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Geometry of distributions and F-Gordon equation

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Abstract

In this paper, we describe the geometry of distributions by their symmetries and present a simplified proof of the Frobenius theorem and some related corollaries. Then, we study the geometry of solutions of the F-Gordon equation, a PDE which appears in differential geometry and relativistic field theory.

Keywords: Distribution, Lie symmetry, Contact geometry, Klein-Gordon equation

Introduction

We begin this paper with the geometry of distributions. The main idea here is the various notions of symmetry and their use in solving a given differential equation. In the 'Tangent and cotangent distribution' section, we introduce the basic notions and definitions.

In the 'Integral manifolds and maximal integral manifolds' section, we describe the relation between differential equations and distributions. In the 'Symmetries' section, we present the geometry of distributions by their symmetries and find out the symmetries of the F-Gordon equation by this machinery. In the 'A proof of the Frobenius theorem' section, we introduce a simplified proof of the Frobenius theorem and some related corollaries. In the 'Symmetries and solutions' section, we describe the relations between symmetries and solutions of a distribution.

In all steps, we study the F-Gordon equation as an application and also a partial differential equation which appears in differential geometry and relativistic field theory. It is a generalized form of the Klein-Gordon equation $u_{tt} - u_{xx} + u = 0$ as well as a relativistic version of the Schrodinger equation, which is used to describe spinless particles. It was named after Walter Gordon and Oskar Klein [1,2].

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Tangent and cotangent distribution

Throughout this paper, M denotes an $(m+n)$-dimensional smooth manifold.

Definition 2.1. A map $\mathbf{D} : M \to TM$ is called an *m-dimensional tangent distribution on M*, or briefly **Tan**^{m}*-distribution*, if

$$\mathbf{D}_x := \mathbf{D}(x) \subseteq T_xM \qquad (x \in M)$$

is an m-dimensional subspace of T_xM. The smoothness of $\mathbf{D}$ means that for each $x \in M$, there exists an open neighborhood U of x and smooth vector fields $X_1, \cdots, X_m$ such that

$$\begin{aligned} \mathbf{D}_y &= \langle X_1(y), \cdots, X_m(y) \rangle \\ &:= span_{\mathbf{R}}\{X_1(y), \cdots, X_m(y)\} \qquad (y \in U) \end{aligned}$$

Definition 2.2. A map $D : M \to T^*M$ is called an *n-dimension cotangent distribution on M*, or briefly **Cot**^{n}*-distribution*, if

$$D_x := D(x) \subseteq T^*_xM \qquad (x \in M)$$

is an n-dimensional subspace of T^*_xM. The smoothness of D means that for each $x \in M$, there exists an open neighborhood U of x and smooth 1-forms $\omega^1, \cdots \omega^n$ such that

$$\begin{aligned} D_y &= \langle \omega^1(y), \cdots, \omega^n(y) \rangle \\ &:= span_{\mathbf{R}}\{\omega^1(y), \cdots, \omega^n(y)\} \qquad (y \in U) \end{aligned}$$

In the sequel, without loss of generality, we can assume that these definitions are globally satisfied.

There is a correspondence between these two types of distributions. For $\mathbf{Tan}^m$-distribution $\mathbf{D}$, there exist nowhere zero smooth vector fields $X_1, \cdots, X_m$ on M such that $\mathbf{D} = \langle X_1, \cdots, X_m \rangle$, and similarly, for $\mathbf{Cot}^n$-distribution D, there exist global smooth 1-forms $\omega^1, \cdots, \omega^n$ on M such that $D = \langle \omega^1, \cdots, \omega^n \rangle$.

Example 2.3. (Cartan distribution) Let $M = \mathbf{R}^{k+1}$. Denote the coordinates in M by $x, p_0, p_1, ..., p_k$, and given a function $f(x, p_0, \cdots, p_{k-1})$, consider the following differential 1-forms

$$\begin{aligned} \omega^0 &= dp_0 - p_1\, dx, \quad \omega^2 = dp_1 - p_2\, dx, \quad \cdots \\ \omega^{k-2} &= dp_{k-2} - p_{k-1}\, dx, \\ \omega^{k-1} &= dp_{k-1} - f(x, p_0, \cdots, p_{k-1})\, dx, \end{aligned}$$

and the distribution $D = \langle \omega^0, \cdots, \omega^{k-1} \rangle$. This is the 1-dimensional distribution, called the *Cartan distribution*. This distribution can also be described by a single vector field X, $\mathbf{D} = \langle X \rangle$, where

$$\begin{aligned} X = \partial_x + p_1\, \partial_{p_0} + p_2\, \partial_{p_1} + \cdots + p_{k-1}\, \partial_{p_{k-2}} \\ + f(x, p_0, \cdots, p_{k-1})\, \partial_{p_{k-1}}. \end{aligned}$$

Example 2.4. (*F*-Gordon equation) Let $F : \mathbf{R}^5 \to \mathbf{R}$ be a differentiable function. The corresponding F-Gordon PDE is $u_{xy} = F(x, y, u, u_x, u_y)$. We construct 7-dimensional sub-manifold M defined by $s = F(x, y, u, p, q)$, of

$$J^2(\mathbf{R}^2, \mathbf{R}) = \{x, y, u, p = u_x, q = u_y, r = u_{xx}, s = u_{xy}, t = u_{yy}\}.$$

Consider the 1-forms

$$\begin{aligned} \omega^1 &= du - p\, dx - q\, dy, \quad \omega^2 = dp - r\, dx - F\, dy, \\ \omega^3 &= dq - F\, dx - t\, dy. \end{aligned}$$

This distribution can also be described by the following vector fields:

$$\begin{aligned} X_1 &= \partial_x + p\, \partial_u + r\, \partial_p + F\, \partial_q, \\ X_2 &= \partial_y + q\, \partial_u + F\, \partial_p + t\, \partial_q, \\ X_3 &= \partial_r, \quad X_4 = \partial_t. \end{aligned}$$

Definition 2.5. Let $\mathbf{D} : M \to TM$ be a $\mathbf{Tan}^m$-distribution and set

$$\mathrm{A}nn\mathbf{D}_x := \{\omega_x \in T_x^*M \mid \omega_x|_{\mathbf{D}_x} = 0\}.$$

It is clear that $\dim \mathrm{A}nn\mathbf{D}_x = n$. A 1-form $\omega \in \Omega^1(M)$ *annihilates* $\mathbf{D}$ on a subset $N \subset M$, if and only if $\omega_x \in \mathrm{A}nn\mathbf{D}_x$ for all $x \in M$.

The set of all differential 1-forms on M which annihilates $\mathbf{D}$, is called *annihilator of* $\mathbf{D}$ and denoted by $\mathrm{A}nn\mathbf{D}$.

Therefore, for each $\mathbf{Tan}^m$-distribution,

$$\mathbf{D} : M \to TM, \quad \mathbf{D} : x \mapsto \mathbf{D}_x,$$

we can construct a $\mathbf{Cot}^n$-distribution

$$D : M \to T^*M, \quad D : x \mapsto D_x = \mathrm{A}nn\, \mathbf{D}_x$$

and vice versa. In the other words, for each $\mathbf{Tan}^m$-distribution $\mathbf{D} = \langle X_1, \cdots, X_m \rangle$, we can construct a $\mathbf{Cot}^n$-distribution $D = \mathrm{A}nn\mathbf{D} = \langle \omega^1, \cdots, \omega^n \rangle$, and vice versa.

Theorem 2.6. *(a)* $\mathbf{D}$ *and its annihilator are modules over* $C^\infty(M)$.
(b) Let X *be a smooth vector field on* M *and* $\omega \in \mathrm{A}nn\mathbf{D}$, *then*

$$\mathbf{L}_X \omega \equiv -\omega \circ \mathbf{L}_X \mod \mathbf{D}$$

Proof. (a) is clear, and for (b), if Y belongs to $\mathbf{D}$, then $\omega(Y) = 0$ and

$$(\mathbf{L}_X\omega)Y = X.(\omega(Y)) - \omega[X.Y] = -\omega[X.Y] = -(\omega \circ \mathbf{L}_X)Y.$$

□

Integral manifolds and maximal integral manifolds

Definition 3.1. Let $\mathbf{D}$ be a distribution. A bijective immersed sub-manifold $N \subset M$ is called an *integral manifold of* $\mathbf{D}$ if one of the following equivalence conditions is satisfied:

(1) $T_xN \subseteq \mathbf{D}_x$, for all $x \in N$.
(2) $N \subseteq \bigcap_{i=1}^n \ker \omega^i$.

Moreover, $N \subset M$ is called *maximal integral manifold* if for each $x \in N$, there exists an open neighborhood U of x such that there is no integral manifold N' containing $N \cap U$.

It is clear that the dimension of maximal integral manifold does not exceed the dimension of the distribution.

Definition 3.2. $\mathbf{D}$ is called a *completely integrable distribution*, or briefly CID, if for all maximal integral manifold N, one of the following equivalence conditions is satisfied:

(1) $\dim N = \dim \mathbf{D}$.
(2) $T_xN = \mathbf{D}_x$ *for all* $x \in N$
(3) $N \subseteq \bigcap_{i=1}^n \ker \omega^i$, *and if* N' *be an integral manifold with* $N \cap N' \neq \emptyset$, *then* $N' \subseteq N$.

In the sequel, the set of all maximal integral manifolds is denoted by $\mathbf{N}$.

Theorem 3.3. $\mathbf{N} = \bigcap_{i=1}^{n} \ker \omega^i$; *that is* $\omega^i|_{\mathbf{N}} = 0$ *for* $i = 1, \cdots, n$.

Example 3.4. **(Continuation of Example 2.3)** If N is an integral curve of the distribution, then x can be chosen as a coordinate on N, and therefore,

$$N = \{(x, h_0(x), h_1(x), \cdots, h_{k-1}(x)) \mid x \in \mathbf{R}\}.$$

Conditions $\omega^0|_N = 0, \cdots, \omega^{k-1}|_N = 0$ imply that $h_1 = h_0'$, $h_2 = h_1', \cdots, h_{k-1} = h_{k-1}'$, or that

$$N = J^{k-1}h = \{(x, h(x), h'(x), \cdots, h^{(k-1)}(x) \mid x \in \mathbf{R}\}$$

for some function $h : \mathbf{R} \to \mathbf{R}$.

The last equation $\omega^{k-1}|_N = 0$ gives us an ordinary differential equation $h^{(k)}(x) = f(x, h(x), h'(x), \cdots, h^{(k-1)}(x))$.

The existence theorem shows us once more that the integral curves do exist, and therefore, the Cartan distribution is a CID.

Example 3.5. **(Continuation of Example 2.4)** This distribution in not a CID because there is no 4-dimensional integral manifold, and $\dim \mathbf{D} = 4$. For, if N be a 4-dimensinal integral manifold of the distribution, then (x, y, u, p) can be chosen as coordinates on N, and therefore,

$$N : \begin{cases} q = h(x, y, u, p), \ r = l(x, y, u, p), \\ t = m(x, y, u, p), \ s = F(x, y, u, p, h). \end{cases}$$

Condition $\omega^1|_N = 0$ implies that $-p\,dx - h(x, y, u, p)\,dy + du = 0$, which is impossible.

By the same reason, we conclude that there is no 3-dimensional integral manifold.

Now, if N be a 2-dimensinal integral manifold of the distribution, then (x, y) can be chosen as coordinates on N, and therefore,

$$N : \begin{cases} u = h(x, y), \ p = l(x, y), \ q = m(x, y), \\ r = n(x, y), \ t = o(x, y), \ s = F(x, y, u, p, q). \end{cases}$$

Conditions $\omega^1|_N = 0$ and $\omega^2|_N = 0$ imply that $l = h_x$, $m = h_y$, $n = l_x = h_{xx}$ and $o = m_y = h_{yy}$.

The last equation $\omega^3|_N = 0$ implies that $h_{xy} = F(x, y, h, h_x, h_y)$. This distribution is not a CID.

Symmetries

In this section, we consider a distribution $\mathbf{D} = \langle X_1, \cdots, X_m\rangle = \langle \omega^1, \cdots, \omega^n\rangle$ on manifold M^{n+m}.

Definition 4.1. A diffeomorphism $F : M \to M$ is called a *symmetry of* $\mathbf{D}$ if $F_*\mathbf{D}_x = \mathbf{D}_{F(x)}$ for all $x \in M$.

Therefore,we have the following theorem.

Theorem 4.2. *The following conditions are equivalent:*

(1) F is a symmetry of $\mathbf{D}$;
(2) $F^*\omega^i$s determine the same distribution $\mathbf{D}$; that is $\mathbf{D} = \langle F^*\omega^1, \cdots, F^*\omega^n\rangle$;
(3) $F^*\omega^i \wedge \cdots \wedge \omega^n = 0$ for $i = 1, \cdots, n$;
(4) $F^*\omega^i = \sum_{j=1}^{n} a_{ij}\,\omega^j$, where $a_{ij} \in C^\infty(M)$;
(5) $(F_*X_i|_x) \in \mathbf{D}_{F(x)}$ for all $x \in M$ and $i = 1, \cdots, n$; and
(6) $F_*X_i = \sum_{j=1}^{n} b_{ij}\,X_j$, where $b_{ij} \in C^\infty(M)$.

Theorem 4.3. *If F be a symmetry of $\mathbf{D}$ and N be an integral manifold, then $F(N)$ is an integral manifold.*

Proof. F is a diffeomorphism; therefore, $F(N)$ is a submanifold of M. From other hand, if $x \in N$, then $\omega^i|_{F(x)} = (F^*\omega^i)|_x = 0$ for all $i = 1, \cdots, n$; therefore, $F(N) = \{F(x) \mid x \in N\}$ is an integral manifold. □

Theorem 4.4. *Let $\mathbf{N}$ be the set of all maximal integral manifolds and $F : M \to M$ be a symmetry, then $F(\mathbf{N}) = \mathbf{N}$.*

Proof. If $x \in \mathbf{N}$, then $\omega^i|_{F(x)} = (F^*\omega^i)|_x = 0$ for all $i = 1, \cdots, n$; therefore, $F(x) \in \mathbf{N}$ and $F(\mathbf{N}) \subset \mathbf{N}$. □

Now, if $y \in \mathbf{N}$, then there exists $x \in M$ such that $F(x) = y$, since F is a diffeomorphism. Therefore, $(F^*\omega^i)|_x = \omega^i|_{F(x)} = \omega^i|_y = 0$ for all $i = 1, \cdots, n$; thus, $x \in \mathbf{N}$ and $\mathbf{N} \subseteq F(\mathbf{N})$.

Definition 4.5. A vector field X on M is called an *infinitesimal symmetry of distribution* $\mathbf{D}$, or briefly a *symmetry of* $\mathbf{D}$, if the flow Fl_t^X of X be a symmetry of $\mathbf{D}$ for all t.

Theorem 4.6. *A vector field $X \in \mathbf{X}(M)$ is a symmetry if and only if*

$$\mathbf{L}_X\omega^i|_{\mathbf{D}} = 0 \quad \textit{for all} \quad i = 1, \cdots, n.$$

Proof. Let X be a symmetry. If $\Omega = \omega^1 \wedge \cdots \wedge \omega^n$, then $\{(\mathrm{Fl}^X)^*\omega^i\} \wedge \Omega = 0$, by condition (3) in Theorem 4.2. Moreover, by the definition $\mathbf{L}_X\omega^i := \frac{d}{dt}\big|_0 (\mathrm{Fl}_t^X)^*\omega^i$, one gets

$$\begin{aligned}(\mathbf{L}_X\omega^i) \wedge \Omega &= \lim_{t\to 0} \frac{1}{t}\left((\mathrm{Fl}_t^X)^*\omega^i - \omega^i\right) \wedge \Omega \\ &= \lim_{t\to 0} \frac{1}{t}\left(\{(\mathrm{Fl}^X)^*\omega^i\} \wedge \Omega - \omega^i \wedge \Omega^1\right) = 0.\end{aligned}$$

□

Therefore $\mathbf{L}_X\omega^i|_{\mathbf{D}} = 0$.

In converse, let $\mathbf{L}_X\,\omega^i|_{\mathbf{D}} = 0$ or $\mathbf{L}_X\,\omega^i = \sum_{j=1}^n b_{ij}\ \omega^j$ for $i = 1,\cdots,n$ and $b_{ij} \in C^\infty(M)$. Now, if $\gamma_i(t) := \{(\mathrm{Fl}_t^X)^*\omega^i\} \wedge \Omega$, then

$$\gamma_i(0) = \{(\mathrm{Fl}_0^X)^*\omega^i\} \wedge \Omega = 0, \tag{1}$$

and

$$\begin{aligned}\gamma_i'(t) &= \frac{d}{dt}\{(\mathrm{Fl}_t^X)^*\omega^i\} \wedge \Omega = ((\mathrm{Fl}_t^X)^*\mathbf{L}_X\omega^i) \wedge \Omega\\ &= (\mathrm{Fl}_t^X)^*\left(\sum b_{ij}\,\omega^i\right) \wedge \Omega = \sum B_{ij}\,\{(\mathrm{Fl}_t^X)^*\omega^j\} \wedge \Omega\end{aligned}$$

where $B_{ij} = (\mathrm{Fl}_t^X)^* b_{ij} = b_{ij} \circ \mathrm{Fl}_t^X$ and

$$\gamma_i'(t) = \sum B_{ij}\,\gamma_i(t)\ \ ,\ \ i = 1,\cdots,n. \tag{2}$$

Therefore, $\gamma = (\gamma_1 \cdots,\gamma_n)$ is a solution of the linear homogeneous system of ODEs (2) with the initial conditions (1), and γ must be identically zero.

Theorem 4.7. *X is symmetry if and only if for all $Y \in \mathbf{D}$, then $[X, Y] \in \mathbf{D}$.*

Proof. By the above theorem, X is a symmetry if and only if for all $\omega \in \mathrm{A}nn\,\mathbf{D}$, then $\mathbf{L}_X\omega \in \mathrm{A}nn\mathbf{D}$.

The Theorem comes from the Theorem 2.6 (b): $\mathbf{L}_X\omega = -\omega \circ \mathbf{L}_X$ on $\mathbf{D}$. In other words, $(\mathbf{L}_X\omega)Y = -\omega[X, Y]$ for all $Y \in \mathbf{D}$. □

Denote by $\mathrm{Sym}_{\mathbf{D}}$ the set of all symmetries of a distribution $\mathbf{D}$.

Example 4.8. **(Continuation of Example** 3.4**)** Let $k = 2$. A vector field $Y = a\,\partial_x + b\,\partial_{p_0} + c\,\partial_{p_1}$ is an infinitesimal symmetry of $\mathbf{D}$ if and only if $L_Y\omega^i \equiv 0 \bmod \mathbf{D}$, for $i = 1, 2$. These give two equations:

$$c = Xb - p_1\,Xa, \quad Xc = f\,Xa + Yf.$$

Example 4.9. **(Continuation of Example** 3.5**)** We consider the point infinitesimal transformation:

$$\begin{aligned}Z = {}& X(x,y,u)\,\partial_x + Y(x,y,u)\,\partial_y + U(x,y,u)\,\partial_u\\ &+P(x,y,u,p,q,r,t)\,\partial_p + Q(x,y,u,p,q,r,t)\,\partial_q\\ &+R(x,y,u,p,q,r,t)\,\partial_r + T(x,y,u,p,q,r,t)\,\partial_t.\end{aligned}$$

Then, Z is an infinitesimal symmetry of $\mathbf{D}$ if and only if $L_Z\omega^i \equiv 0 \bmod \mathbf{D}$, for $i = 1, 2, 3$. These give ten equations:

$$\begin{aligned}&P_r = P_t = Q_r = Q_t = 0,\\ &p^2X_y + qpY_y + qU_x + pQ = pqX_x + qP + q^2Y_x + pU_y,\\ &rpX_y + FY_y + qP_x + (qr - pF)P_p + (qF - pt)P_q\\ &\qquad + pXF_x + pYF_y + pUF_u + pPF_p + pQF_q = qrX_x\\ &\qquad + qFY_x + pP_y + qR,\\ &pFX_y + ptY_y + (qr-pF)Q_p + (qF-pt)Q_q + qQ_x + pT\\ &\qquad = qFX_x + qtY_x + pQ_y + qPF_p + qYF_y + qXF_x\\ &\qquad + qQF_q + qUF_u,\\ &Q_y + qQ_u + FQ_p + tQ_q = tY_y + tqY_u + FX_y + qFX_u + T,\\ &U_y + qU_u = pX_y + pqX_u + qY_y + q^2Y_u + Q,\\ &P_y + qP_u + FP_p + tP_q = rX_y + qrX_u + FY_y + qFY_u\\ &\qquad + (XF_x + YF_y + UF_u + PF_p + QF_q).\end{aligned}$$

Complicated computations using Maple show that

$$\begin{aligned}P = {}& -pX_x - p^2X_u - qY_x - pqY_u + U_x + pU_u,\\ Q = {}& \frac{1}{p}\left(pqX_x - p^2X_y + q^2Y_x - pqY_y - qU_x + pU_y + qP\right),\\ R = {}& \frac{1}{q}\left(\left(pqX_x - p^2X_y + q^2Y_x - pqY_y - qU_x + pU_y + qP\right).F_q\right.\\ &+ \left((qr - pF).P_p + (qF - pt).P_q + qP_x - pP_y\right.\\ &+ pXF_x + pYF_y + pUF_u + pPF_p + (pY_y - qY_x).F\\ &\left.\left.-pr.(qX_x - pX_y)\right)\right),\\ T = {}& \frac{1}{p^3}\left((p^2t + q^2r - 2pqF)P + p^2(pt + q^2F_q)X_x\right.\\ &+p^2(qr - 2pF - pqF_q)X_y + q(q^2(r + pF_q)\\ &+3p(pt - qF))Y_x - p^2(q^2F_q + 2pt + qF)Y_y\\ &-(q^2r + pq^2F_q + p^2t + 2pq)U_x + p^2qF_qU_y\\ &-pq^2P_x + p^2qP_y + pq(pF - qr)P_p + pq(pt - qF)P_q\\ &+pq(pXF_x + pYF_y + pUF_u + pPF_p + qPF_q)\\ &-p^2q^2X_{xx} + 2p^3qX_{xy} - p^4X_{yy} - pq^3Y_{xx} + 2p^2q^2Y_{xy}\\ &\left.-p^3qY_{yy} + pq^2U_{xx} - 2p^2qU_{xy} + p^3U_{yy}\right),\end{aligned}$$

and $X = X(x, u - qy)$, $Y = Y(y, u - px)$, and $U(x, y, u)$ must satisfy in PDE:

$$\begin{aligned}&(pF_p - F)X_x + p(pF_p - 2F)X_u + (qF_q - F)Y_y + q(qF_q - 2F)Y_u\\ &\quad - F_pU_x - F_qU_y + (F - pF_p - qF_q)U_u + U_{xy} + qU_{xu}\\ &\quad + pU_{yu} + pqU_{uu} = XF_x + YF_y + UF_u.\end{aligned}$$

A proof of the Frobenius theorem

Theorem 5.1. *Let $X \in Sym_{\mathbf{D}} \cap \mathbf{D}$ and N be maximal integral manifold. Then, X is tangent to N.*

Proof. Let $X(x) \notin T_xN$. Then, there exists an open set U of x and sufficiently small ε such that $\bar{N} := \bigcup_{-\varepsilon<t<\varepsilon} \mathrm{F}l_t^X (N \cap U)$ is a smooth sub-manifold of M.

Since $X \in \mathbf{D}$, So $\bar{N}$ is an integral manifold.

Since $X \in \mathrm{Sym}_{\mathbf{D}}$, so tangent to $\mathrm{F}l_t^X (N \cap U)$ belongs to $\mathbf{D}$, for all $-\varepsilon < t < \varepsilon$.

On the other hand, tangent spaces to $\bar{N}$ are sums of tangent spaces to $\mathrm{F}l_t^X(N \cap U)$ and the 1-dimensional subspace generated by X, but both of them belong to $\mathbf{D}$, and their means are $\bar{N} \subset N$. □

Theorem 5.2. *If* $X \in \mathbf{D} \cap Sym_{\mathbf{D}}$ *and* N *be a maximal integral manifold, then* $\mathrm{F}l_t^X(N) = N$ *for all* t.

Theorem 5.3 (**Frobenious**). *A distribution* $\mathbf{D}$ *is completely integrable, if and only if it is closed under Lie bracket. In other words,* $[X, Y] \in \mathbf{D}$ *for each* $X, Y \in \mathbf{D}$.

Proof. Let N be a maximal integral manifold with $T_xN = \mathbf{D}_x$. Therefore, for all $X, Y \in \mathbf{D}$, X and Y are tangent to N, and so $[X, Y]$ is also tangent to N.

On the other hand, let for all $X, Y \in \mathbf{D}$, their $[X, Y] \in \mathbf{D}$. By the Theorem, all $x \in \mathbf{D}$ is a symmetry too, and so all $X \in \mathbf{D}$ is tangent to N, and this means $T_x N = \mathbf{D}_x$, for all $x \in N$. □

Theorem 5.4. *A distribution* $\mathbf{D}$ *is completely integrable if and only if* $\mathbf{D} \subset Sym_{\mathbf{D}}$.

Theorem 5.5. *Let* $\mathbf{D} = \langle \omega^1, \cdots, \omega^n \rangle$ *be a completely integrable distribution and* $X \in \mathbf{D}$. *Then, the differential 1-forms* $(\mathrm{F}l_t^X)^*\omega^1, \cdots, (\mathrm{F}l_t^X)^*\omega^n$ *vanish on* $\mathbf{D}$ *for all* t.

Proof. If $\mathbf{D}$ is completely integrable, then X is a symmetry. Hence,

$$(\mathrm{F}l_t^X)^*\omega^i = \sum_j a_{ij}\, \omega^j.$$

□

Symmetries and solutions

Definition 6.1. If an (infinitesimal) symmetry X belongs to the distribution $\mathbf{D}$, then it is called a *characteristic symmetry*. Denote by $\mathrm{Char}(\mathbf{D}) := S_{\mathbf{D}} \cap \mathbf{D}$ the set of all characteristic symmetries [3,4].

It is shown that $\mathrm{Char}(\mathbf{D})$ is an ideal of the Lie algebra $S_{\mathbf{D}}$ and is a module on $\mathbf{C}^\infty(M)$. Thus, we can define the quotient Lie algebra

$$\mathrm{Shuf}(\mathbf{D}) := \mathrm{Sym}_{\mathbf{D}}/\mathrm{Char}(\mathbf{D}).$$

Definition 6.2. Elements of *Shuf*$(\mathbf{D})$ are called *shuffling symmetries of* $\mathbf{D}$.

Any symmetry $X \in Sym_{\mathbf{D}}$ generates a flow on $\mathbf{N}$ (the set of all maximal integral manifolds of $\mathbf{D}$), and, in fact, the characteristic symmetries generate trivial flows. In other words, classes X mod $Char(\mathbf{D})$ mix or 'shuffle' the set of all maximal manifolds.

Example 6.3. **(Continuation of Example** 4.8**)** Let $k = 2$. In this case,

$$\partial_x \equiv -p_1\, \partial_{p_0} - f\, \partial_{p_1} \mod Char(\mathbf{D}).$$

Therefore, $Shuf(\mathbf{D})$ is spanned by $Z = (b - ap_1)\, \partial_{p_0} + (c - af)\, \partial_{p_1}$, where

$$c = Xb - p_1\, Xa, \quad Xc = f\, Xa + Yf.$$

Example 6.4. **(Continuation of Example** 4.9**)** In this case, we have

$$\begin{aligned} \partial_x &\equiv -p\, \partial_u - r\, \partial_p - F\, \partial_q, \quad &\partial_r \equiv 0,\\ \partial_y &\equiv -q\, \partial_u - F\, \partial_p - t\, \partial_q, \quad &\partial_t \equiv 0, \end{aligned}$$

in $Shuf(\mathbf{D})$. Therefore, $Shuf(\mathbf{D})$ is spanned by

$$\begin{aligned} W = {} & (U - pX - qY)\, \partial_u + (P - rX - FY)\, \partial_p \\ & + (Q - FX - tY)\, \partial_q, \end{aligned}$$

where

$$\begin{aligned} P &= -pX_x - p^2X_u - qY_x - pqY_u + U_x + pU_u,\\ Q &= \frac{1}{p}\Big(pqX_x - p^2X_y + q^2Y_x - pqY_y - qU_x + pU_y + qP\Big), \end{aligned}$$

and $X = X(x, u - qy)$, $Y = Y(y, u - px)$, and $U(x, y, u)$ must satisfy in PDE:

$$\begin{aligned} &(pF_p - F)X_x + p(pF_p - 2F)X_u + (qF_q - F)Y_y\\ &\quad + q(qF_q - 2F)Y_u,\\ &\quad - F_pU_x - F_qU_y + (F - pF_p - qF_q)U_u + U_{xy}\\ &\quad + qU_{xu} + pU_{yu} + pqU_{uu} = XF_x + YF_y + UF_u. \end{aligned} \tag{3}$$

Example 6.5. **(Quasilinear Klein-Gordon Equation)** In this example, we find the shuffling symmetries of the quasilinear Klein-Gordon equation

$$u_{tt} - \alpha^2\, u_{xx} + \gamma^2\, u = \beta\, u^3$$

as an application of the previous example, where α, β, and γ are real constants. The equation can be transformed by defining $\xi = \frac{1}{2}(x - \alpha t)$ and $\eta = \frac{1}{2}(x + \alpha t)$. Then, by the chain rule, we obtain $\alpha^2\, u_{\xi\eta} + \gamma^2\, u = \beta\, u^3$. This equation reduces to

$$u_{xy} = au + bu^3, \tag{4}$$

by $t = y$, $a = -(\gamma/\alpha)^2$, and $b = \beta/\alpha^2$.

By solving the PDE (3), we conclude that $Shuf(\mathbf{D})$ is spanned by the three following vector fields:

$$\begin{aligned}X_1 &= (px - qy)\,\partial_u - (p + yu^2(a + bu) - rx)\,\partial_p\\ &\quad + (q + xu^2(a + bu) - ty)\,\partial_q,\\ X_2 &= q\,\partial_u + u^2(a + bu)\,\partial_p + t\,\partial_q,\\ X_3 &= p\,\partial_u + r\,\partial_p + u^2(a + bu)\,\partial_q.\end{aligned}$$

For example, we have

$$\begin{aligned}\mathrm{Fl}_s^{X_3}(x, y, u, p, q, r, t) &= \Big(x, y, u + sp + \frac{s^2}{2}r, p + sr, q\\ &\quad + s.u^2(a + bu) + \frac{s^2}{40}.up(2a + 3bu) + \frac{s^3}{42}.(14ap^2\\ &\quad + 42bup^2 + 14aur + 21bu^2r) + \frac{s^4}{4}.p(bp^2 + ar\\ &\quad + 3bur) + \frac{s^5}{20}.r(6bp^2 + ar + 3bur) + \frac{s^6}{8}.bpr^2\\ &\quad + \frac{s^7}{56}.br^3, r, t\Big),\end{aligned}$$

and if $u = h(x,y)$ be a solution of (4), then $\mathrm{Fl}_s^{X_3}(x, y, h, h_x, h_y, h_{xx}, h_{yy})$ is also a new solution of (4), for sufficiently small $s \in \mathbf{R}$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
In this paper, RA and MN described the geometry of distributions by their symmetries. Also, a simplified proof of the Frobenius theorem as well as some related corollaries are presented. Moreover, MN and RA studied the geometry of solutions of the *F*-Gordon equation. Both authors read and approved the final manuscript.

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References
1. Barone, A, Esposito, F, Magee, CJ, Scott, AC: Theory and applications of the Sine-Gordon equation. Riv. Nuovo Cim. **1**, 227–267 (1971)
2. Kragh, H: Equation with the many fathers. The Klein-Gordon equation in 1926. Am. J. Phys. **52**(11), 1024–1033 (1984)
3. Alekseevskij, DV, Lychagin, VV, Vinogradov, AM: Basic Ideas and Concepts of Differential Geometry, Vol. I. Springer, New York (1991)
4. Kushner, A, Lychagin, VV, Rubtsov, V: Contact Geometry and Non-linear Differential Equations. Cambridge University Press, Cambridge (2007)

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