s} a_{\pi(n)}^{\sigma(s)} = \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} a_n^s$.*

Lemma 2. *Let M be an Orlicz function with the Δ_2-condition and $w = \{w_i\}_{i=1}^{\infty} \in \Lambda$. Let $\ell_M(w)$ be equipped with Luxemburg's or Amemiya's norm. Then,*
(1) For every $x \in \ell_M(w)$, such that $\widetilde{M}_w(\lambda x) < \infty$, for every $\lambda > 0$, there are $y, z \in \ell_M(w)$, such that $|y| \wedge |z| = 0$, $\widetilde{M}_w(\lambda y) = \widetilde{M}_w(\lambda z) = \widetilde{M}_w(\lambda x)$, for any $\lambda > 0$
(2) For every $x \in S_{(\ell_M(w), \|\cdot\|)}$, there are y, z, such that $|y| \wedge |z| = 0$, $y, z \in S_{(\ell_M(w), \|\cdot\|)}$
(3) For every $x \in S_{(\ell_M(w), \|\cdot\|\|)}$, there are y, z, such that $|y| \wedge |z| = 0$, $y, z \in S_{(\ell_M(w), \|\cdot\|\|)}$.

Proof. (1) Let $x = \{x_n\}_{n=1}^{\infty} \in \ell_M(w)$ be arbitrarily chosen. By $M \in \Delta_2$, it follows that $\widetilde{M}_w(\lambda x) < \infty$ for every $\lambda > 0$. By $w \in \Lambda$, it follows that we can choose two subsequences $v = \{v_i\}_{i=1}^{\infty}, u = \{u_i\}_{i=1}^{\infty}$ of w, such that $v, u \in \Lambda, v \cap u = \emptyset, v \cup u = w$.

By Lemma 1, there are sequences of naturals $\{m_i^{(s)}\}_{i=1}^{\infty}, \{k_i^{(s)}\}_{i=1}^{\infty}, \{\alpha_i^{(s)}\}_{i=1}^{\infty}, \{\beta_i^{(s)}\}_{i=1}^{\infty}, s \in \mathbb{N}$, such that

$$1 \le m_1^{(1)} \le k_1^{(1)}, \ 1 \le \alpha_1^{(1)} \le \beta_1^{(1)}$$

$$k_{n-1}^{(1)} < m_1^{(n)}, \ m_i^{(s)} \le k_i^{(s)}, \ k_i^{(s)} < m_{i+1}^{(s-1)}$$

for $n, i, s \in \mathbb{N}, n \ge 2, i + s = n + 1$

$$\beta_{n-1}^{(1)} < \alpha_1^{(n)}, \alpha_i^{(s)} \le \beta_i^{(s)} \beta_i^{(s)} < \alpha_{i+1}^{(s-1)}$$

for $n, i, s \in \mathbb{N}, n \ge 2, i + s = n + 1$

and there hold the equalities

$$\sum_{s=1}^{\infty} \sum_{j=m_i^{(s)}}^{k_i^{(s)}} v_j = w_i = \sum_{s=1}^{\infty} \sum_{j=\alpha_i^{(s)}}^{\beta_i^{(s)}} u_j,$$

for every $i \in \mathbb{N}$.

Put

$$y_n = \sum_{s=1}^{\infty} \sum_{j=m_n^{(s)}}^{k_n^{(s)}} x_n e_j, \quad z_n = \sum_{s=1}^{\infty} \sum_{j=\alpha_n^{(s)}}^{\beta_n^{(s)}} x_n e_j$$

and $y = \sum_{n=1}^{\infty} y_n$, $z = \sum_{n=1}^{\infty} z_n$. We will show that $\widetilde{M}_w(\lambda y) = \widetilde{M}_w(\lambda z) = \widetilde{M}_w(\lambda x)$, for any $\lambda > 0$.

Let $\lambda > 0$ and put $a_n^s(\lambda) = M(\lambda x_n) \sum_{j=m_n^{(s)}}^{k_n^{(s)}} v_j$ for $n, s \in \mathbb{N}$. Let us consider the infinite matrix

$$a_1^1(\lambda) \;\; a_1^2(\lambda) \;\; a_1^3(\lambda) \;\; \ldots \;\; a_1^s(\lambda) \;\; \ldots$$

$$a_2^1(\lambda) \;\; a_2^2(\lambda) \;\; a_2^3(\lambda) \;\; \ldots \;\; a_2^s(\lambda) \;\; \ldots$$

$$\ldots\ldots \;\; \ldots\ldots \;\; \ldots\ldots \;\; \ldots \;\; \ldots\ldots \;\; \ldots$$

$$a_n^1(\lambda) \;\; a_n^2(\lambda) \;\; a_n^3(\lambda) \;\; \ldots \;\; a_n^s(\lambda) \;\; \ldots$$

$$\ldots\ldots \;\; \ldots\ldots \;\; \ldots\ldots \;\; \ldots \;\; \ldots\ldots \;\; \ldots$$

For every $n \in \mathbb{N}$, the equality $\sum_{s=1}^{\infty} a_n^s(\lambda) = M(\lambda x_n) \sum_{s=1}^{\infty} \sum_{j=m_n^{(s)}}^{k_n^{(s)}} v_j = w_n M(\lambda x_n)$ holds and thus $\sum_{n=1}^{\infty} \sum_{s=1}^{\infty} a_n^s(\lambda) = \sum_{n=1}^{\infty} w_n M(\lambda x_n) < \infty$ for every $\lambda > 0$. By $a_n^s(\lambda) \geq 0$ for every $n, s \in \mathbb{N}$ and Theorem 3, it follows that for any two permutations $\pi, \sigma : \mathbb{N} \to \mathbb{N}$, the series $\sum_{n,s}^{\infty} a_{\pi(n)}^{\sigma(s)}(\lambda)$ is convergent and there hold the equalities

$$\sum_{n,s}^{\infty} a_{\pi(n)}^{\sigma(s)}(\lambda) = \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} a_n^s(\lambda) = \widetilde{M}_w(\lambda x).$$

Consequently, there exist two permutations $\pi, \sigma : \mathbb{N} \to \mathbb{N}$, such that we can write the chain of equalities

$$\begin{aligned}
\widetilde{M}_w(\lambda y) &= \sum_{p=2}^{\infty} \sum_{n=1}^{p-1} a_n^{p-n}(\lambda) \\
&= \sum_{p=2}^{\infty} \sum_{n=1}^{p-1} \sum_{j=m_n^{(s)}}^{k_n^{(s)}} v_j M(\lambda x_n) \\
&= \sum_{n,s}^{\infty} a_{\pi(n)}^{\sigma(s)}(\lambda) \\
&= \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} a_n^s(\lambda) = \widetilde{M}_w(\lambda x).
\end{aligned} \tag{2}$$

Similarly, if we put $b_n^s(\lambda) = M(\lambda x_n) \sum_{j=\alpha_n^{(s)}}^{\beta_n^{(s)}} v_j$ for $n, s \in \mathbb{N}$, we get the chain of equalities

$$\begin{aligned}
\widetilde{M}_w(\lambda z) &= \sum_{p=2}^{\infty} \sum_{n=1}^{p-1} b_n^{p-n}(\lambda) \\
&= \sum_{p=2}^{\infty} \sum_{n=1}^{p-1} \sum_{j=\alpha_n^{(s)}}^{\beta_n^{(s)}} u_j M(\lambda x_n) \\
&= \sum_{n,s}^{\infty} b_{\pi(n)}^{\sigma(s)}(\lambda) \\
&= \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} b_n^s(\lambda) = \widetilde{M}_w(\lambda x).
\end{aligned} \tag{3}$$

(2) If $\lambda = 1$, then by $\widetilde{M}_w(y) = \widetilde{M}_w(z) = \widetilde{M}_w(x) = 1$ we get that if $x \in S_{(\ell_M(w), \|\cdot\|)}$ then $y, z \in S_{(\ell_M(w), \|\cdot\|)}$.

(3) If $x \in S_{(\ell_M(w), \||\cdot\||)}$, then by (1) it follows that $\frac{1}{\lambda}(1 + \widetilde{M}_w(\lambda y)) = \frac{1}{\lambda}(1 + \widetilde{M}_w(\lambda z)) = \frac{1}{\lambda}(1 + \widetilde{M}_w(\lambda x))$ for any $\lambda > 0$. Therefore, if $\||x\|| = 1$, then $\||y\|| = \||z\|| = \||x\|| = 1$. $\qquad \square$

Lemma 3. *Let M be an Orlicz function with the Δ_2-condition and $w = \{w_i\}_{i=1}^{\infty} \in \Lambda$. Then, for every $x \in S_{(\ell_M(w), \||\cdot\||)}$ and $k > 1$, there exists a unique $d_{x,k} > 1$, such that $\widetilde{M}_w\left(\frac{kx}{d_{x,k}}\right) = \frac{k-1}{2}$.*

Proof. Let $x \in S_{(\ell_M(w), \||\cdot\||)}$ and $k > 1$ be arbitrarily chosen and fixed, then we define the function $S(d) : [1, +\infty) \to \mathbb{R}$ by $S(d) = \widetilde{M}_w\left(\frac{kx}{d}\right) = \sum_{i=1}^{\infty} w_i M\left(\frac{kx_i}{d}\right)$. By the inequality $M\left(\frac{kx_i}{d}\right) \leq M(kx_i)$ and the convergence of the series $\sum_{i=1}^{\infty} w_i M(kx_i)$, it follows that $\sum_{i=1}^{\infty} w_i M\left(\frac{kx_i}{d}\right)$ is uniformly convergent on $[1, +\infty)$. Thus, S is a continuous function. It is easy to see that S is a strictly decreasing function on $[1, +\infty)$. Therefore, by the inequalities

$$f(1) = \widetilde{M}_w(kx) \geq \||kx\||_M - 1 = k - 1 > \frac{k-1}{2}$$

and

$$\lim_{d \to +\infty} f(d) = \lim_{d \to +\infty} \widetilde{M}_w\left(\frac{kx}{d}\right) = 0 < \frac{k-1}{2},$$

we get that there is a unique $d_{x,k} > 0$, such that $f(d_{x,k}) = \widetilde{M}_w\left(\frac{kx}{d_{x,k}}\right) = \frac{k-1}{2}$. $\qquad \square$

Lemma 4. *([10]) For a Köthe sequences space $(X, \|\cdot\|)$, the Riesz angle $a(X)$ can be expressed as*

$$a(X) = \sup\{\|(|x| \vee |y|)\| : x, y \in S_X, |x| \wedge |y| = 0\},$$

where $|x| \wedge |y| = \min\{|x|, |y|\}$.

Proof of Theorem 1 (1). Let $x = \{x_n\}_{n=1}^{\infty} \in S_{(\ell_M(w), \|\cdot\|)}$ be arbitrarily chosen. By $w = \{w_i\}_{i=1}^{\infty} \in \Lambda$, it follows that we can choose two subsequences $v = \{v_i\}_{i=1}^{\infty}$, $u = \{u_i\}_{i=1}^{\infty}$ of w, such that $v, u \in \Lambda$, $v \cap u = \emptyset$, $v \cup u = w$.

It follows from Lemma 1 that there exist sequences of naturals $\{m_i^{(s)}\}_{i=1}^{\infty}, \{k_i^{(s)}\}_{i=1}^{\infty}, \{\alpha_i^{(s)}\}_{i=1}^{\infty}, \{\beta_i^{(s)}\}_{i=1}^{\infty}, s \in \mathbb{N}$, such that

$$1 \leq m_1^{(1)} \leq k_1^{(1)}, \;\; 1 \leq \alpha_1^{(1)} \leq \beta_1^{(1)}$$

$$k_{n-1}^{(1)} < m_1^{(n)}, \;\; m_i^{(s)} \leq k_i^{(s)}, \;\; k_i^{(s)} < m_{i+1}^{(s-1)}$$

for $n, i, s \in \mathbb{N}, n \geq 2, i + s = n + 1,$

$$\beta_{n-1}^{(1)} < \alpha_1^{(n)}, \;\; \alpha_i^{(s)} \leq \beta_i^{(s)}, \;\; \beta_i^{(s)} < \alpha_{i+1}^{(s-1)}$$

for $n, i, s \in \mathbb{N}, n \geq 2, i + s = n + 1$

and there holds the equalities

$$\sum_{s=1}^{\infty} \sum_{j=m_i^{(s)}}^{k_i^{(s)}} v_j = w_i = \sum_{s=1}^{\infty} \sum_{j=\alpha_i^{(s)}}^{\beta_i^{(s)}} u_j$$

for every $i \in \mathbb{N}$.

We can put

$$y_n = \sum_{s=1}^{\infty} \sum_{j=m_n^{(s)}}^{k_n^{(s)}} x_n e_j, \ z_n = \sum_{s=1}^{\infty} \sum_{j=\alpha_n^{(s)}}^{\beta_n^{(s)}} x_n e_j$$

and $y = \sum_{n=1}^{\infty} y_n, z = \sum_{n=1}^{\infty} z_n$.

Using Lemma 2, we get that $y, z \in S_{(\ell_M(w), \|\cdot\|)}$ and $\widetilde{M}_w(\lambda x) = \widetilde{M}_w(\lambda y) = \widetilde{M}_w(\lambda z)$ for any $\lambda \in \mathbb{R}$.

By the choice of the subsequences $v, u \subset w$, we have that $|y| \wedge |z| = 0$ and $\|(|y| \vee |z|)\| = \|y + z\|$. Therefore, we can write the chain of equalities:

$$1 = \widetilde{M}_w \left(\frac{y+z}{\|y+z\|} \right)$$

$$= \widetilde{M}_w \left(\frac{y}{\|y+z\|} \right) + \widetilde{M}_w \left(\frac{z}{\|y+z\|} \right)$$

$$= 2\widetilde{M}_w \left(\frac{x}{\|(|y| \vee |z|)\|} \right).$$

Consequently, it follows that for every $x \in S_{(\ell_M(w), \|\cdot\|)}$, there exists $k_x = \|(|y| \vee |z|)\|$, such that $\widetilde{M}_w \left(\frac{x}{k_x} \right) = \frac{1}{2}$. By Lemma 4, we get the inequality

$$a(X) \geq \sup \left\{ k_x : \widetilde{M}_w \left(\frac{x}{k_x} \right) = \frac{1}{2}, x \in S_X \right\},$$

where $X = (\ell_M(w), \|\cdot\|)$.

On the other hand, let us put

$$D = \sup \left\{ k_x : \widetilde{M}_w \left(\frac{x}{k_x} \right) = \frac{1}{2}, x \in S_{(\ell_M(w), \|\cdot\|)} \right\}.$$

It follows from Lemma 4 that for every $\varepsilon > 0$, there are $x, y \in S_{(\ell_M(w), \|\cdot\|)}, |x| \wedge |y| = 0$, such that $\|(|x| \vee |y|)\| > a(\ell_M(w)) - \varepsilon$.

Since

$$\widetilde{M}_w \left(\frac{(|x| \vee |y|)}{d} \right) = \widetilde{M}_w \left(\frac{x}{d} \right) + \widetilde{M}_w \left(\frac{y}{d} \right)$$

$$\leq \frac{1}{2} + \frac{1}{2} = 1,$$

we get the inequality $\|(|x| \vee |y|)\| \leq d$, which implies $a(\ell_M(w)) \leq d + \varepsilon$. By the arbitrariness of $\varepsilon > 0$, we obtain that $d \geq a(\ell_M(w))$.

Proof of Theorem 2. Let us denote

$$d = \sup_{\||x\||=1} \inf_{k>1} \left\{ d_{x,k} : \widetilde{M}_w \left(\frac{kx}{d_{x,k}} \right) = \frac{k-1}{2} \right\}. \quad (4)$$

For any $\varepsilon > 0$, there exist $x = \{x_n\}_{n=1}^{\infty} \in S_{(\ell_M(w), \||\cdot\||)}$ and $k > 1$, such that $d_{x,k} \geq d - \varepsilon$.

By $w = \{w_i\}_{i=1}^{\infty} \in \Lambda$, it follows that we can choose two subsequences $v = \{v_i\}_{i=1}^{\infty}, u = \{u_i\}_{i=1}^{\infty}$ of w, such that $v, u \in \Lambda, v \cap u = \emptyset, v \cup u = w$.

It follows from Lemma 1 that there exist sequences of naturals $\{m_i^{(s)}\}_{i=1}^{\infty}, \{k_i^{(s)}\}_{i=1}^{\infty}, \{\alpha_i^{(s)}\}_{i=1}^{\infty}, \{\beta_i^{(s)}\}_{i=1}^{\infty}, s \in \mathbb{N}$, such that

$$1 \leq m_1^{(1)} \leq k_1^{(1)}, \ 1 \leq \alpha_1^{(1)} \leq \beta_1^{(1)}$$

$$k_{n-1}^{(1)} < m_1^{(n)}, \ m_i^{(s)} \leq k_i^{(s)}, \ k_i^{(s)} < m_{i+1}^{(s-1)}$$

for $n, i, s \in \mathbb{N}, n \geq 2, i + s = n + 1$,

$$\beta_{n-1}^{(1)} < \alpha_1^{(n)}, \ \alpha_i^{(s)} \leq \beta_i^{(s)}, \ \beta_i^{(s)} < \alpha_{i+1}^{(s-1)}$$

for $n, i, s \in \mathbb{N}, n \geq 2, i + s = n + 1$,

and there hold the equalities

$$\sum_{s=1}^{\infty} \sum_{j=m_i^{(s)}}^{k_i^{(s)}} v_j = w_i = \sum_{s=1}^{\infty} \sum_{j=\alpha_i^{(s)}}^{\beta_i^{(s)}} u_j$$

for every $i \in \mathbb{N}$.

We can put

$$y_n = \sum_{s=1}^{\infty} \sum_{j=m_n^{(s)}}^{k_n^{(s)}} x_n e_j, z_n = \sum_{s=1}^{\infty} \sum_{j=\alpha_n^{(s)}}^{\beta_n^{(s)}} x_n e_j$$

and $y = \sum_{n=1}^{\infty} y_n, z = \sum_{n=1}^{\infty} z_n$.

Using Lemma 2, we see that $y, z \in S_{(\ell_M(w), \||\cdot\||)}$, and $\widetilde{M}_w(\lambda x) = \widetilde{M}_w(\lambda y) = \widetilde{M}_w(\lambda z)$ for any $\lambda \in \mathbb{R}$. Let us put

$$D_1 = \inf_{0 < k \leq 1} \frac{1}{k} \left(1 + \widetilde{M}_w \left(\frac{k(y+z)}{d - \varepsilon} \right) \right)$$

and

$$D_2 = \inf_{k > 1} \frac{1}{k} \left(1 + \widetilde{M}_w \left(\frac{k(y+z)}{d - \varepsilon} \right) \right)$$

Therefore, by the chain of inequalities

$$D_2 = \inf_{k>1} \frac{1}{k} \left(1 + \widetilde{M}_w \left(\frac{k(y+z)}{d-\varepsilon} \right) \right)$$

$$= \inf_{k>1} \frac{1}{k} \left(1 + 2\widetilde{M}_w \left(\frac{kx}{d-\varepsilon} \right) \right)$$

$$\geq \inf_{k>1} \frac{1}{k} \left(1 + 2\widetilde{M}_w \left(\frac{kx}{d_{x,k}} \right) \right)$$

$$= \inf_{k>1} \frac{1}{k} \left(1 + 2\frac{k-1}{2} \right) = 1$$

and

$$\left\| \left\| \frac{y+z}{d-\varepsilon} \right\| \right\| = \min\{D_1, D_2\}$$

$$\geq \min\{1, D_2\} = 1,$$

we get that $\||y + z\|| \geq d - \varepsilon$. By the arbitrariness of $\varepsilon > 0$ and Lemma 4, we obtain the inequality $a((\ell_M(w), \||\cdot\||)) \geq d$.

On the other hand, for any $\varepsilon > 0$, there are $x, y \in S_{(\ell_M(w), \||\cdot\||)}, |x| \wedge |y| = 0$, such that there holds the

inequality $|||(|x| \vee |y|)||| > a((\ell_M(w), ||| \cdot |||)) - \varepsilon$. It follows from (4) and Lemma 3 that for every $\varepsilon > 0$, there are $k, h > 1$, such that $d_{x,k} < d + \varepsilon$, $d_{y,h} < d + \varepsilon$, where $d_{x,k}$ and $d_{y,h}$ are the solutions of the equations $\widetilde{M}_w \left(\frac{kx}{d_{x,k}} \right) = \frac{k-1}{2}$ and $\widetilde{M}_w \left(\frac{hy}{d_{y,h}} \right) = \frac{h-1}{2}$, respectively. WLOG, we may assume that $1 < h \leq k$. By the chain of inequalities

$$\left| \left| \left| \frac{(|x| \vee |y|)}{d+\varepsilon} \right| \right| \right| \leq \frac{1}{h} \left(1 + \widetilde{M}_w \left(\frac{h(|x| \vee |y|)}{d+\varepsilon} \right) \right)$$

$$= \frac{1}{h} \left(1 + \widetilde{M}_w \left(\frac{hx}{d+\varepsilon} \right) + \widetilde{M}_w \left(\frac{hy}{d+\varepsilon} \right) \right)$$

$$= \frac{1}{h} + \frac{1}{h} \widetilde{M}_w \left(\frac{hx}{d_{h,x}} \right) + \frac{1}{h} \widetilde{M}_w \left(\frac{hy}{d_{k,y}} \right)$$

$$\leq \frac{1}{h} + \frac{1}{h} \widetilde{M}_w \left(\frac{hx}{d_{h,x}} \right) + \frac{1}{k} \widetilde{M}_w \left(\frac{ky}{d_{k,y}} \right)$$

$$= \frac{1}{h} + \frac{1}{h} \frac{h-1}{2} + \frac{1}{k} \frac{k-1}{2}$$

$$= 1 + \frac{1}{2h} - \frac{1}{2k} < \frac{3}{2},$$

we obtain the inequality $|||(|x| \vee |y|)||| \leq \frac{3}{2} d + \frac{3}{2}\varepsilon$ and hence $a(\ell_M(w)) < \frac{3}{2} d + \frac{5\varepsilon}{2}$. Since $\varepsilon > 0$ is arbitrarily chosen, it follows that $a(\ell_M(w)) \leq \frac{3}{2} d$.

For the estimation of the Riesz angle in weighted Orlicz sequence spaces, we will need some well-known indices. For an Orlicz function M, we consider the index function $G_M(u) = \frac{M^{-1}(u)}{M^{-1}(2u)}, u \in (0, +\infty)$ [20]. Following [10,21], we define the indices:

$$\alpha_M^0 = \liminf_{u \to 0} G_M(u),$$

$$\alpha_M^{+\infty} = \liminf_{u \to \infty} G_M(u),$$

$$\alpha_M^{0,+\infty} = \min\{\alpha_M^0, \alpha_M^{+\infty}\}, \qquad (5)$$

$$\widetilde{\alpha}_M = \inf \left\{ \frac{M^{-1}(u)}{M^{-1}(2u)} : u \in (0, +\infty) \right\}.$$

Let us mention that for an Orlicz sequence space ℓ_M, only the behavior of the Orlicz function M at zero is important, and therefore, the above indices are defined only at zero in [10,21].

Theorem 4. *Let M be an Orlicz function with the Δ_2-condition and $w = \{w_i\}_{i=1}^{\infty} \in \Lambda$. Then,*

$$\frac{1}{\alpha_M^{0,\infty}} \leq a((\ell_M(w), \| \cdot \|)) = \frac{1}{\widetilde{\alpha}_M}.$$

Proof. We will prove first that $a((\ell_M(w), \| \cdot \|)) = \frac{1}{\widetilde{\alpha}_M}$.

We chose arbitrary $x = \{x_i\}_{i=1}^{\infty} \in S_{(\ell_M(w), \|\cdot\|)}$ and put $u_i = \frac{1}{2} M(x_i)$. It is easy to check the equality

$M \left(x_i . G_M \left(\frac{M(x_i)}{2} \right) \right) = \frac{1}{2} M(x_i)$. For any $u_i \in (0, +\infty)$, the inequality $\widetilde{\alpha}_M \leq G_M(u_i)$ holds. Then,

$$\widetilde{M}_w(\widetilde{\alpha}_M x) = \sum_{i=1}^{\infty} w_i M(\widetilde{\alpha}_M x_i)$$

$$\leq \sum_{i=1}^{\infty} w_i M(x_i G_M(u_i))$$

$$= \sum_{i=1}^{\infty} w_i M \left(x_i G_M \left(\frac{M(x_i)}{2} \right) \right) \qquad (6)$$

$$= \sum_{i=1}^{\infty} \frac{w_i M(x_i)}{2} = \frac{1}{2}.$$

For every $x = \{x_i\}_{i=1}^{\infty} \in S_{(\ell_M(w), \|\cdot\|)}$, there exists k_x, such that $\widetilde{M}_w \left(\frac{x}{k_x} \right) = \frac{1}{2}$, and thus by (6), it follows the inequality $k_x \leq \frac{1}{\widetilde{\alpha}_M}$. Therefore,

$$a(X) = \sup \left\{ k_x : \widetilde{M}_w \left(\frac{x}{k_x} \right) = \frac{1}{2}, x \in S_X \right\}$$

$$\leq \frac{1}{\widetilde{\alpha}_M},$$

where $X = (\ell_M(w), \| \cdot \|)$.

Now, we will prove that $a((\ell_M(w), \| \cdot \|)) \geq \frac{1}{\widetilde{\alpha}_M}$.

For any $u \in (0, +\infty)$, there are sequences of naturals $\{p_n\}_{n=1}^{\infty}, \{q_n\}_{n=1}^{\infty}$, such that $p_n \leq q_n < p_{n+1}$, for $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \sum_{i=p_n}^{q_n} w_i = \frac{1}{u}$.

Put $x = \sum_{n=1}^{\infty} \sum_{i=p_n}^{q_n} M^{-1}(u) e_i$. It is easy to see that $\widetilde{M}_w(x) = 1$ and thus $x \in S_{(\ell_M(w), \|\cdot\|)}$. By the equality

$$\widetilde{M}_w \left(\frac{x}{\frac{M^{-1}(u)}{M^{-1}(u/2)}} \right) = \frac{1}{2},$$

we get that for any $u \in (0, +\infty)$, there holds the inequality

$$\frac{M^{-1}(u)}{M^{-1}(u/2)} \leq a((\ell_M(w), \| \cdot \|)),$$

and therefore,

$$\frac{1}{\widetilde{\alpha}_M} = \sup \left\{ \frac{M^{-1}(u)}{M^{-1}(u/2)} : u \in (0, +\infty) \right\}$$

$$\leq a(\ell_M(w)).$$

Thus, we have proven that

$$a((\ell_M(w), \| \cdot \|)) = \frac{1}{\widetilde{\alpha}_M}. \qquad (7)$$

The proof of the inequality $\frac{1}{\alpha_M^{0,\infty}} \leq a((\ell_M(w), \| \cdot \|))$ follows directly by equality (7) and the inequality $\alpha_M^{0,\infty} \geq \widetilde{\alpha}_M$. □

Theorem 5. *Let M be an Orlicz function with the Δ_2-condition and $w = \{w_i\}_{i=1}^{\infty} \in \Lambda$. Then,*

$$\frac{1}{\alpha_M^{0,\infty}} \leq a((\ell_M(w), ||| \cdot |||)) \leq \frac{3}{2 \widetilde{\alpha}_M}.$$

Proof. (i) We will prove first that

$$a((\ell_M(w), ||| \cdot |||)) \leq \frac{3}{2 \widetilde{\alpha}_M}.$$

Let us choose arbitrary $x = \{x_i\}_{i=1}^{\infty} \in S_{(\ell_M(w), ||| \cdot |||)}$. By $M \in \Delta_2$ and the equality

$$1 = |||x||| = \inf \left\{ \frac{1}{k} \left(1 + \widetilde{M}_w(kx) \right) : k > 0 \right\},$$

it follows that there exists $k_0 > 0$, such that the equality $\widetilde{M}_w(k_0 x) = k_0 - 1$ holds [20].

We put $u_i = \frac{1}{2} M(k_0 x_i)$. Then, similarly to (6), we can write the inequality:

$$\begin{aligned}
\widetilde{M}_w(k_0 \widetilde{\alpha}_M x) &= \sum_{i=1}^{\infty} w_i M(k_0 \widetilde{\alpha}_M x_i) \\
&\leq \sum_{i=1}^{\infty} w_i M(k_0 x_i G_M(u_i)) \\
&= \sum_{i=1}^{\infty} w_i M \left(k_0 x_i G_M \left(\frac{M(k_0 x_i)}{2} \right) \right) \\
&= \sum_{i=1}^{\infty} \frac{w_i M(k_0 x_i)}{2} = \frac{k_0 - 1}{2}
\end{aligned}$$

and consequently we get that $d_{x,k_0} < \frac{1}{\widetilde{\alpha}_M}$, where $d_{x,k}$ is the solution of the equation $\widetilde{M}_w \left(\frac{kx}{d_{x,k}} \right) = \frac{k-1}{2}$, $k > 1$. Thus, for any $k > 1$, there holds the inequality

$$\inf\{d_{x,k} : k > 1\} < d_{x,k_0} < \frac{1}{\widetilde{\alpha}}. \tag{8}$$

By the arbitrary choice of $x = \{x_i\}_{i=1}^{\infty} \in S_{(\ell_M(w), ||| \cdot |||)}$ and (8), we obtain that

$$\begin{aligned}
a((\ell_M(w), ||| \cdot |||)) &\leq \frac{3}{2} \sup_{|||x|||=1} \inf\{d_{x,k} : k > 0\} \\
&\leq \frac{3}{2 \widetilde{\alpha}_M}.
\end{aligned}$$

From the inequalities $\|\cdot\| \leq ||| \cdot ||| \leq 2\|\cdot\|$, it follows that for any $x, y \in S_{(\ell_M(w), ||| \cdot |||)}$ with $|x| \wedge |y| = 0$, there holds the inequality

$$\begin{aligned}
a(X) &= \sup\{|||(|x| \vee |y|)||| : |||x|||, |||y||| = 1\} \\
&\geq \sup\{\|(|x| \vee |y|)\| : \|x\|, \|y\| \leq 1\} \\
&= a((\ell_M(w), \|\cdot\|)),
\end{aligned}$$

$$\tag{9}$$

where $X = (\ell_M(w), ||| \cdot |||)$.

Therefore, we get that

$$a((\ell_M(w), ||| \cdot |||)) \geq a((\ell_M(w), \|\cdot\|)) = \frac{1}{\widetilde{\alpha}_M}.$$

(ii) By (9), we obtain the inequality

$$a((\ell_M(w), ||| \cdot |||)) \geq a((\ell_M(w), \|\cdot\|)) \geq \frac{1}{\alpha_M^{0,\infty}}.$$

$\square$

Definition 2. We say that the Orlicz function M satisfies the ∇_2 condition if there exists $l > 1$, such that $M(x) \leq \frac{1}{2l} M(lx)$, for every $x \in [0, +\infty)$, and we denote this by $M \in \nabla_2$.

The function M^{-1} is a concave function and thus $\frac{M^{-1}(t)}{M^{-1}(2t)} \geq \frac{1}{2}$. According to [20] (p. 22), $M \in \nabla_2$ iff $\liminf_{t \to +\infty} \frac{M^{-1}(t)}{M^{-1}(2t)} > \frac{1}{2}$ and $\liminf_{t \to 0} \frac{M^{-1}(t)}{M^{-1}(2t)} > \frac{1}{2}$, i.e.

$$M \in \nabla_2 \Leftrightarrow \alpha_M^0 > 1/2 \quad \text{and} \quad \alpha_M^{+\infty} > 1/2. \tag{10}$$

Corollary 1. *Let M be an Orlicz function with $M \in \Delta_2$ and $w \in \Lambda$ is a weight sequence. Then,*

(a) $M \notin \nabla_2$ iff $a((\ell_M(w), \|\cdot\|)) = 2$

(b) $M \in \nabla_2$ iff $a((\ell_M(w), \|\cdot\|)) < 2$.

Proof. (a) Let $M \notin \nabla_2$. Then from (10), it follows that $\alpha_M^0 = 1/2$ or $\alpha_M^{+\infty} = 1/2$, and thus by Theorem 4, it follows that $a((\ell_M(w), \|\cdot\|)) \geq 2$. Therefore by the inequalities $1 \leq a((\ell_M(w), \|\cdot\|)) \leq 2$, it follows that $a(\ell_M(w), \|\cdot\|) = 2$.

Let $a((\ell_M(w), \|\cdot\|)) = 2$. There are three cases: $\alpha_M^0 = 1/2$, $\alpha_M^{+\infty} = 1/2$ or there exists $t_0 \in (0, +\infty)$, such that $\frac{M^{-1}(t_0)}{M^{-1}(2t_0)} = 1/2$.

Let $\alpha_M^0 = 1/2$ or $\alpha_M^{+\infty} = 1/2$ holds; then by (10), it follows that $M \notin \nabla_2$.

There exists $t_0 \in (0, +\infty)$, such that $\frac{M^{-1}(t_0)}{M^{-1}(2t_0)} = 1/2$. Then, we can write the equality $\frac{M^{-1}(2t_0) - M^{-1}(t_0)}{2t_0 - t_0} = \frac{M^{-1}(t_0)}{t_0}$, and consequently by the concavity of the function M^{-1}, it follows that the points $(0, 0)$, $(t_0, M^{-1}(t_0))$ and $(2t_0, M^{-1}(2t_0))$ lie on a line. Thus, the function M^{-1} is linear on the segment $[0, 2t_0]$ and therefore $\alpha_M^0 = 1/2$. Therefore by (10), it follows that $M \notin \nabla_2$.

(b) The proof follows directly from (a). Indeed, let $M \in \nabla_2$ holds, but $a((\ell_M(w), \|\cdot\|)) < 2$ does not hold. Then $a((\ell_M(w), \|\cdot\|)) = 2$ and by (a), it follows that $M \notin \nabla_2$ which is a contradiction.

Let $a((\ell_M(w), \|\cdot\|)) < 2$ holds, but $M \in \nabla_2$ does not hold. Then $M \notin \nabla_2$ and by (a), it follows that $a((\ell_M(w), \|\cdot\|)) = 2$ which is a contradiction. $\square$

For the next Corollary, we will need the indices form [22].

We put $F_M(t) = \frac{tp(t)}{M(t)}$, $t \in (0, +\infty)$, where p is the right derivative of M. Let us define

$$A_M^0 = \liminf_{t \to 0^+} F_M(t), \quad B_M^0 = \limsup_{t \to 0^+} F_M(t)$$

$$A_M^{+\infty} = \liminf_{t \to +\infty} F_M(t), \quad B_M^{+\infty} = \limsup_{t \to +\infty} F_M(t)$$

The above indices are connected by the formulas ([11], p.149; [20], p.27).

$$\frac{1}{A_M^0} + \frac{1}{B_N^0} = \frac{1}{A_N^0} + \frac{1}{B_M^0} = 1$$

and

$$\frac{1}{A_M^{+\infty}} + \frac{1}{B_N^{+\infty}} = \frac{1}{A_N^{+\infty}} + \frac{1}{B_M^{+\infty}} = 1,$$

where N is the complementary function to M.

The inequalities

$$2^{-1/A_M^0} \le \alpha_M^0 \le 2^{-1/B_M^0} \tag{11}$$

$$2^{-1/A_M^{+\infty}} \le \alpha_M^{+\infty} \le 2^{-1/B_M^{+\infty}} \tag{12}$$

hold [23]. Let us mention that inequalities (11) are proven in [23]. The proof of inequalities (12) is similar. We are sure that inequalities (12) are proven somewhere. Just for completeness, we will prove (12) using the technique from [23].

If $B_M^{+\infty} = \infty$, then clearly $\alpha_M^{+\infty} = \liminf_{u \to \infty} \frac{M^{-1}(u)}{M^{-1}(2u)} \le 1 = 2^{-1/B_M^{+\infty}}$. Assume that $B_M^{+\infty} < \infty$. For any $\varepsilon > 0$, there exists $t_0 > 0$ such that $\frac{tp(t)}{M(t)} = F_M(t) < B_M^{+\infty} + \varepsilon$ for every $t \in [t_0, +\infty)$. Then for any $t_0 \le t_1 < t_2 < +\infty$, we have

$$\log \frac{M(t_2)}{M(t_1)} = \int_{t_1}^{t_2} \frac{p(t)}{M(t)} dt \le \int_{t_1}^{t_2} \frac{B_M^{+\infty} + \varepsilon}{t} dt$$

$$= \log \left(\frac{t_2}{t_1} \right)^{B_M^{+\infty} + \varepsilon}.$$

We put $t_1 = M^{-1}(u)$ and $t_2 = M^{-1}(2u)$. Thus for any $u \in [M(t_0), +\infty)$, there holds the inequality

$$\frac{M^{-1}(u)}{M^{-1}(2u)} \le 2^{-1/B_M^{+\infty} + \varepsilon}.$$

By the arbitrary choice of $\varepsilon > 0$, it follows the proof of the right side of the inequality (12). The proof of the left side is similar.

If $\lim_{t \to 0^+} F_M(t)$ exists, we denote it by C_M^0 and if $\lim_{t \to +\infty} F_M(t)$ exists we denote it by $C_M^{+\infty}$. We put $C_M^{0,+\infty} = \min\{C_M^0, C_M^{+\infty}\}$.

Corollary 2. *Let M be an Orlicz function with the Δ_2-condition and $w \in \Lambda$ is a weight sequence. Then,*

(a) If F_M is an increasing function on $(0, +\infty)$, then
$$a(\ell_M(w), \| \cdot \|) = 2^{1/C_M^0}$$

(b) If F_M is a decreasing function on $(0, +\infty)$, then
$$a(\ell_M(w), \| \cdot \|) = 2^{1/C_M^{+\infty}}$$

(c) If there is $t_0 \in (0, +\infty)$, such that F_M is increasing on $(0, M^{-1}(t_0))$ and decreasing on $(M^{-1}(t_0), +\infty)$, then
$$a(\ell_M(w), \| \cdot \|) = 2^{1/C_M^{0,+\infty}}$$

Proof. (a) If F_M is an increasing function in $(0, +\infty)$, $C_M^0 = \lim_{t \to 0+} F_M(t)$ exists and $G_M(u) = \frac{M^{-1}(u)}{M^{-1}(2u)}$ is increasing in $(0, +\infty)$ [21]. Then from (11), we get

$$\alpha_M^0 = 2^{-\frac{1}{C_M^0}} = \lim_{t \to 0} G(u) = \widetilde{\alpha}_M$$

and therefore $a(\ell_M(w), \| \cdot \|) = 2^{1/C_M^0}$.

(b) If F_M is a decreasing function in $(0, +\infty)$, then $C_M^{+\infty} = \lim_{t \to +\infty} F_M(t)$ exists, and $G_M(u) = \frac{M^{-1}(u)}{M^{-1}(2u)}$ is decreasing in $(0, +\infty)$ [21]. Then by (11), we obtain

$$\alpha_M^{+\infty} = 2^{-\frac{1}{C_M^{+\infty}}} = \lim_{t \to +\infty} G(u) = \widetilde{\alpha}_M$$

and therefore $a(\ell_M(w), \| \cdot \|) = 2^{1/C_M^{+\infty}}$.

(c) If F_M is an increasing function in $(0, M^{-1}(t_0))$, then $C_M^0 = \lim_{t \to 0+} F_M(t)$ exists, and $G_M(u) = \frac{M^{-1}(u)}{M^{-1}(2u)}$ is increasing in $(0, t_0/2)$. If F_M is a decreasing function in $(M^{-1}(t_0), +\infty)$, then $C_M^{+\infty} = \lim_{t \to +\infty} F_M(t)$ exists, and $G_M(u) = \frac{M^{-1}(u)}{M^{-1}(2u)}$ is decreasing in $(t_0/2, +\infty)$ [21].

From (11), it follows that $\alpha_M^0 = 2^{-\frac{1}{C_M^0}}$, $\alpha_M^{+\infty} = 2^{-\frac{1}{C_M^{+\infty}}}$ and hence $\widetilde{\alpha}_M = 2^{-\frac{1}{C_M^{0,+\infty}}}$. By Theorem 4, we get that $a(\ell_M(w), \| \cdot \|) = 2^{1/C_M^{0,+\infty}}$. □

Conclusions

Example 1. Let $M_1(t) = 2|t|^p + |t|^{2p}$, $p \in [1, +\infty)$. Then, $F_{M_1}(t) = 2p \left(1 - \frac{1}{t^p + 2} \right)$ for $t \in [0, +\infty)$. The function F_{M_1} is an increasing function and $\lim_{t \to 0} F_{M_1}(t) = p$. By Corollary 2, we get that $a(\ell_{M_1}(w), \| \cdot \|) = \sqrt[p]{2}$.

Example 2. Let $M_2(t) = \frac{|t|^p}{\log(1+|t|)}$, $p \in [1, +\infty)$. Then, $F_{M_2}(t) = p - \frac{t}{(1+t)\log(1+t)}$ for $t \in [0, +\infty)$. The function F_{M_2} is an increasing function and $\lim_{t \to 0} F_{M_2}(t) = p - 1$. By Corollary 2, we get that $a(\ell_{M_2}(w), \| \cdot \|) = 2^{\frac{1}{p-1}}$.

Example 3. Let $M_3(t) = |t|^p \log^r(1 + |t|)$, $p \in [1, +\infty)$, $r \in (0, +\infty)$. Then,

$$F_{M_3}(t) = p + \frac{rt}{(1+t)\log(1+t)} \text{ for } t \in [0, +\infty).$$

The function F_{M_3} is a decreasing function and $\lim_{t \to +\infty} F_{M_3}(t) = p$. By Corollary 2, we get that $a(\ell_{M_3}(w), \|\cdot\|) = 2^{1/p}$.

Example 4. Let $q \geq 2$ and $p \in [q-1, 2q-1]$. We define the function

$$M_4(t) = \begin{cases} |t|^p(1 + \log|t|), & |t| \geq 1 \\[2mm] \frac{2q-p-1}{q}|t|^q + \frac{p-q+1}{q}|t|^{2q}, & |t| \leq 1. \end{cases}$$

The function M_4 is an Orlicz function. Then,

$$F_{M_4}(t) = \begin{cases} p + \frac{1}{1+\log t}, & t \geq 1 \\[2mm] 2q\left(1 - \frac{a}{2(a+bt^q)}\right), & t \in [0,1], \end{cases}$$

where $a = \frac{2q-p-1}{q}$ and $b = \frac{p-q+1}{q}$. The index function F_{M_4} is increasing on $[0,1]$ and is decreasing on $[0, +\infty)$, $\lim_{t \to 0} F_{M_4}(t) = q$ and $\lim_{t \to +\infty} F_{M_4}(t) = p$. By Corollary 2, we get that $a(\ell_{M_4}(w), \|\cdot\|) = 2^{\frac{1}{\min\{p,q\}}}$.

Example 5. Let $M_5(t) = (1 + |t|)\log(1 + |t|) - |t|$. Then $F_{M_5}(t) = \frac{t\log(1+t)}{(1+t)\log(1+t)-t}$ is a decreasing function on $(0, +\infty)$ and $\lim_{t \to +\infty} F_{M_5}(t) = 1$. Thus, $a(\ell_{M_5}(w), \|\cdot\|) = 2$. By Corollary 1 it follows that $M_6 \notin \nabla_2$.

Competing interests
The author declare that he have no competing interests.

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