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An inverse problem related to a half-linear eigenvalue problem

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Abstract

We study an inverse problem on the half-linear Dirichlet eigenvalue problem $-(|y'(x)|^{p-2}y'(x))' = (p-1)\lambda r(x)|y(x)|^{p-2}y(x)$, where $p > 1$ with $p \neq 2$ and r is a positive function defined on $[0,1]$. Using eigenvalues and nodal data (the lengths of two consecutive zeros of solutions), we reconstruct $r^{-1/p}(x)$ and its derivatives. Our method is based on (Law and Yang in Inverse Probl. 14:299-312, 779-780, 1998; Shen and Tsai in Inverse Probl. 11:1113-1123, 1995), and our result extends the result in (Shen and Tsai in Inverse Probl. 11:1113-1123, 1995) for the linear case to the half-linear case.

MSC: 34A55; 34B24; 47A75

1 Introduction

The subject under investigation is the half-linear eigenvalue problem consisting of

$$\begin{cases} -(|y'(x)|^{p-2}y'(x))' = (p-1)\lambda r(x)|y(x)|^{p-2}y(x), \\ y(0) = y(1) = 0, \end{cases} \tag{1}$$

where $p > 1$ with $p \neq 2$, and r is a positive function defined on $[0,1]$. By [1–4], it is well known that the problem (1) has countably many eigenpairs $\{(\lambda_n, y_n(x)) : n \in \mathbb{N}\}$, and the eigenfunction $y_n(x)$ has exactly $n-1$ nodal points in $(0,1)$, say $0 = x_0^{(n)} < x_1^{(n)} < \cdots < x_{n-1}^{(n)} < x_n^{(n)} = 1$. In this paper, we intend to give the representation of the function $r(x)$ and its derivatives in (1) by using eigenvalues and nodal points. This formation is treated as the reconstruction formula. Such a problem is called an inverse nodal problem and has attracted researchers' attention. Readers can refer to [5–7] for the linear case ($p = 2$), and to [8, 9] for the general case ($p > 1$).

In [8, 9], inverse nodal problems on

$$-\left(\left|y'(x)\right|^{p-2}y'(x)\right)' = (p-1)\left(\lambda - q(x)\right)\left|y(x)\right|^{p-2}y(x) \tag{2}$$

are considered. The authors in [8] studied (2) with Dirichlet boundary conditions

$$y(0) = y(1) = 0,$$

while the authors in [9] studied (2) with eigenparameter dependent boundary conditions

$$y(0) = 0, \qquad \alpha y'(1) + \lambda y(1) = 0 \quad \text{for } \alpha \neq 0.$$

Both of them first used the modified Prüfer substitution to derive the asymptotic expansion of eigenvalues and nodal points and then gave the reconstruction formula of $q(x)$ by using the nodal data. Note that the authors did not deal with the derivatives of $q(x)$ in [8, 9]. Besides, the author in [10] considered the same issue for the p-Laplacian energy-dependent Sturm-Liouville problem,

$$-\left(\left|y'(x)\right|^{p-2}y'(x)\right)' = (p-1)\left[\lambda^2 - q(x) - 2\lambda r(x)\right]\left|y(x)\right|^{p-2}y(x)$$

coupling with the Dirichlet boundary conditions.

In [6], Shen and Tsai studied the inverse nodal problem on the string equation $y'' + \lambda r(x)y = 0$ with Dirichlet boundary conditions. They employed the standard difference operator $\triangle$ to give a reconstruction formulas for $r^{-1/2}(x)$ and its derivatives. On the other hand, Law and Yang [7] not only studied the inverse nodal problems for the linear problem

$$-y'' + q(x)y = \lambda y$$

with separated boundary conditions

$$\begin{cases} y(0)\cos\alpha + y'(0)\sin\alpha = 0, \\ y(1)\cos\beta + y'(1)\sin\beta = 0, \end{cases} \tag{3}$$

where $0 \leq \alpha, \beta < \pi$, and gave a reconstruction formulas for $q(x)$ and its derivatives, but they also mentioned that the formulas for $r(x)$ and its derivatives in the string equation with (3) are still valid. They applied the difference quotient operator δ in the formulas.

The main aim and methods of this study are basically the same as the ones in [6, 7]. Here we employ a modified Prüfer substitution on (1) derived by the generalized sine function $S_p(x)$. The well-known properties of $S_p(x)$ can be referred to [1, 2, 4], *etc.* It shall be mentioned that S_p is not C^2 at odd multiples of $\pi_p/2$ as $p > 2$, and not C^3 at even multiples of $\pi_p/2$ as $1 < p < 2$. These lead to that the reconstruction formulas for Nth derivatives, $N \geq 3$, in [6, 7] cannot be extended to the half-linear case ($p \neq 2$) in this article.

Denote by π_p the first zero of $S_p(x)$ in the positive axis. Define $f(x) = r^{-1/p}(x)$, $j_n(x) \equiv \max\{k : x_k^{(n)} \leq x\}$, and the nodal length $\ell_k^{(n)} \equiv x_{k+1}^{(n)} - x_k^{(n)}$ for $k = 0, 1, 2, \ldots, n-1$. The following is our first result.

Theorem 1 *Consider* (1) *and suppose r is continuous on* $[0,1]$. *For each $x \in [0,1)$, let $j = j_n(x)$ for the sake of simplicity. Then the following asymptotic formula is valid*:

$$f(x) = \frac{\lambda_n^{1/p}\ell_j^{(n)}}{\pi_p} + o(1). \tag{4}$$

Moreover, if $r \in C^1[0,1]$, the error term can be replaced by $O(\frac{1}{n})$.

Now, define the difference operator $\triangle$ and the difference quotient operator δ as follows:

$$\triangle a_j \equiv a_{j+1} - a_j, \qquad \delta a_j \equiv \frac{a_{j+1} - a_j}{x_{j+1} - x_j} = \frac{\triangle a_j}{\ell_j} \quad \text{and} \quad \delta^k a_j \equiv \frac{\delta^{k-1}a_{j+1} - \delta^{k-1}a_j}{\ell_j}.$$

This δ-operator discretizes the differential operator in a nice way. It resembles the difference quotient operator in finite difference. For the derivatives of $f(x)$, we have the following result.

Theorem 2 *Consider* (1) *and suppose* r *is* C^3 *on* $[0,1]$. *For each* $x \in [0,1)$, *let* $j = j_n(x)$ *for the sake of simplicity. Then as* $n \to \infty$,

$$f^{(k)}(x) = \frac{\lambda_n^{1/p}}{\pi_p}\delta^k \ell_j^{(n)} + O\left(\frac{1}{n}\right) \quad \text{for } k = 1,2. \tag{5}$$

This paper is organized as follows. In Section 2, we give the asymptotic estimates for eigenvalues. This step makes us know such quantities well. It is necessary to specify the orders of the expansion terms in the proofs of the main results. In Section 3, the proofs of the main theorems are given.

2 Asymptotic estimates for eigenvalues

Before we prove the main results, we derive the eigenvalue expansion. Note that if $r(x) \in C[0,1]$, the last term in (6) can be replaced by $o(n)$ (*cf.* [2, p.171]). In [3, Theorem 2.5], they proved the error term is $o(1)$ if $r \in C^1[0,1]$. The smoothness of $r(x)$ increases, and the smaller error can be derived.

Theorem 3 *Suppose that* $r(x)$ *is a positive* C^2*-function defined on* $[0,1]$. *Then the asymptotic estimate yields*

$$\lambda_n^{1/p}\int_0^1 r^{1/p}(t)\,dt = n\pi_p + O\left(\frac{1}{n}\right). \tag{6}$$

Proof Let $y(x)$ be a solution of (1) and define a Prüfer-type substitution

$$\lambda^{1/p} r^{1/p}(x) y(x) = \rho(x) S_p\bigl(\theta(x)\bigr), \qquad y'(x) = \rho(x) S_p'\bigl(\theta(x)\bigr). \tag{7}$$

Then a direct calculation yields

$$\theta'(x) = \lambda^{1/p} r^{1/p}(x) + \frac{r'(x)}{pr(x)} S_p\bigl(\theta(x)\bigr)\bigl|S_p'\bigl(\theta(x)\bigr)\bigr|^{p-2} S_p'\bigl(\theta(x)\bigr), \tag{8}$$

$$\rho'(x) = \frac{r'(x)\rho(x)}{pr(x)}\bigl|S_p\bigl(\theta(x)\bigr)\bigr|^p. \tag{9}$$

With each eigenvalue λ_n of (1), one can associate a Prüfer angle $\theta_n(x) \equiv \theta(x;\lambda_n)$ via (7) if one also specifies the initial condition $\theta_n(0) = 0$ for $n = 1,2,3,\ldots$. In particular, $\theta_n(1) = n\pi_p$. Integrating both sides of (8) over $[0,1]$, one obtains

$$n\pi_p = \lambda_n^{1/p}\int_0^1 r^{1/p}(t)\,dt + \int_0^1 \frac{r'(t)}{pr(t)} S_p\bigl(\theta_n(t)\bigr)\bigl|S_p'\bigl(\theta_n(t)\bigr)\bigr|^{p-2} S_p'\bigl(\theta_n(t)\bigr)\,dt. \tag{10}$$

Let $g(\tau) \equiv S_p(\tau)|S_p'(\tau)|^{p-2}S_p'(\tau)$. Note that if $\theta_n'(x) = 0$ is valid in some subinterval of $(0,1)$, the term $S_p(\theta_n(x))|S_p'(\theta_n(x))|^{p-2}S_p'(\theta_n(x))$ will be constant in this subinterval. This implies that the function $r(x)$ depends on λ_n in this subinterval from (8). This will contradict our original problem. Hence, the points satisfying $\theta_n'(x) = 0$ shall be isolated. Then, in (8)

$$\begin{aligned}\frac{r'(t)}{pr(t)}g(\theta_n(t)) &= \frac{r'(t)}{pr(t)}g(\theta_n(t))\frac{\theta_n'(t)}{\lambda_n^{1/p}r^{1/p}(t)+\frac{r'(t)}{pr(t)}g(\theta_n(t))}\\ &= \frac{r'(t)g(\theta_n(t))\theta_n'(t)}{p\lambda_n^{1/p}r^{\frac{p+1}{p}}(t)}\left[\frac{1}{1+\frac{r'(t)g(\theta_n(t))}{p\lambda_n^{1/p}r^{\frac{p+1}{p}}(t)}}\right]\end{aligned}$$

for $\theta_n' \neq 0$. Let $f(t) = r^{-1/p}(t)$. Then $f'(t) = \frac{-r'(t)}{pr^{\frac{p+1}{p}}(t)}$. Dropping the function variable t, one has the following:

$$\frac{r'}{pr}g(\theta_n) = \frac{-f'g(\theta_n)\theta_n'}{\lambda_n^{1/p}}\left[\frac{1}{1-\frac{f'}{\lambda_n^{1/p}}g(\theta_n)}\right] = -\sum_{k=0}^{\infty}\left[\frac{f'g(\theta_n)}{\lambda_n^{1/p}}\right]^{k+1}\theta_n'. \tag{11}$$

Define $G_1^{(n)}(t) = \int_0^{\theta_n(t)} g(\tau)\,d\tau$. Then $G_1^{(n)}(0) = G_1^{(n)}(1) = 0$ since $g(\tau)$ is a π_p-periodic function. Moreover, by integration by parts, (11) implies that

$$\int_0^1 \lambda_n^{-1/p}f'(t)g(\theta_n(t))\theta_n'(t)\,dt = -\lambda_n^{-1/p}\int_0^1 f''(t)G_1^{(n)}(t)\,dt = O(\lambda_n^{-1/p}), \tag{12}$$

for sufficiently large n. Substituting (11)-(12) into (10), the eigenvalue estimates (6) can be derived. □

3 Proofs of Theorems 1-2

Proof of Theorem 1 Let $j = j_n(x)$. By the Sturm comparison theorem for the p-Laplacian (*cf.* [4, 11, 12] *etc.*), one has

$$\frac{\pi_p}{(\lambda_n r_{M_j})^{1/p}} \le \ell_j^{(n)} \le \frac{\pi_p}{(\lambda_n r_{m_j})^{1/p}},$$

where r_{M_j} and r_{m_j} are the maximal and minimal values of r on $[x_j^{(n)}, x_{j+1}^{(n)}]$, respectively. Then

$$r_{M_j}^{-1/p} \le \frac{\lambda_n^{1/p}\ell_j^{(n)}}{\pi_p} \le r_{m_j}^{-1/p}. \tag{13}$$

In particular, for $0 \le j \le n-1$, we have

$$\ell_j^{(n)} = O\left(\frac{1}{n}\right). \tag{14}$$

Moreover, by the continuity of $f(x)$ and (13), there is an $\xi_j^{(n)} \in (x_j^{(n)}, x_{j+1}^{(n)})$ such that

$$\frac{\lambda_n^{1/p}\ell_j^{(n)}}{\pi_p} = f(\xi_j^{(n)}). \tag{15}$$

On the other hand, by the mean value theorem and (14), one also has

$$f(x) - f\left(\xi_j^{(n)}\right) = f'(y_j)\left(x - \xi_j^{(n)}\right) = O\left(\frac{1}{n}\right) \tag{16}$$

for some y_j between x and $\xi_j^{(n)}$. Therefore, (15) and (16) complete the proof. □

Before we prove Theorem 2, one has to derive the asymptotic behavior of $f(x)$ at the nodal points. Note that the series in (11) is uniformly convergent on $[0,1]$. Set $h(x) = r^{1/p}(x) = \frac{1}{f(x)}$. Integrating (8) over $[x_j^{(n)}, x_{j+1}^{(n)}]$ and applying (11), one has

$$\pi_p = \lambda_n^{1/p} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} h(t)\,dt - \sum_{k=1}^{\infty} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} \left[\lambda_n^{-1/p} f'(t) g\left(\theta_n(t)\right)\right]^k \theta_n'(t)\,dt. \tag{17}$$

By the Taylor expansion theorem and integration by parts, we find

$$\begin{aligned}
\pi_p &= \lambda_n^{1/p} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} \left[h\left(x_j^{(n)}\right) + h'\left(x_j^{(n)}\right)\left(t - x_j^{(n)}\right) + \frac{h''(y_j^{(n)})}{2!}\left(t - x_j^{(n)}\right)^2\right] dt \\
&\quad - \lambda_n^{-1/p}\left[f'(t) G_1^{(n)}(t)\right]_{x_j^{(n)}}^{x_{j+1}^{(n)}} + \lambda_n^{-1/p} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} f''(t) G_1^{(n)}(t)\,dt \\
&\quad - \sum_{m=2}^{\infty} \lambda_n^{-m/p}\left[\left(f'(t)\right)^m G_m^{(n)}(t)\right]_{x_j^{(n)}}^{x_{j+1}^{(n)}} + \sum_{m=2}^{\infty} \lambda_n^{-m/p} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} \left[\left(f'(t)\right)^m\right]' G_m^{(n)}(t)\,dt
\end{aligned}$$

for some $y_j^{(n)} \in (x_j^{(n)}, x_{j+1}^{(n)})$, where $G_k^{(n)}(t) = \int_0^{\theta_n(t)} (g(\tau))^k\,d\tau$. Note that

$$G_1^{(n)}\left(x_j^{(n)}\right) = G_1^{(n)}\left(x_{j+1}^{(n)}\right) = 0.$$

Hence,

$$\begin{aligned}
\pi_p &= \lambda_n^{1/p} h\left(x_j^{(n)}\right) \ell_j^{(n)} + \frac{1}{2!}\lambda_n^{1/p} h'\left(x_j^{(n)}\right)\left(\ell_j^{(n)}\right)^2 + \frac{1}{3!}\lambda_n^{1/p} h''\left(y_j^{(n)}\right)\left(\ell_j^{(n)}\right)^3 \\
&\quad + \sum_{m=1}^{\infty} \lambda_n^{-m/p} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} \left(\left[f'(t)\right]^m\right)' G_m^{(n)}(t)\,dt - \sum_{m=2}^{\infty} \lambda_n^{-m/p}\left[f'\left(x_{j+1}^{(n)}\right)\right]^m G_m^{(n)}\left(x_{j+1}^{(n)}\right) \\
&\quad + \sum_{m=2}^{\infty} \lambda_n^{-m/p}\left[f'\left(x_j^{(n)}\right)\right]^m G_m^{(n)}\left(x_j^{(n)}\right).
\end{aligned} \tag{18}$$

Multiplying (18) by $\frac{f(x_j^{(n)})}{\pi_p}$, one has

$$\begin{aligned}
f\left(x_j^{(n)}\right) &= \frac{\lambda_n^{1/p} \ell_j^{(n)}}{\pi_p} + \frac{1}{2\pi_p}\lambda_n^{1/p} f\left(x_j^{(n)}\right) h'\left(x_j^{(n)}\right)\left(\ell_j^{(n)}\right)^2 \\
&\quad + \frac{1}{6\pi_p}\lambda_n^{1/p} f\left(x_j^{(n)}\right) h''\left(y_j^{(n)}\right)\left(\ell_j^{(n)}\right)^3 \\
&\quad + \sum_{m=1}^{\infty} \frac{\lambda_n^{-m/p}}{\pi_p} f\left(x_j^{(n)}\right) \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} \left[\left(f'(t)\right)^m\right]' G_m^{(n)}(t)\,dt
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{m=2}^{\infty}\frac{\lambda_n^{-m/p}}{\pi_p}f\bigl(x_j^{(n)}\bigr)\bigl[f'\bigl(x_{j+1}^{(n)}\bigr)\bigr]^m G_m^{(n)}\bigl(x_{j+1}^{(n)}\bigr)\\
&+\sum_{m=2}^{\infty}\frac{\lambda_n^{-m/p}}{\pi_p}f\bigl(x_j^{(n)}\bigr)\bigl[f'\bigl(x_j^{(n)}\bigr)\bigr]^m G_m^{(n)}\bigl(x_j^{(n)}\bigr).
\end{aligned}
$$

For convenience, we denote

$$
f\bigl(x_j^{(n)}\bigr)\equiv\frac{\lambda_n^{1/p}\ell_j^{(n)}}{\pi_p}+A_j+B_j+C_j+D_j+E_j, \tag{19}
$$

where

$$
\begin{aligned}
A_j&=\frac{1}{2\pi_p}\lambda_n^{1/p}f\bigl(x_j^{(n)}\bigr)h'\bigl(x_j^{(n)}\bigr)\bigl(\ell_j^{(n)}\bigr)^2,\\
B_j&=\frac{1}{6\pi_p}\lambda_n^{1/p}f\bigl(x_j^{(n)}\bigr)h''\bigl(y_j^{(n)}\bigr)\bigl(\ell_j^{(n)}\bigr)^3,\\
C_j&=\sum_{m=1}^{\infty}\frac{\lambda_n^{-m/p}}{\pi_p}f\bigl(x_j^{(n)}\bigr)\int_{x_j^{(n)}}^{x_{j+1}^{(n)}}\bigl[\bigl(f'(t)\bigr)^m\bigr]'G_m^{(n)}(t)\,dt,\\
D_j&=-\sum_{m=2}^{\infty}\frac{\lambda_n^{-m/p}}{\pi_p}f\bigl(x_j^{(n)}\bigr)\bigl[f'\bigl(x_{j+1}^{(n)}\bigr)\bigr]^m G_m^{(n)}\bigl(x_{j+1}^{(n)}\bigr),\\
E_j&=\sum_{m=2}^{\infty}\frac{\lambda_n^{-m/p}}{\pi_p}f\bigl(x_j^{(n)}\bigr)\bigl[f'\bigl(x_j^{(n)}\bigr)\bigr]^m G_m^{(n)}\bigl(x_j^{(n)}\bigr).
\end{aligned}
$$

Remark 1 Note that the function $G_m^{(n)}$ is defined by the integral of $g(\tau)\equiv S_p(\tau)|S_p'(\tau)|^{p-2}\times S_p'(\tau)$. By the unlike property of S_p, $G_m^{(n)}$ is not a smooth function. This is the main reason that the result in [7, Theorem 5.1] does not hold in our case, $p\neq 2$.

Then the following lemmas are necessary to the proof of Theorem 2 and the superscript will be dropped for the sake of convenience, $x_j=x_j^{(n)}$ and $\ell_j=\ell_j^{(n)}$.

Lemma 1 *Let $f=r^{-1/p}\in C^3[0,1]$. Then $\delta^k f(x_j)=O(1)$ as $k=1,2,3$ and $\delta^k f(x_j)=O(n^{k-3})$ as $k\geq 4$. Moreover, if x_j is replaced by $y_j\in(x_j,x_{j+1})$, the above result is still valid. Furthermore, $\delta^k(\ell_j)^m=O(\frac{1}{n^m})$ as $k=1,2,3$.*

Proof The first part of the proof is followed by the mean value theorem and the asymptotic estimates for nodal length (14), *i.e.*,

$$
\delta f(x_j)=\frac{f(x_{j+1})-f(x_j)}{\ell_j}=f'(y_j)=O(1)\quad\text{for some } y_j\in(x_j,x_{j+1}).
$$

It is also valid for $k=2,3$ by similar arguments. On the other hand, for $k=4$, we find

$$
\delta^4 f(x_j)=\frac{\delta^3 f(x_{j+1})-\delta^3 f(x_j)}{\ell_j}=\frac{O(1)}{\ell_j}=O(n),
$$

and it is also valid for $k > 4$. Finally, by (15) and (16), we find $f(\xi_j) = f(x_j) + O(\frac{1}{n})$ and

$$\delta^k(\ell_j)^m = \frac{\pi_p^m}{\lambda_n^{m/p}}\delta^k\big[f(\xi_j)\big]^m = O\left(\frac{1}{n^m}\right) \quad \text{for } k = 1,2,3. \qquad \square$$

Lemma 2 *Let $u \in C^2$. Then, for $k = 1,2$,*

$$\delta^k\big[u(x_j)(\ell_j)^2\big] = O\left(\frac{1}{n^2}\right).$$

Moreover, if $v \in C^1$, then for $k = 1,2$

$$\delta^k\big[v(x_j)(\ell_j)^3\big] = O\left(\frac{1}{n^{4-k}}\right).$$

Proof First applying the identity $\delta(a_jb_j) = a_{j+1}\delta b_j + b_j\delta a_j$, (14) and Lemma 1, one can obtain

$$\begin{aligned}
&\delta\big[u(x_j)(\ell_j)^2\big] = (\ell_{j+1})^2\delta u(x_j) + u(x_j)\delta(\ell_j)^2 = O\left(\frac{1}{n^2}\right),\\
&\delta^2\big[u(x_j)(\ell_j)^2\big] = (\ell_{j+2})^2\delta^2\big[u(x_j)\big] + 2\delta\big[(\ell_{j+1})^2\big]\delta\big[u(x_j)\big] + u(x_j)\delta^2\big[(\ell_j)^2\big] = O\left(\frac{1}{n^2}\right).
\end{aligned}$$

The second part is similar to the one of the first part. So it is omitted here. $\square$

The following corollary is similar to [7, Lemma 2.3]. We give the proof for the convenience of the readers.

Corollary 1 *Let $q \in C^1$. Define $Q(x_j) = \int_{x_j}^{x_{j+1}} q(t)\,dt$. Then, for $k = 1,2$,*

$$\delta^k Q(x_j) = O\left(\frac{1}{n^{2-k}}\right).$$

Proof By the mean value theorem for integrals, for every j there exists some $y_j \in (x_j, x_{j+1})$ such that $Q(x_j) = q(y_j)\ell_j$. Then applying Lemmas 1-2, we complete the proof. $\square$

Proof of Theorem 2 Recall (19). By Theorem 3 and Lemma 2, one has $\delta^k A_j = O(\frac{1}{n})$ and $\delta^k B_j = O(\frac{1}{n^{3-k}})$ for $k = 1,2$. By Theorem 3 and Corollary 1, one can obtain $\delta^k C_j = O(\frac{1}{n^{3-k}})$ for $k = 1,2$. By Theorem 3, Lemma 1 and the definition of $G_m^{(n)}$, one has $\delta^k D_j$ and $\delta^k E_j$ are $O(\frac{1}{n^{3-k}})$ for $k = 1,2$. Hence, one can find

$$\delta^k f\big(x_j^{(n)}\big) = \frac{\lambda_n^{1/p}}{\pi_p}\delta^k\ell_j^{(n)} + O\left(\frac{1}{n}\right) \tag{20}$$

for $k = 1,2$. To complete the proof, it suffices to show that

$$f^{(k)}(x) = \delta^k f\big(x_j^{(n)}\big) + O\left(\frac{1}{n}\right) \tag{21}$$

for sufficiently large n. Obviously, (21) holds for $k = 0$ by (14) and (16). When $k = 1, 2$, by the Taylor theorem,

$$f^{(k-1)}\left(x_{j+1}^{(n)}\right) = f^{(k-1)}\left(x_j^{(n)}\right) + f^{(k)}\left(x_j^{(n)}\right)\ell_j^{(n)} + \frac{f^{(k+1)}(\xi_{k+1,j}^{(n)})}{2}\left(\ell_j^{(n)}\right)^2$$

for some $\xi_{k+1,j}^{(n)} \in (x_j^{(n)}, x_{j+1}^{(n)})$. Thus,

$$f^{(k)}\left(x_j^{(n)}\right) = \delta f^{(k-1)}\left(x_j^{(n)}\right) + \frac{1}{2} f^{(k+1)}\left(\xi_{k+1,j}^{(n)}\right)\ell_j^{(n)}, \tag{22}$$

for $k = 1, 2$; *i.e.*, by the mean value theorem and (22),

$$f'(x) = f'\left(x_j^{(n)}\right) + O\left(\frac{1}{n}\right) = \delta f\left(x_j^{(n)}\right) + O\left(\frac{1}{n}\right). \tag{23}$$

Successively, we have

$$\begin{aligned} f''(x) &= \delta f'\left(x_j^{(n)}\right) + O\left(\frac{1}{n}\right) \\ &= \delta\left[\delta f\left(x_j^{(n)}\right) + \frac{1}{2} f''\left(\xi_{2,j}^{(n)}\right)\ell_j^{(n)}\right] + O\left(\frac{1}{n}\right) \\ &= \delta^2 f\left(x_j^{(n)}\right) + O\left(\frac{1}{n}\right). \end{aligned} \tag{24}$$

Therefore, substituting (20) into (23)-(24), this completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The author WCW was a major contributor in writing the manuscript. The author YHC performed the literature review. Both of them performed the final editing of the manuscript. They also gave their final approval of the version to be submitted and any revised version.

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