Boundary Value Problems
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Abstract elliptic operators appearing in atmospheric dispersion

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Abstract
In this paper, the boundary value problem for the differential-operator equation with principal variable coefficients is studied. Several conditions for the separability and regularity in abstract L^p-spaces are given. Moreover, sharp uniform estimates for the resolvent of the corresponding elliptic differential operator are shown. It is implies that this operator is positive and also is a generator of an analytic semigroup. Then the existence and uniqueness of maximal regular solution to nonlinear abstract parabolic problem is derived. In an application, maximal regularity properties of the abstract parabolic equation with variable coefficients and systems of parabolic equations are derived in mixed $L^{\mathbf{p}}$-spaces.

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1 Introduction

It is well known that many classes of PDEs, pseudo DEs and integro DEs can be expressed as a differential-operator equation (DOE). DOEs have been studied extensively in the literature (see [1–22] and the references therein). Note the regularity results for the PDE studied *e.g.* in [11, 23–25]. The main goal of the present paper is to discuss the maximal regularity properties of nonlocal boundary value problems (BVPs) for the following DOE:

$$\sum_{k=1}^{n} a_k(x)\frac{\partial^2 u}{\partial x_k^2} + A(x)u + \sum_{k=1}^{n} A_k(x)\frac{\partial u}{\partial x_k} = f(x), \quad x \in G \subset R^n. \tag{1.1}$$

Afterwards, the well-posedness of initial and BVP (IBVP) for the following abstract parabolic equation:

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{n} a_k(x)\frac{\partial^2 u}{\partial x_k^2} + A(x)u + \sum_{k=1}^{n} A_k(x)\frac{\partial u}{\partial x_k} = f(x,t), \quad t \in (0,T), x \in G,$$

is derived, where $a_k(x)$ are complex-valued functions, A and A_k are linear operators in a Banach space E, $u(x)$ and $f(x)$, respectively, are an E-valued unknown and data function. By using this, we obtain the existence and uniqueness result of IBVP for the following nonlinear parabolic equation:

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{n} a_k(x) \frac{\partial^2 u}{\partial x_k^2} + B(t,x,u)u = F(t,x,u,\nabla u).$$

Finally, we discuss the application of the above result to systems of parabolic PDEs. Particularly, we consider the system that serves as a model of systems used to describe photochemical generation and atmospheric dispersion of ozone and other pollutants. The model of the process is given by the atmospheric reaction-advection-diffusion system having the form

$$\begin{aligned}\frac{\partial u_i}{\partial t} = \sum_{k=1}^{3} \left[a_{ki}(x) \frac{\partial^2 u_i}{\partial x_k^2} + b_{ki}(x) \frac{\partial}{\partial x_k}(u_i \omega_k) \right] \\ + \sum_{k=1}^{3} d_k u_k + f_i(u) + g_i, \quad x \in D, t \in (0,T),\end{aligned}$$

where

$$\begin{aligned}&D = \{x = (x_1,x_2,x_3), 0 < x_k < b_k,\}, \\ &u_i = u_i(t,x), \quad i,k = 1,2,3, \qquad u = u(t,x) = (u_1,u_2,u_3),\end{aligned}$$

and the state variables u_i represent concentration densities of the chemical species involved in the photochemical reaction. The relevant chemistry of the chemical species involved in the photochemical reaction appears in the nonlinear functions $f_i(u)$ with the terms g_i, representing elevated point sources, and where $a_{ki}(x)$, $b_{ki}(x)$ are real-valued functions. The advection terms $\omega = \omega(x) = (\omega_1(x), \omega_2(x), \omega_3(x))$ describe transport of the velocity vector field of atmospheric currents or wind; see [4] and references therein.

2 Definitions, notations, and background

Let E be a Banach space. $L^p(\Omega;E)$ denotes the space of strongly measurable E-valued functions that are defined on the measurable subset $\Omega \subset R^n$ with the norm

$$\|f\|_{L^p} = \|f\|_{L^p(\Omega;E)} = \left(\int_\Omega \|f(x)\|_E^p \, dx \right)^{\frac{1}{p}}, \quad 1 \le p < \infty.$$

The Banach space E is called an UMD-space if the Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} \, dy$$

is bounded in $L^p(R,E)$, $p \in (1,\infty)$ (see, *e.g.*, [26]). UMD-spaces include *e.g.* L^p, l_p spaces and Lorentz spaces L_{pq}, $p,q \in (1,\infty)$.

Let

$$S_\psi = \{\lambda \in \mathbb{C}, |\arg \lambda| \le \psi, 0 \le \psi < \pi\}, \qquad S_{\psi,\varkappa} = \{\lambda \in S_\psi, |\lambda| > \varkappa > 0\}.$$

A linear operator A is said to be ψ-positive in a Banach space E with bound $M > 0$ if $D(A)$ is dense on E and $\|(A+\lambda I)^{-1}\|_{L(E)} \leq M(1+|\lambda|)^{-1}$ for any $\lambda \in S_\psi$, $0 \leq \psi < \pi$, where I is the identity operator in E, and $L(E)$ is the space of bounded linear operators in E. It is well known [25, §1.15.1] that there exist fractional powers A^θ of a positive operator A. Let $E(A^\theta)$ denote the space $D(A^\theta)$ endowed with the norm

$$\|u\|_{E(A^\theta)} = \left(\|u\|^p + \left\|A^\theta u\right\|^p\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, 0 < \theta < \infty.$$

Let E_1 and E_2 be two Banach spaces. By $(E_1, E_2)_{\theta,p}$, $0 < \theta < 1$, $1 \leq p \leq \infty$, will be denoted the interpolation spaces obtained from $\{E_1, E_2\}$ by the K-method [25, §1.3.2].

Let $\mathbb{N}$ denote the set of natural numbers. A set $\Phi \subset B(E_1, E_2)$ is called R-bounded (see, *e.g.*, [3]) if there is a positive constant C such that for all $T_1, T_2, \dots, T_m \in \Phi$ and $u_1, u_2, \dots, u_m \in E_1$, $m \in \mathbb{N}$,

$$\int_\Omega \left\| \sum_{j=1}^m r_j(y) T_j u_j \right\|_{E_2} dy \leq C \int_\Omega \left\| \sum_{j=1}^m r_j(y) u_j \right\|_{E_1} dy,$$

where $\{r_j\}$ is a sequence of independent symmetric $\{-1, 1\}$-valued random variables on Ω. The smallest C for which the above estimate holds is called a R-bound of the collection Φ and denoted by $R(\Phi)$.

Since we will consider the problem with spectral parameter, we need the concept of the uniform R-boudedness of a parameter-dependent family of operators. A set $\Phi_h \subset B(E_1, E_2)$ is called the uniform R-bounded with respect to the parameter $h \in Q \subset \mathbb{C}$ if there is a constant M independent on h such that

$$\int_\Omega \left\| \sum_{j=1}^m r_j(y) T_j(h) u_j \right\|_{E_2} dy \leq M \int_\Omega \left\| \sum_{j=1}^m r_j(y) u_j \right\|_{E_1} dy$$

for all $T_1(h), T_2(h), \dots, T_m(h) \in \Phi_h$ and $u_1, u_2, \dots, u_m \in E_1$, $m \in \mathbb{N}$ and $\{r_j\}$. It is implied that $\sup_{h \in Q} R(\Phi_h) \leq M$.

The ψ-positive operator A is said to be R-positive in a Banach space E if the set $L_A = \{\xi(A+\xi)^{-1} : \xi \in S_\psi\}$, $0 \leq \psi < \pi$, is R-bounded.

The operator $A(t)$ is said to be ψ-positive in E uniformly with respect to t with bound $M > 0$ if $D(A(t))$ is independent of t, $D(A(t))$ is dense in E and $\|(A(t)+\lambda)^{-1}\| \leq \frac{M}{1+|\lambda|}$ for all $\lambda \in S_\psi$, $0 \leq \psi < \pi$, where M does not depend on t and λ.

Let E_0 and E be two Banach spaces. E_0 is continuously and densely embedded into E. Let Ω be a domain in R^n and m be a positive integer. $W^{m,p}(\Omega; E_0, E)$ denotes the space of all functions $u \in L^p(\Omega; E_0)$ that have generalized derivatives $\frac{\partial^m u}{\partial x_k^m} \in L^p(\Omega; E)$ with the norm

$$\|u\|_{W^{m,p}(\Omega;E_0,E)} = \|u\|_{L^p(\Omega;E_0)} + \sum_{k=1}^n \left\| \frac{\partial^m u}{\partial x_k^m} \right\|_{L^p(\Omega;E)} < \infty.$$

For $n = 1$, $\Omega = (a, b)$, $a, b \in R$, the space $W^{m,p}(\Omega; E_0, E)$ will be denoted by $W^{m,p}(a, b; E_0, E)$. For $E_0 = E$ the space $W^{m,p}(\Omega; E_0, E)$ is denoted by $W^{m,p}(\Omega; E)$.

Sometimes we use one and the same symbol C without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say α, we write C_α.

The embedding theorems in vector-valued spaces play a key role in the theory of DOEs. For estimating lower order derivatives we use following embedding theorems from [17].

Theorem A_1 *Suppose the following conditions are satisfied*:

(1) *E is a UMD-space and A is an R-positive operator in E*;

(2) *$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is an n-tuple of nonnegative integer numbers and m is a positive integer such that*

$$\varkappa = \sum_{k=1}^{n} \frac{|\alpha|}{m} \leq 1, \quad 0 \leq \mu \leq 1 - \varkappa, 1 < p < \infty;$$

(3) *h is a positive parameter with $0 < h \leq h_0$, where h_0 is a fixed positive number*;

(4) *$\Omega \subset R^n$ is a region such that there exists a bounded linear extension operator from $W^{m,p}(\Omega; E(A), E)$ to $W^{m,p}(R^n; E(A), E)$.*

Then the embedding $D^\alpha W^{m,p}(\Omega; E(A), E) \subset L^p(\Omega; E(A^{1-\varkappa-\mu}))$ is continuous and for $u \in W^{m,p}(\Omega; E(A), E)$ the following uniform estimate holds:

$$\left\|D^\alpha u\right\|_{L^p(\Omega;E(A^{1-\varkappa-\mu}))} \leq h^\mu \|u\|_{W^{m,p}(\Omega;E(A),E)} + h^{-(1-\mu)} \|u\|_{L^p(\Omega;E)}.$$

Remark 2.1 If $\Omega \subset R^n$ is a region satisfying the strong m-horn condition (see [27, §7]), $E = R$, $A = I$, then for $p \in (1, \infty)$ there exists a bounded linear extension operator from $W^{m,p}(\Omega) = W^{m,p}(\Omega; R, R)$ to $W^{m,p}(R^n) = W^{m,p}(R^n; R, R)$.

Theorem A_2 *Suppose all conditions of Theorem* A_1 *are satisfied and $0 < \mu \leq 1 - \varkappa$. Moreover, let Ω be a bounded region and $A^{-1} \in \sigma_\infty(E)$. Then the embedding*

$$D^\alpha W^{m,p}\bigl(\Omega; E(A), E\bigr) \subset L^p\bigl(\Omega; E\bigl(A^{1-\varkappa-\mu}\bigr)\bigr)$$

is compact.

Theorem A_3 *Suppose all conditions of Theorem* A_1 *are satisfied. Let $0 < \mu \leq 1 - \varkappa$. Then the embedding*

$$D^\alpha W^{m,p}\bigl(\Omega; E(A), E\bigr) \subset L^p\bigl(\Omega; \bigl(E(A), E\bigr)_{\varkappa,p}\bigr)$$

is continuous and there exists a positive constant C_μ such that for all $u \in W_p^l(\Omega; E(A), E)$ the uniform estimate holds:

$$\left\|D^\alpha u\right\|_{L^p(\Omega;(E(A),E)_{\varkappa,p})} \leq C_\mu \bigl[h^\mu \|u\|_{W^{m,p}(\Omega;E(A),E)} + h^{-(1-\mu)} \|u\|_{L^p(\Omega;E)}\bigr].$$

From [14, Theorem 2.1] we obtain the following.

Theorem A_4 *Let E be a Banach space, A be a φ-positive operator in E with bound M, $0 \leq \varphi < \pi$. Let m be a positive integer, $1 < p < \infty$ and $\alpha \in (\frac{1}{2p}, \frac{1}{2p} + m)$. Then, for $\lambda \in S_\varphi$ an operator $-A_\lambda^{\frac{1}{2}}$ generates a semigroup $e^{-xA_\lambda^{\frac{1}{2}}}$ which is holomorphic for $x > 0$. Moreover,* *there exists a positive constant* C (*depending only on* M, φ, m, α, *and* p) *such that for every* $u \in (E, E(A^m))_{\frac{\alpha}{m}-\frac{1}{2mp},p}$ *and* $\lambda \in S_\varphi$,

$$\int_0^\infty \left\| A_\lambda^\alpha e^{-xA_\lambda^{\frac{1}{2}}} u \right\|^p dx \le C\left[\|u\|^p_{(E,E(A^m))_{\frac{\alpha}{m}-\frac{1}{2mp},p}} + |\lambda|^{\alpha p-\frac{1}{2}} \|u\|_E^p\right].$$

3 Boundary value problems for abstract elliptic equations with constant coefficients

Consider first the BVP for the constant coefficients DOE

$$\sum_{k=1}^n a_k \frac{\partial^2 u}{\partial x_k^2} + (A+\lambda)u = f(x), \quad x \in G, \tag{3.1}$$

$$\sum_{i=0}^{m_{kj}} \left[\alpha_{kji} \frac{\partial^i u}{\partial x_k^i}(G_{k0}) + \beta_{kji} \frac{\partial^i u}{\partial x_k^i}(G_{kb})\right] = 0, \quad x^{(k)} \in G_k, j = 1,2, \tag{3.2}$$

where

$$x = (x_1, x_2, \ldots, x_n) \in G = \prod_{k=1}^n (0, b_k), \quad m_{kj} \in \{0,1\},$$

$$x^{(k)} = (x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in G_k = \prod_{j \neq k} (0, b_j),$$

$$G_{k0} = (x_1, x_2, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n), \qquad G_{kb} = (x_1, x_2, \ldots, x_{k-1}, b_k, x_{k+1}, \ldots, x_n);$$

here A is a linear operator in a Banach space E, a_k are complex numbers, and λ is a complex parameter.

Let $\alpha_{kj} = \alpha_{kjm_k}$ and $\beta_{kj} = \beta_{kjm_k}$. Let ω_{ki}, $i = 1,2$ denote the roots of the equations

$$a_k \omega^2 = 1, \quad k = 1,2,\ldots,n$$

and

$$\eta_k = \begin{vmatrix} \alpha_{k1}(-\omega_{k1})^{m_{k1}} & \beta_{k1}\omega_{k1}^{m_{k1}} \\ \alpha_{k2}(-\omega_{k1})^{m_{k2}} & \beta_{k2}\omega_{k2}^{m_{k2}} \end{vmatrix}.$$

Condition 3.1 Assume;

(1) E is a UMD-space and A is a uniformly R-positive operator in E for $\varphi \in [0, \pi)$;
(2) $a_k \neq 0$, $a_k \in S(\varphi_0) \cap \mathbb{C}/\mathbb{R}_+$, $\varphi + \varphi_0 < \pi$;
(3) $|\alpha_{kj}| + |\beta_{kj}| > 0$, $\eta_k \neq 0$, $k = 1,2,\ldots,n$, $j = 1,2$.

The main result of this section is the following.

Theorem 3.1 *Assume Condition* 3.1 *is satisfied. Then problem* (3.1)-(3.2) *has a unique solution* $u \in W^{2+m,p}(G; E(A), E)$ *for* $f \in W^{m,p}(G; E)$, $p \in (1, \infty)$, $\lambda \in S_{\psi,\varkappa}$ *with sufficiently large* $|\lambda|$ *and the following uniform coercive estimate holds*:

$$\sum_{k=1}^n \sum_{i=0}^{m+2} |\lambda|^{1-\frac{i}{m+2}} \left\| \frac{\partial^i u}{\partial x_k^i} \right\|_{L^p(G;E)} + \|Au\|_{L^p(G;E)} \le C\|f\|_{W^{m,p}(G;E)}. \tag{3.3}$$

For proving Theorem 3.1, we consider the BVP for the ordinary DOE

$$(L+\lambda)u = au^{(2)}(x) + (A+\lambda)u(x) = f(x), \quad x \in (0,1), \tag{3.4}$$

$$L_k u = \sum_{i=0}^{m_k} \bigl[\alpha_{ki} u^{(i)}(0) + \beta_{ki} u^{(i)}(1)\bigr] = f_k, \quad k = 1,2,$$

where $f \in L_p(0,1;E)$, $f_k \in E_k = (E(A),E)_{\theta_k,p}$, $\theta_k = \frac{1}{2}(m_k + \frac{1}{p})$, $p \in (1,\infty)$, $m_k \in \{0,1\}$, α_{ki}, β_{ki} are complex numbers, a is a complex number, λ is a complex parameter, and A is a linear operator in E. Let us first consider the corresponding homogeneous problem:

$$(L+\lambda)u = au^{(2)}(x) + (A+\lambda)u(x) = 0, \tag{3.5}$$

$$L_k u = \sum_{i=0}^{m_k} \bigl[\alpha_{ki} u^{(i)}(0) + \beta_{ki} u^{(i)}(1)\bigr] = f_k, \quad k = 1,2. \tag{3.6}$$

Let ω_i, $i = 1,2$ be roots of equations $a\omega^2 = 1$. We put $\alpha_k = \alpha_{km_k}$, $\beta_k = \beta_{km_k}$ and

$$\eta_k = \begin{vmatrix} \alpha_1(-\omega_1)^{m_1} & \beta_1 \omega_1^{m_1} \\ \alpha_2(-\omega_1)^{m_{k2}} & \beta_2 \omega_2^{m_2} \end{vmatrix}.$$

Condition 3.2 Assume the following conditions are satisfied:

(1) $a \neq 0$, $a \in S(\varphi_0) \cap \mathbb{C}/\mathbb{R}_+$, for $\varphi + \varphi_0 < \pi$, $p \in (1,\infty)$;
(2) $\eta = (-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \neq 0$, $|\alpha_k| + |\beta_k| > 0$;
(3) A is a R-positive operator in a UMD-space E, m is a nonnegative integer.

Theorem 3.2 *Let Condition* 3.2 *hold. Then, problem* (3.5)-(3.6) *has a unique solution* $u \in W^{m+2,p}(0,1;E(A),E)$ *for* $f_k \in E_k$, $\lambda \in S_\psi$ *with sufficiently large* $|\lambda|$ *and the following coercive uniform estimate holds*:

$$\sum_{i=0}^{m+2} |\lambda|^{1-\frac{i}{m+2}} \bigl\|u^{(i)}\bigr\|_{L^p(0,1;E)} + \|Au\|_{L^p(0,1;E)} \le M \sum_{k=1}^{2} \bigl(\|f_k\|_{E_k} + |\lambda|^{1-\theta_k}\|f_k\|_E\bigr). \tag{3.7}$$

Proof In a similar way as in [10, Theorem 3.1] we obtain the representation of the solution of (3.4):

$$\begin{aligned} u(x) &= \bigl\{e^{-xA_\lambda^{\frac{1}{2}}}\bigl[C_{11} + d_{11}(\lambda)\bigr] + e^{-(1-x)A_\lambda^{\frac{1}{2}}}\bigl[C_{12} + d_{12}(\lambda)\bigr]\bigr\} A_\lambda^{-\frac{m_1}{2}} f_1 \\ &\quad + \bigl\{e^{-xA_\lambda^{\frac{1}{2}}}\bigl[C_{21} + d_{21}(\lambda)\bigr] + e^{-(1-x)A_\lambda^{\frac{1}{2}}}\bigl[C_{22} + d_{22}(\lambda)\bigr]\bigr\} A_\lambda^{-\frac{m_2}{2}} f_2, \end{aligned} \tag{3.8}$$

where C_{ij} and d_{ij} are uniformly bounded operators. Then in view of the positivity of A we obtain from (3.8)

$$\begin{aligned} &\sum_{i=0}^{m+2} |\lambda|^{1-\frac{i}{m+2}} \bigl\|u^{(i)}\bigr\|_{L^p(0,1;E)} + \|Au\|_{L^p(0,1;E)} \\ &\quad \le C \sum_{k=1}^{2} \Biggl[\sum_{i=0}^{m+2} |\lambda|^{1-\frac{i}{m+2}} \bigl\|A_\lambda^{-\frac{(m_k-i)}{2}} \bigl[e^{-(1-x)A_\lambda^{\frac{1}{2}}} + e^{-xA_\lambda^{\frac{1}{2}}}\bigr] f_k\bigr\|_{L^p(0,1;E)} \end{aligned}$$

$$+ \left\| A A_{\lambda}^{-\frac{m_k}{2}} e^{-x A_{\lambda}^{\frac{1}{2}}} f_k \right\|_{L^p(0,1;E)} \Bigg]$$

$$\leq C, \tag{3.9}$$

$$\sum_{k=1}^{2} \sum_{i=0}^{m+2} |\lambda|^{1-\frac{i}{m+2}} \left\| A_{\lambda}^{-(1-\frac{m m_k}{2(m+2)}+\frac{i}{2})} \right\| \left\| A_{\lambda}^{1-\frac{m_k}{m+2}} e^{-x A_{\lambda}^{\frac{1}{2}}} f_k \right\|_{L^p(0,1;E)}$$

$$+ \left\| A A_{\lambda}^{-\frac{m_k}{m+2}} \left[e^{-x A_{\lambda}^{\frac{1}{2}}} + e^{-(1-x) A_{\lambda}^{\frac{1}{2}}} \right] f_k \right\|_{L^p(0,1;E)}$$

$$\leq C \sum_{k=1}^{2} \left[\left(\int_0^1 \left\| A_{\lambda}^{1-\frac{m_k}{m+2}} \left[e^{-x A_{\lambda}^{\frac{1}{2}}} + e^{-(1-x) A_{\lambda}^{\frac{1}{2}}} \right] f_k \right\|^p dx \right)^{\frac{1}{p}} \right.$$

$$\left. + \left(\int_0^1 \left\| A A_{\lambda}^{-\frac{m_k}{m+2}} \left[e^{-x A_{\lambda}^{\frac{1}{2}}} + e^{-(1-x) A_{\lambda}^{\frac{1}{2}}} \right] f_k \right\|^p dx \right)^{\frac{1}{p}} \right]. \tag{3.10}$$

By virtue of Theorem A_4 we obtain

$$\left(\int_0^1 \left\| A_{\lambda}^{-\frac{m_k}{m+2}} \left[e^{-x A_{\lambda}^{\frac{1}{2}}} + e^{-(1-x) A_{\lambda}^{\frac{1}{2}}} \right] f_k \right\|^p dx \right)^{\frac{1}{p}} \leq M_1 \sum_{k=1}^{2} \left[\|f_k\|_{E_k} + |\lambda|^{1-\theta_k} \|f_k\| \right]. \tag{3.11}$$

Moreover, due to the positivity of the operator A and the estimate (3.11), in view of Theorem A_4 we get the uniform estimate

$$\sum_{k=1}^{2} \left(\int_0^1 \left\| A A_{\lambda}^{-\frac{m_k}{m+2}} \left[e^{-x A_{\lambda}^{\frac{1}{2}}} + e^{-(1-x) A_{\lambda}^{\frac{1}{2}}} \right] f_k \right\|^p dx \right)^{\frac{1}{p}} \leq M_2 \sum_{k=1}^{2} \left[\|f_k\|_{E_k} + |\lambda|^{1-\theta_k} \|f_k\| \right]. \tag{3.12}$$

Hence, from (3.9)-(3.12) we obtain (3.7). □

Theorem 3.3 *Assume Condition* 3.2 *to hold. Then the operator* $u \to \{(L+\lambda)u, L_1 u, L_2 u\}$ *for* $\lambda \in S_{\psi,\varkappa}$ *and for sufficiently large* $\varkappa > 0$ *is an isomorphism from*

$$W^{m+2,p}\big(0,1;E(A),E\big) \text{ onto } W^{m,p}(0,1;E) \times E_1 \times E_2.$$

Moreover, the following uniform coercive estimate holds:

$$\sum_{i=0}^{m+2} |\lambda|^{1-\frac{i}{m+2}} \left\| u^{(i)} \right\|_{L^p(0,1;E)} + \|Au\|_{L^p(0,1;E)}$$

$$\leq C \left[\|f\|_{W^{m,p}(0,1;E)} + \sum_{k=1}^{2} \left(\|f_k\|_{E_k} + |\lambda|^{1-\theta_k} \|f_k\|_E \right) \right]. \tag{3.13}$$

Proof The uniqueness of solution of problem (3.4) is obtained from Theorem 3.3. Let us define

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in [0,1], \\ 0 & \text{if } x \notin [0,1]. \end{cases}$$

We now show that problem (3.4) has a solution $u \in W^{m+2,p}(0,1;E(A),E)$ for all $f \in W^{m,p}(0,1;E), f_k \in E_k$ and $u = u_1 + u_2$, where u_1 is the restriction on $[0,1]$ of the solution of the equation

$$(L+\lambda)u=\bar{f}(x), \quad x\in R=(-\infty,\infty) \tag{3.14}$$

and u_2 is a solution of the problem

$$(L+\lambda)u=0, \qquad L_k u=f_k-L_k u_1. \tag{3.15}$$

A solution of (3.14) is given by

$$u(x)=F^{-1}L^{-1}(\lambda,\xi)F\bar{f}=\frac{1}{2\pi}\int_{\infty}^{\infty}e^{i\xi x}L^{-1}(\lambda,\xi)(F\bar{f})(\xi)\,d\xi,$$

where

$$L(\lambda,\xi)=A-a\xi^2+\lambda.$$

It follows from the above expression that

$$\begin{aligned}&\sum_{i=0}^{m+2}|\lambda|^{1-\frac{i}{m+2}}\left\|u^{(i)}\right\|_{L^p(R;E)}+\|Au\|_{L^p(R;E)}\\&\quad=\sum_{i=0}^{m+2}|\lambda|^{1-\frac{i}{m+2}}\left\|F^{-1}\xi^iL^{-1}(\lambda,\xi)F\bar{f}\right\|_{L^p(R;E)}+\left\|F^{-1}AL^{-1}(\lambda,\xi)F\bar{f}\right\|_{L^p(R;E)}.\end{aligned} \tag{3.16}$$

It is sufficient to show that the operator functions

$$\Psi_\lambda(\xi)=AL^{-1}(\lambda,\xi)\left(1+\xi^m\right)^{-1}, \qquad \sigma_\lambda(\xi)=\sum_{i=0}^{m+2}|\lambda|^{1-\frac{i}{m+2}}\xi^iL^{-1}(\lambda,\xi)\left(1+\xi^m\right)^{-1}$$

are Fourier multipliers in $L^p(R;E)$ uniformly in λ. Actually, due to $\varphi+\varphi_0<\pi$ and the positivity of A we have

$$\begin{aligned}&\left\|L^{-1}(\lambda,\xi)\right\|\le M\left(1+|a|\xi^2+|\lambda|\right)^{-1},\\&\left\|\Psi_\lambda(\xi)\right\|=\left\|A\left[A+\lambda+|a|\xi^2\right]^{-1}\right\|\le C_1.\end{aligned} \tag{3.17}$$

It is clear that

$$\xi\frac{d}{d\xi}\Psi_\lambda(\xi)=2a\xi^2AL^{-2}(\lambda,\xi)=\left[2a\xi^2L^{-1}(\lambda,\xi)\right]AL^{-1}(\lambda,\xi).$$

In a similar way we see that the sets

$$\left\{2a\xi^2\left[A-a\xi^2+\lambda\right]^{-1}:\xi\in R\backslash\{0\}\right\}, \qquad \left\{A\left[A-a\xi^2+\lambda\right]^{-1}:\xi\in R\backslash\{0\}\right\}$$

are R-bounded. Then, in view of Kahane's contraction principle and from the product properties of the collection of R-bounded operators (see *e.g.* [3], Lemma 3.5, Proposition 3.4) we obtain

$$R\left\{\xi\frac{d}{d\xi}\Psi_\lambda(\xi):\xi\in R\backslash\{0\}\right\}\le C.$$

Namely, the R-bound of the set $\{\xi \frac{d}{d\xi}\Psi_\lambda(\xi) : \xi \in R\backslash\{0\}\}$ is independent of λ. Moreover, it is clear that

$$\left\|\sigma_\lambda(\xi)\right\|_{B(E)} \le C|\lambda| \sum_{i=0}^{m+2} \left[|\xi||\lambda|^{-\frac{1}{m+2}}\right]^i \left(1+\xi^m\right)^{-1} \left\|L^{-1}(\lambda,\xi)\right\|_{B(E)}.$$

Hence, by using the well-known inequality $y^i \le C(1+y^l)$, $y \ge 0$, $i \le l$ for $y = (|\lambda|^{-\frac{1}{2}}|\xi|)$ and $l = m+2$ we get the estimate

$$\left|\sum_{i=0}^{m+2} |\lambda|^{1-\frac{i}{m+2}}\xi^i\right| \le C|\lambda|\left(1+|\lambda|^{-1}|a|\xi^{m+2}\right). \tag{3.18}$$

From (3.17) and (3.18) we have the uniform estimate

$$\left\|\sigma_\lambda(\xi)\right\|_{B(E)} \le C.$$

Due to R-positivity of the operator A, the set

$$\left\{\left(|\lambda| - a\xi^2\right)L^{-1}(\lambda,\xi) : \xi \in R\backslash\{0\}\right\}$$

is R-bounded. Then, by estimate (3.17) and by Kahane's contraction principle we obtain the R-boundedness of the set $\{\sigma_\lambda(\xi) : \xi \in R\backslash\{0\}\}$. In a similar way we obtain the uniform estimates

$$\left\|\frac{d}{d\xi}\Psi_\lambda(\xi)\right\|_{B(E)} \le C_1, \qquad \left\|\frac{d}{d\xi}\sigma_\lambda(\xi)\right\|_{B(E)} \le C_2.$$

Consider the set

$$\sigma_1(\lambda,\xi) = \left\{\xi\frac{d}{d\xi}\sigma_\lambda(\xi) : \xi \in R\backslash\{0\}\right\},$$
$$\Psi_1(\lambda,\xi) = \left\{\xi\frac{d}{d\xi}\Psi_\lambda(\xi) : \xi \in R\backslash\{0\}\right\}.$$

Due to the R-positivity of the operator A, in view of estimate (3.17), by virtue of Kahane's contraction principle, from the additional and product properties of the collection of R-bounded operators, for $\xi_1, \xi_2, \ldots, \xi_\mu \in R$, $u_1, u_2, \ldots, u_\mu \in E$, and the independent symmetric $\{-1,1\}$-valued random variables $r_j(y)$, $j = 1, 2, \ldots, \mu$, $\mu \in \mathbb{N}$ we obtain the uniform estimate

$$\begin{aligned}
&\int_\Omega \left\|\sum_{j=1}^{\mu} r_j(y)\sigma_1\left(\lambda,\xi^{(j)}\right)u_j\right\|_E dy \\
&\quad\le C\int_\Omega \left\|\sum_{j=1}^{\mu} \sigma_1\left(\lambda,\xi^{(j)}\right)r_j(y)u_j\right\|_E dy \\
&\quad\le C\sup_\lambda R\left(\left\{\xi\frac{d}{d\xi}\sigma_\lambda(\xi) : \xi \in R\backslash\{0\}\right\}\right)\int_\Omega \left\|\sum_{j=1}^{\mu} r_j(y)u_j\right\|_E dy \le C.
\end{aligned}$$

In a similar way, the above estimate is obtained for Ψ_1. So, by [21, Theorem 3.4] it follows that $\Psi_\lambda(\xi)$ and $\sigma_\lambda(\xi)$ are the uniform collection of multipliers in $L^p(R;E)$. Then, by using the equality (3.16) we see that problem (3.14) has a solution $u \in W^{m+2,p}(R;E(A),E)$ and the following uniform estimate holds:

$$\sum_{i=0}^{m+2} |\lambda|^{1-\frac{i}{m+2}} \left\| u^{(i)} \right\|_{L^p(R;E)} + \|Au\|_{L^p(R;E)} \le C\|\bar{f}\|_{L^p(R;E)}. \tag{3.19}$$

Let u_1 be the restriction of u on $(0,1)$. Then the estimate (3.17) implies that $u_1 \in W^{m+2,p}(0,1;E(A),E)$. By virtue of the trace theorem (see *e.g.* [25, §1.8.2]) we get

$$u_1^{(m_k)}(\cdot) \in \left(E(A);E\right)_{\theta_k,p}, \quad k = 1,2.$$

Hence, $L_k u_1 \in E_k$. Thus, by virtue of Theorem 3.2, problem (3.15) has a unique solution $u_2(x)$ that belongs to the space $W^{m+2,p}(0,1;E(A),E)$. Moreover, we have

$$\begin{aligned}
&\sum_{i=0}^{m+2} |\lambda|^{1-\frac{i}{m+2}} \left\| u_2^{(i)} \right\|_{L^p(0,1;E)} + \|Au_2\|_{L^p(0,1;E)} \\
&\quad \le C\sum_{k=1}^{2} \left[\|f_k\|_{E_k} + |\lambda|^{1-\theta_k} \|f_k\|_E + \left\| u_1^{(m_k)} \right\|_{C([0,1];E_k)} + |\lambda|^{1-\theta_k} \|u_1\|_{C([0,1];E)} \right].
\end{aligned} \tag{3.20}$$

From (3.19) we obtain

$$\sum_{i=0}^{m+2} |\lambda|^{1-\frac{i}{m+2}} \left\| u_1^{(i)} \right\|_{L^p(0,1;E)} + \|Au_1\|_{L^p(0,1;E)} \le C\|f\|_{W^{m,p}(0,1;E)}. \tag{3.21}$$

Therefore, by Theorem A_3 and by estimate (3.21) we obtain

$$\begin{aligned}
\left\| u_1^{(m_k)}(\cdot) \right\|_{E_k} &\le C\left(\left\| u_1^{(m+2)} \right\|_{L^p(0,1;E)} + \|Au_1\|_{L^p(0,1;E)} \right) \\
&\le C\|f\|_{W^{m,p}(0,1;E)}.
\end{aligned} \tag{3.22}$$

Hence, in view of Theorem 3.2 and estimates (3.20)-(3.22) we get

$$\begin{aligned}
&\sum_{i=0}^{m+2} |\lambda|^{1-\frac{i}{m+2}} \left\| u_2^{(i)} \right\|_{L^p(0,1;E)} + \|Au_2\|_{L^p(0,1;E)} \\
&\quad \le C\left(\|f\|_{L^p(0,1;E)} + \sum_{k=1}^{2} \left(\|f_k\|_{E_k} + |\lambda|^{1-\theta_k} \|f_k\|_E \right) \right).
\end{aligned} \tag{3.23}$$

Finally, from (3.21) and (3.23) we obtain (3.13). □

Now, by using of Theorems 3.2, 3.3 we can prove the main result of this section.

Proof of Theorem 3.1 Let $G_2 = (0,b_1) \times (0,b_2)$. It is clear that

$$W^{m,p}(G_2;E) = W^{m,p}(0,b_1;X_0,X) = W^{m,p}(0,b_1;X) \cap L^p(0,b_1;X_0),$$

where

$$X_0 = W^{m,p}(0,b_2;E), \qquad X = L^p(0,b_2;E).$$

Let us consider the BVP

$$a_1 \frac{\partial^2 u}{\partial x_1^2} + a_2 \frac{\partial^2 u}{\partial x_2^2} + (A+\lambda)u(x_1,x_2) = f(x_1,x_2), \qquad L_{kj}u = 0, \quad k,j = 1,2, \tag{3.24}$$

where L_{kj} are defined by equalities (3.6). Problem (3.24) can be expressed as the following BVP for the ordinary DOE:

$$Lu = a_1 \frac{d^2 u}{dx_1^2} + (B_2 + \lambda)u(x_1) = f(x_1), \quad x_1 \in (0,b_1), \qquad L_{k1}u = 0, \tag{3.25}$$

where B_2 is the operator in X defined by

$$B_2 u = a_2 \frac{d^2 u}{dx_2^2} + Au(x_2), \qquad D(B_{\varepsilon_2}) = W^{2,p}\big(0,b_2;E(A),E,L_{2k}\big),$$

respectively. Since X_0 and X are UMD-spaces (see *e.g.* [1, Theorem 4.5.2]), by virtue of Theorem 3.3 we obtain the result that problem (3.25) has a unique solution $u \in W^{2,p}(0,b_1;D(B_2),X_0)$, $u \in W^{m+2,p}(0,b_1;D(B_2),X)$ for $f \in L^p(0,b_1;X_0)$ and $f \in W^{m,p}(0,b_1;X)$, respectively. Moreover, for $\lambda \in S_{\psi,\varkappa}$ and sufficiently large $\varkappa > 0$ the following coercive uniform estimates hold:

$$\begin{aligned} &\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \big\| u^{(i)} \big\|_{L^p(0,b_1;X_0)} + \|B_2 u\|_{L^p(0,b_1;X_0)} \leq C\|f\|_{L^p(0,b_1;X_0)}, \\ &\sum_{i=0}^{m+2} |\lambda|^{1-\frac{i}{m+2}} \big\| u^{(i)} \big\|_{L^p(0,b_1;X)} + \|B_2 u\|_{L^p(0,b_1;X)} \leq C\|f\|_{W^{m,p}(0,b_1;X)}. \end{aligned} \tag{3.26}$$

From (3.26) we find that problem (3.24) has a unique solution,

$$u \in W^{m+2,p}\big(G_2;E(A),E\big) \quad \text{for } W^{m,p}(G_2;E,)$$

and the following uniform coercive estimates hold:

$$\sum_{i=0}^{m+2} |\lambda|^{1-\frac{i}{m+2}} \big\| u^{(i)} \big\|_{L^p(0,b_1;X)} + \|B_2 u\|_{L^p(0,b_1;X)} \leq C\|f\|_{W^{m,p}(G_2;E)}. \tag{3.27}$$

By applying Theorem 3.3, for $f_k = 0$ and $E = X$ we get the following uniform estimate:

$$\sum_{j=0}^{m+2} |\lambda|^{1-\frac{i}{m+2}} \big\| u^{(i)} \big\|_X + \|Au\|_X \leq C\|B_2 u\|_{W^{m,p}(0,b_2;E)}. \tag{3.28}$$

From estimates (3.27)-(3.28) we obtain the assertion for problem (3.24). Then by continuing this process n times we obtain the conclusion. □

4 Boundary value problems for abstract elliptic equations with variable coefficients

Consider the BVP for DOE with variable coefficients

$$\sum_{k=1}^{n} a_k(x)\frac{\partial^2 u}{\partial x_k^2} + \big(A(x)+\lambda\big)u + \sum_{k=1}^{n} A_k(x)\frac{\partial u}{\partial x_k} = f(x), \quad x \in G, \tag{4.1}$$
$$\sum_{i=0}^{m_{kj}} \left[\alpha_{kji}\frac{\partial^i u}{\partial x_k^i}(G_{k0}) + \beta_{kji}\frac{\partial^i u}{\partial x_k^i}(G_{kb})\right] = 0, \quad x^{(k)} \in G_k, j = 1, 2,$$

where G, G_k, G_{k0}, G_{kb}, $x^{(k)}$ are defined as in (3.1)-(3.2), $a_k(x)$ are complex-valued continuous functions, $A(x)$ and $A_k(x)$ are linear operators in a Banach space E for $x \in G$, and $u(x)$ and $f(x)$, respectively, are E-valued unknown and data functions. We will derive in this section the maximal regularity properties of problem (4.1).

Nonlocal BVPs for DOEs investigated, *e.g.*, in [2, 4, 13–19, 22, 28]. Let $\alpha_{kj} = \alpha_{kjm_k}$ and $\beta_{kj} = \beta_{kjm_k}$. Let $\omega_{ki} = \omega_{ki}(x)$, $i = 1, 2$, be roots of the equations

$$a_k(x)\omega^2 = 1, \quad k = 1, 2, \ldots, n$$

and

$$\eta_k(x) = \begin{vmatrix} \alpha_{k1}(-\omega_{k1})^{m_{k1}} & \beta_{k1}\omega_{k1}^{m_{k1}} \\ \alpha_{k2}(-\omega_{k1})^{m_{k2}} & \beta_{k2}\omega_{k2}^{m_{k2}} \end{vmatrix}.$$

Condition 4.1 Assume:

(1) E is a UMD-space and $A(x)$ is a uniformly R-positive operator in E for $\varphi \in [0, \pi)$;

(2) $a_k(x) \in C^{(m)}(\bar{G})$, $a_k(G_{i0}) = a_k(G_{ib})$, $a_k \neq 0$, $a_k \in S(\varphi_0) \cap \mathbb{C}/\mathbb{R}_+$ for all $x \in G$, $\varphi + \varphi_0 < \pi$;

(3) $A(x)A^{-1}(\bar{x}) \in C^{(m)}(\bar{G}; L(E))$, $A(G_{k0}) = A(G_{kb})$, $A_k(x)A^{-(\frac{1}{2}-\nu)}(x) \in C^{(m)}(\bar{G}; L(E))$, $0 < \nu < \frac{1}{2}$, $m \in \mathbb{N}$;

(4) $|\alpha_{kjm_j}| + |\beta_{kjm_j}| > 0$, $\eta_k(x) \neq 0$, $k, i = 1, 2, \ldots, n$, $j = 1, 2$, $p \in (1, \infty)$.

Remark 4.1 Let $l_k = 2m_k$, $k = 1, 2, \ldots, n$ and $a_k = (-1)^{m_k} b_k(x)$, where b_k are real-valued positive functions and m_k are natural numbers. Then Condition 4.1 is satisfied.

Remark 4.2 The periodicity conditions are given due to nonlocality of boundary conditions. For local boundary conditions these assumptions are not required.

Let $X = W^{m,p}(G; E)$ and $Y = W^{m+2,p}(G; E(A), E)$. Consider the operator O in X generated by problem (4.1), *i.e.*,

$$D(O) = W^{m+2,p}\big(G; E(A), E, L_{kj}\big), \qquad Ou = \sum_{k=1}^{n} a_k(x)\frac{\partial^2 u}{\partial x^2} + A(x) + \sum_{k=1}^{n} A_k(x)\frac{\partial u}{\partial x_k}.$$

The main result is the following.

Theorem 4.1 *Assume Condition* 4.1 *is satisfied. Then problem* (4.1) *has a unique solution* $u \in Y$ *for* $f \in X$, $\lambda \in S_{\psi,\varkappa}$ *and the following uniform coercive estimate holds*:

$$\sum_{k=1}^{n}\sum_{i=0}^{m+2}|\lambda|^{1-\frac{i}{m+2}}\left\|\frac{\partial^i u}{\partial x_k^i}\right\|_{L^p(G;E)} + \|Au\|_{L^p(G;E)} \leq C\|f\|_X. \tag{4.2}$$

Proof First we will show the uniqueness of solution. For this aim we use microlocal analysis. Let $D_1, D_2, \dots, D_N$ be rectangular regions with sides parallel to the coordinate planes covering G and let $\varphi_1, \varphi_2, \dots, \varphi_N$ be a corresponding partition of unity, *i.e.*, $\varphi_j \in C_0^\infty(G)$, $\sigma_j = \operatorname{supp}\varphi_j \subset D_j$ and $\sum_{j=1}^N \varphi_j(x) = 1$, where $C_0^\infty(G)$ denotes the space of all infinitely differentiable functions on G with compact support. Now for $u \in W^{2+m,p}(G; E(A), E, L_{ki})$ being a solution of (4.1) and $u_j(x) = u(x)\varphi_j(x)$ we get

$$(L+\lambda)u_j = \sum_{k=1}^n a_k(x)\frac{\partial^2 u_j}{\partial x_k^2} + \bigl(A(x)+\lambda\bigr)u_j(x) = f_j(x), \qquad L_{ki}u_j = 0, \quad i = 1,2, \tag{4.3}$$

where

$$\begin{aligned} f_j(x) = f(x)\varphi_j(x) &+ \sum_{k=1}^n a_k(x)\left[C_0^1 u\frac{\partial\varphi_j}{\partial x_k} + C_1^1\varphi_j\frac{\partial u}{\partial x_k}\right] \\ &- \sum_{k=1}^n \varphi_j(x)A_k(x)\frac{\partial u}{\partial x_k}, \quad j = 1,2,\dots,N. \end{aligned} \tag{4.4}$$

Freezing the coefficients of (4.3), extending $u_j(x)$ outside of σ_j up to D_j, we obtain the BVP

$$\begin{aligned} &\sum_{k=1}^n a_k(x_{0j})\frac{\partial^2 u_j}{\partial x_k^2} + \bigl(A(x_{0j})+\lambda\bigr)u_j(x) = F_j(x), \quad x \in D_j, \\ &L_{ki}u_j = 0, \quad i = 1,2, k = 1,2,\dots,n, \end{aligned} \tag{4.5}$$

where

$$F_j = f_j + \bigl[A(x_{0j}) - A(x)\bigr]u_j + \sum_{k=1}^n \bigl[a_k(x_{0j}) - a(x)\bigr]\frac{\partial^2 u_j}{\partial x_k^2}, \tag{4.6}$$

and C_i^1 are the usual binomial coefficients. It is clear that $F_j \in W^{m,p}(D_j; E) = X_j$. By applying Theorem 3.1 we obtain the following *a priori* estimate:

$$\sum_{k=1}^{n}\sum_{i=0}^{m+2}|\lambda|^{1-\frac{i}{m+2}}\left\|\frac{\partial^i u_j}{\partial x_k^i}\right\|_{L^p(D_j;E)} + \|Au_j\|_{L^p(D_j;E)} \leq C\|F_j\|_{X_j} \tag{4.7}$$

for the solution $u \in Y_j = W^{2+m,p}(D_j; E(A), E)$ of (4.5) on the domains D_j containing the boundary points. In a similar way we obtain the same estimates for the domains $D_j \subset G$. In view of F_j, by Theorem A_1, in view of the continuity of coefficients, choosing diameters of $\operatorname{supp}\varphi_j$ sufficiently small we see that for all small δ there is a positive continuous function $C(\delta)$ so that

$$\|F_j\|_{X_j} \leq \|f\cdot\varphi_j\|_{X_j} + \delta\|u_j\|_{Y_j} + C(\delta)\|u_j\|_{X_j}. \tag{4.8}$$

Consequently, from (4.6)-(4.8) we have

$$\begin{aligned}&\sum_{k=1}^{n}\sum_{i=0}^{m+2}|\lambda|^{1-\frac{i}{m+2}}\left\|\frac{\partial^i u_j}{\partial x_k^i}\right\|_{L^p(D_j;E)}+\|Au_j\|_{L^p(D_j;E)}\\&\quad\leq C\|f\|_{X_j}+\delta\|u_j\|_{Y_j}+M(\delta)\|u_j\|_{X_j}.\end{aligned}\tag{4.9}$$

Choosing $\varepsilon_k<1$ from (4.9) we obtain

$$\sum_{k=1}^{n}\sum_{i=0}^{m+2}|\lambda|^{1-\frac{i}{m+2}}\left\|\frac{\partial^i u_j}{\partial x_k^i}\right\|_{L^p(D_j;E)}+\|Au_j\|_{L^p(D_j;E)}\leq C\bigl[\|f\|_{X_j}+\|u_j\|_{X_j}\bigr].\tag{4.10}$$

Since $u(x)=\sum_{j=1}^{N}u_j(x)$ and by (4.10) we find that the solution of (4.1) satisfies the estimate (4.2). It is clear that

$$\|u\|_X=\frac{1}{|\lambda|}\bigl\|(O+\lambda)u-Ou\bigr\|_X\leq\frac{1}{|\lambda|}\bigl[\bigl\|(O+\lambda)u\bigr\|_X+\|Ou\|_X\bigr].$$

Hence, by using the definition of Y and applying Theorem A_1 we obtain

$$\|u\|_X\leq\frac{C}{|\lambda|}\bigl[\bigl\|(O+\lambda)u\bigr\|_X+\|u\|_Y\bigr].$$

From the above estimate we have

$$\sum_{k=1}^{n}\sum_{i=0}^{m+2}|\lambda|^{1-\frac{i}{m+2}}\left\|\frac{\partial^i u}{\partial x_k^i}\right\|_{L^p(G;E)}+\|Au\|_{L^p(G;E)}\leq C\bigl\|(O+\lambda)u\bigr\|_X.\tag{4.11}$$

The estimate (4.11) implies that uniqueness of the solution of problem (4.1). It implies that the operator $O+\lambda$ has a bounded inverse in its rank space. We need to show that this rank space coincides with the space X, *i.e.*, we have to show that for all $f\in X$ there is a unique solution of problem (4.1). We consider the smooth functions $g_j=g_j(x)$ with respect to φ_j on D_j that equal 1 on supp φ_j, where supp $g_j\subset D_j$ and $|g_j(x)|<1$. Let us construct for all j the functions u_j that are defined on the regions $\Omega_j=G\cap D_j$ and satisfying problem (4.1). Problem (4.1) can be expressed as

$$\begin{aligned}&\sum_{k=1}^{n}a_k(x_{0j})\frac{\partial^2 u_j}{\partial x_k^2}+\bigl(A(x_{0j})+\lambda\bigr)u_j(x)\\&\quad=g_j\left\{f+\bigl[A(x_{0j})-A(x)\bigr]u_j+\sum_{k=1}^{n}\bigl[a_k(x_{0j})-a(x)\bigr]\frac{\partial^2 u_j}{\partial x_k^2}+\sum_{k=1}^{n}A_k(x)\frac{\partial u_j}{\partial x_k}\right\},\end{aligned}\tag{4.12}$$

$$L_{ki}u_j=0,\quad i=1,2,k=1,2,\ldots,n,x\in D_j.$$

Consider operators $O_{j\lambda}=O_j+\lambda$ in X_j that are generated by the problem

$$\sum_{k=1}^{n}a_k(x_{0j})\frac{\partial^2 u}{\partial x_k^2}+\bigl(A(x_{0j})+\lambda\bigr)u(x)=f(x),\quad x\in D_j,\tag{4.13}$$

$$L_{ki}u=0,\quad i=1,2.$$

By virtue of Theorem 3.1, the local operators $O_{j\lambda}$ have bounded inverses $O_{j\lambda}^{-1}$ from X_j to Y_j and for all $f \in X_j$ we have the following uniform estimate:

$$\sum_{k=1}^{n} \sum_{i=0}^{m+2} |\lambda|^{1-\frac{i}{m+2}} \left\| \frac{\partial^i}{\partial x_k^i} O_{j\lambda}^{-1} f \right\|_{L^p(D_j;E)} + \left\| A O_{j\lambda}^{-1} f \right\|_{L^p(D_j;E)} \leq C \|f\|_{X_j}. \tag{4.14}$$

Extending the solutions u_j of (4.13) to zero on the outside of σ_j and using the substitutions $u_j = O_{j\lambda}^{-1} \upsilon_j$ we obtain the equations

$$\upsilon_j = K_{j\lambda} \upsilon_j + g_j f, \quad j = 1, 2, \ldots, N, \tag{4.15}$$

where $K_{j\lambda} = K_{j\lambda}(\varepsilon)$ are bounded linear operators in X_j defined by

$$\begin{aligned} K_{j\lambda} = g_j \Bigg\{ & f + \big[A(x_{0j}) - A(x)\big] O_{j\lambda}^{-1} \\ & + \sum_{k=1}^{n} \big[a_k(x_{0j}) - a_k(x)\big] \frac{\partial^2}{\partial x_k^2} O_{j\lambda}^{-1} - \sum_{k=1}^{n} A_k(x) \frac{\partial}{\partial x_k} D^\alpha O_{j\lambda}^{-1} \Bigg\}. \end{aligned}$$

In fact, due to the smoothness of the coefficients of the expression $K_{j\lambda}$ and in view of the estimate (4.14) for sufficiently large $|\lambda|$ there is a sufficiently small $\delta > 0$ such that

$$\begin{aligned} & \big\| \big[A(x_{0j}) - A(x)\big] O_{j\lambda}^{-1} \upsilon_j \big\|_{L^p(D_j;E)} \leq \delta \|\upsilon_j\|_{L^p(D_j;E)}, \\ & \sum_{k=1}^{n} \left\| \big[a_k(x_{0j}) - a_k(x)\big] \frac{\partial^2}{\partial x_k^2} O_{j\lambda}^{-1} \upsilon_j \right\|_{L^p(D_j;E)} \leq \delta \|\upsilon_j\|_{L^p(D_j;E)}. \end{aligned}$$

Moreover, by Theorem A_1 we find that for all $\delta > 0$ there is a constant $C(\delta) > 0$ such that

$$\sum_{k=1}^{n} \sum_{i=0}^{m+2} |\lambda|^{1-\frac{i}{m+2}} \left\| \sum_{k=1}^{n} A_k(x) \frac{\partial}{\partial x_k} O_{j\lambda}^{-1} \upsilon_j \right\|_{L^p(D_j;E)} \leq \delta \|\upsilon_j\|_{Y_j} + C(\delta) \|\upsilon_j\|_{X_j}.$$

Hence, for $|\arg \lambda| \leq \varphi$ with sufficiently large $|\lambda|$ there is a $\gamma \in (0,1)$ such that $\|K_{j\lambda}\| < \gamma$. Consequently, (4.15) for all j have a unique solution $\upsilon_j = [I - K_{j\lambda}]^{-1} g_j f$. Moreover,

$$\|\upsilon_j\|_{X_j} = \big\| [I - K_{j\lambda}]^{-1} g_j f \big\|_{X_j} \leq \|f\|_{X_j}.$$

Thus, $[I - K_{j\lambda}]^{-1} g_j$ are bounded linear operators from X to X_j. Thus, the functions $u_j = O_{j\lambda}^{-1} [I - K_{j\lambda}]^{-1} g_j f$ are solutions of (4.12). Consider the linear operator U in $L^p(G;E)$ defined by

$$\begin{aligned} & D(U) = W^{m+2,p}\big(G; E(A), E, L_{kj}\big), \quad j = 1, 2, k = 1, 2, \ldots, n, \\ & Uf = \sum_{j=1}^{N} \varphi_j(y) U_{j\lambda} f = O_{j\lambda}^{-1} [I - K_{j\lambda}]^{-1} g_j f, \quad j = 1, 2, \ldots, N. \end{aligned}$$

It is clear from the constructions $U_{j\lambda}$ and from the estimate (4.14) that the operators $U_{j\lambda}$ are bounded linear from X to Y_j and for $|\arg\lambda| \leq \varphi$ and sufficiently large $|\lambda|$ we have

$$\sum_{k=1}^{n}\sum_{i=0}^{m+2}|\lambda|^{1-\frac{i}{m+2}}\left\|\frac{\partial^i}{\partial x_k^i}U_{j\lambda}f\right\|_{L^p(D_j;E)} + \|AU_{j\lambda}f\|_{L^p(D_j;E)} \leq C\|f\|_X. \tag{4.16}$$

Therefore, U is a bounded linear operator in X. By the construction of the solution operators $U_{j\lambda}$ of the local problems (4.12), we get

$$\begin{aligned}(O+\lambda)u &= \sum_{j=1}^{N}(O+\lambda)(\varphi_j U_{j\lambda}f)\\ &= \sum_{j=1}^{N}\bigl[\varphi_j(O+\lambda)(U_{j\lambda}f) + \Phi_{j\lambda}f\bigr] = \sum_{j=1}^{N}\varphi_j g_j f + \sum_{j=1}^{N}\Phi_{j\lambda}f = f + \sum_{j=1}^{N}\Phi_{j\lambda}f,\end{aligned}$$

where $\Phi_{j\lambda}$ are bounded linear operators defined by

$$\begin{aligned}\Phi_{j\lambda}f &= \sum_{k=1}^{n}a_k(x)\left[C_0^1 U_{j\lambda}f\frac{\partial\varphi_j}{\partial x_k} + C_1^1\varphi_j\frac{\partial U_{j\lambda}f}{\partial x_k}\right]\\ &\quad + \sum_{k=1}^{n}A_k(x)\left[C_0^1 U_{j\lambda}f\frac{\partial\varphi_j}{\partial x_k} + C_1^1\varphi_j\frac{\partial U_{j\lambda}f}{\partial x_k}\right], \quad j = 1,2,\ldots,N.\end{aligned} \tag{4.17}$$

Indeed, by Theorem A_1, estimate (4.16), and from the expression $\Phi_{j\lambda}$ we find that the operators $\Phi_{j\lambda}$ are bounded linear from X to X and for $|\arg\lambda| \leq \varphi$ with sufficiently large $|\lambda|$ there is an $\delta \in (0,1)$ such that $\|\Phi_{j\lambda}\| < \delta$. Therefore, there exists a bounded linear invertible operator $(I + \sum_{j=1}^{N}\Phi_{j\lambda})^{-1}$, *i.e.*, we infer for all $f \in X$ that the BVP (3.1) has a unique solution

$$u(x) = (O+\lambda)^{-1}f = \sum_{j=1}^{N}\varphi_j O_{j\lambda}^{-1}[I-K_{j\lambda}]^{-1}g_j\left(I + \sum_{j=1}^{N}\Phi_{j\lambda}\right)^{-1}f.$$

□

Let $B_p = L(X)$.

Remark 4.3 Theorem 4.1 implies that the resolvent $(O+\lambda)^{-1}$ satisfies the sharp uniform estimate

$$\sum_{k=1}^{n}\sum_{i=0}^{m+2}|\lambda|^{1-\frac{i}{m+2}}\left\|\frac{\partial^i}{\partial x_k^i}(O+\lambda)^{-1}\right\|_{B_p} + \left\|A(O+\lambda)^{-1}\right\|_{B_p} \leq C,$$

for $|\arg\lambda| \leq \varphi$ and $\varphi \in [0,\pi)$.

5 Abstract Cauchy problem for parabolic equation

Consider now the initial BVP for the following parabolic equation with variable coefficients, *i.e.*,

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{n}a_k(x)\frac{\partial^2 u}{\partial x_k^2} + \sum_{k=1}^{n}A_k(x)\frac{\partial u}{\partial x_k} + A(x)u + du = f(x,t), \tag{5.1}$$

$$\sum_{i=0}^{m_{kj}}\left[\alpha_{kji}\frac{\partial^i u}{\partial x_k^i}(G_{k0},t)+\beta_{kji}\frac{\partial^i u}{\partial x_k^i}(G_{kb},t)\right]=0, \tag{5.2}$$

$$u(x,0)=0, \quad t\in(0,T), x\in G, x^{(k)}\in G_k, j=1,2,$$

where $A(x)$ and $A_k(x)$ are linear operator functions in a Banach space E, a_k are complex-valued functions, λ is a complex parameter, $d>0$, $m_{kj}\in\{0,1\}$ and G, G_k, G_{k0}, G_{kb} are the domains defined in (3.1)-(3.2).

For $\mathbf{p}=(p,p_1)$, $G_+=(0,T)\times G$, $L^{\mathbf{p}}(G_+;E)$ will denote the space of all E-valued $\mathbf{p}$-summable functions with mixed norm (see *e.g.* [27]), *i.e.*, the space of all measurable functions f defined on G_+, for which

$$\|f\|_{L^{\mathbf{p}}(G_+;E)}=\left(\int_{R_+}\left(\int_G\|f(x,y)\|_E^p\,dx\right)^{\frac{p_1}{p}}dt\right)^{\frac{1}{p_1}}<\infty.$$

Analogously, $W^{m,\mathbf{p}}(G_+,E(A),E)$ denotes the Sobolev space with the corresponding mixed norm (see [27] for the scalar case).

In this section, we obtain the existence and uniqueness of the maximal regular solution of problem (5.1)-(5.2) in mixed $L^{\mathbf{p}}$ norms. Let O denote the differential operator in $L^p(G;E)$ generated by (4.1) for $\lambda=0$.

Theorem 5.1 *Let all conditions of Theorem* 4.1 *hold for* $m=0$ *and* $\varphi\in(\frac{\pi}{2},\pi)$. *Then*:
(a) *the operator O is an R-positive in $L^p(G;E)$*;
(b) *the operator O is a generator of an analytic semigroup.*

Proof In fact, by virtue of Theorem 4.1 we see that for $f\in L^p(G;E)$ the BVP (4.1) have a unique solution expressed in the form

$$u(x)=(O+\lambda)^{-1}f=\sum_{j=1}^N\varphi_jO_{j\lambda}^{-1}[I-K_{j\lambda}]^{-1}g_j\left(I+\sum_{j=1}^N\Phi_{j\lambda}\right)^{-1}f,$$

where $O_{j\lambda}=O_j+\lambda$ are local operators generated by BVPs with constant coefficients of type (3.1)-(3.2) and $K_{j\lambda}$ and $\Phi_{j\lambda}$ are uniformly bounded operators defined in the proof of Theorem 4.1. By virtue of [2, Theorem 5.1] the operators O_j are R-positive. Then by using the above representation and by virtue of Kahane's contraction principle, and the product and additional properties of the collection of R-bounded operators (see *e.g.* [3, Lemma 3.5, Proposition 3.4]) we obtain the assertions. □

Theorem 5.2 *Let all conditions of Theorem* 5.1 *hold*. *Then for $f\in L^{\mathbf{p}}(G_+;E)$ problem* (5.1)-(5.2) *has a unique solution $u\in W^{1,2,\mathbf{p}}(G_+;E(A),E)$ and for sufficiently large $d>0$ the following coercive estimate holds*:

$$\left\|\frac{\partial u}{\partial t}\right\|_{L^{\mathbf{p}}(G_+;E)}+\sum_{k=1}^n\left\|\frac{\partial^2 u}{\partial x_k^2}\right\|_{L^{\mathbf{p}}(G_+;E)}+\|Au\|_{L^{\mathbf{p}}(G_+;E)}\le C\|f\|_{L^{\mathbf{p}}(G_+;E)}.$$

Proof Problem (5.1)-(5.2) can be expressed as the following Cauchy problem:

$$\frac{du}{dt}+Ou(t)=f(t), \qquad u(0)=0. \tag{5.3}$$

Theorem 5.1 implies that the operator O is R-positive and also is a generator of an analytic semigroup in $F = L_p(G;E)$. Then by virtue of [23] or [21, Theorem 4.2] we see that for $f \in L^{p_1}((0,T);F)$ problem (5.3) has a unique solution $u \in W^{1,p_1}((0,T);D(O),F)$ and the following uniform estimate holds:

$$\left\| \frac{du}{dt} \right\|_{L^{p_1}(0,T;F)} + \|Ou\|_{L^{p_1}(0,T;F)} \leq C\|f\|_{L^{p_1}(0,T;F)}. \tag{5.4}$$

Since $L^{p_1}(0,T;F) = L^{\mathbf{p}}(G_+;E)$, by Theorem 4.1 we have $\|Ou\|_{L^{p_1}(R_+;F)} = D(O)$. This relation and the estimate (5.4) implies the assertion. □

6 Nonlinear abstract parabolic problem

Consider the following nonlinear parabolic problem:

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{n} a_k(x) \frac{\partial^2 u}{\partial x_k^2} + B(t,x,u)u = F(t,x,u,\nabla u), \tag{6.1}$$

$$L_{k1}u = \sum_{i=0}^{m_{k1}} \alpha_{ki} \frac{\partial^i u}{\partial x_k^i}(G_{k0},t) = 0, L_{k2}u = \sum_{i=0}^{m_{k2}} \beta_{ki} \frac{\partial^i u}{\partial x_k^i}(G_{kb},t) = 0, \tag{6.2}$$

$$u(x,0) = 0, \quad t \in (0,T), x \in G, x^{(k)} \in G_k,$$

where a_k are complex-valued functions, α_{ki}, β_{ki} are complex numbers, $m_k \in \{0,1\}$ and G, G_k, G_{k0}, G_{kb} are domains defined in (3.1)-(3.2).

Let $G_T = (0,T) \times G$. Moreover, we let

$$G_0 = \prod_{k=1}^{n}(0,b_{0k}), \qquad G = \prod_{k=1}^{n}(0,b_k), \quad b_k \in (0,b_{0k}], T \in (0,T_0),$$

$$G_k = (0,b_1) \times \cdots \times (0,b_{k-1}) \times (0,b_{k+1}) \times \cdots \times (0,b_n),$$

$$B_{ki} = \left(W^{2,p}\left(G_k,E(A),E\right),L^p(G_k;E)\right)_{\eta_i,p}, \quad \eta_i = \frac{i+\frac{1}{p}}{2},$$

$$B_0 = \prod_{k=1}^{n}\prod_{i=0}^{1} B_{ki}, \quad x^{(k)} = (x_1,x_2,\ldots,x_{k-1},x_{k+1},\ldots,x_n).$$

Remark 6.1 By virtue of [25, §1.8] the operators $u \to \frac{\partial^i u}{\partial x_k^i}|_{x_{k=0}}$ are continuous from $W^{2,p}(G;E(A),E)$ onto B_{ki} and there are the constants C_1 and C_0 such that for $w \in W^{2,p}(G;E(A),E)$, $W = \{w_{ki}\}$, $w_{ki} = \frac{\partial^i w}{\partial x_k^i}$, $i = 0,1$, $k = 1,2,\ldots,n$,

$$\left\| \frac{\partial^i w}{\partial x_k^i} \right\|_{B_{ki},\infty} = \sup_{x \in G} \left\| \frac{\partial^i w}{\partial x_k^i} \right\|_{B_{ki}} \leq C_1 \|w\|_{W^{2,p}(G;E(A),E)},$$

$$\|W\|_{0,\infty} = \sup_{x \in G} \sum_{k,i} \|w_{ki}\|_{B_{ki}} \leq C_0 \|w\|_{W^{2,p}(G;E(A),E)}.$$

Condition 6.1 Suppose the following hold:

(1) E is an UMD-space;

(2) a_k are continuous functions on $\bar{G}$, $a_k(\cdot) \in S(\varphi_1) \cap \mathbb{C}/\mathbb{R}_+$, $\alpha_{km_{k1}} \neq 0$, $\beta_{km_{k1}} \neq 0$, $k = 1, 2, \dots, n$, where $\varphi + \varphi_1 < \pi$;

(3) there exist $\Phi_{ki} \in B_{ki}$, such that the operator $B(t, x, \Phi)$ for $\Phi = \{\Phi_{kj}\} \in B_0$ is R-positive in E uniformly with respect to $x \in G$ and $t \in [0, T]$; moreover,

$$B(t, x, \Phi)B^{-1}(t^0, x^0, \Phi) \in C(\bar{G}; L(E)), \quad t^0 \in (0, T), x^0 \in G;$$

(4) $A = B(t^0, x^0, \Phi)$: $G_T \times B_0 \to L(E(A), E)$ is continuous; moreover, for each positive r there is a positive constant $L(r)$ such that

$$\left\| \left[B(t, x, U) - B(t, x, \bar{U})\right]\upsilon \right\|_E \leq L(r) \|U - \bar{U}\|_{B_0} \|A\upsilon\|_E$$

for $t \in (0, T)$, $x \in G$, $U, \bar{U} \in B_0$, $\bar{U} = \{\bar{u}_{kj}\}$, $\bar{u}_{kj} \in B_{kj}$, $\|U\|_{B_0}, \|\bar{U}\|_{B_0} \leq r$, $\upsilon \in D(A)$;

(5) the function F: $G_T \times B_0 \to E$ such that $F(\cdot, U)$ is measurable for each $U \in B_0$ and $F(t, x, \cdot)$ is continuous for a.a. $t \in (0, T)$, $x \in G$; moreover, $\|F(t, x, U) - F(t, x, \bar{U})\|_E \leq C\|U - \bar{U}\|_{B_0}$ for a.a. $t \in (0, T)$, $x \in G$, $U, \bar{U} \in B_0$ and $\|U\|_{B_0}, \|\bar{U}\|_{B_0} \leq r$; $f(\cdot) = F(\cdot, 0) \in L^p(G_T; E)$.

By reasoning as in [20, Theorem 5.1] we obtain the following result.

Theorem 6.1 *Let Condition* 6.1 *be satisfied. Then there are* $T \in (0, T_0)$ *and* $b_k \in (0, b_{0k})$ *such that problem* (6.1)-(6.2) *has a unique solution belonging to* $W^{1,2,p}(G_T; E(A), E)$.

7 The mixed value problem for system of parabolic equations

Consider the initial and BVP for the system of nonlinear parabolic equations

$$\frac{\partial u_m}{\partial t} + \sum_{k=1}^{n} a_k(x) \frac{\partial^2 u_m}{\partial x_k^2} + \sum_{j=1}^{N} d_{mj}(x) u_j(x, t) + \sum_{k=1}^{n} \sum_{j=1}^{N} b_{kj}(x) \frac{\partial u_j}{\partial x_k} = F_m(x, t, u), \tag{7.1}$$

$$\sum_{i=0}^{m_{k1}} \alpha_{ki} u_m^{(i)}(G_{k0}, t) = 0, \qquad \sum_{i=0}^{m_{k2}} \beta_{ki} u_m^{(i)}(G_{kb}, t) = 0, \tag{7.2}$$

$$u_m(x, 0) = \varphi_m(x), \quad x \in G, t \in (0, T), m = 1, 2, \dots, N, N \in \mathbb{N}, \tag{7.3}$$

where $u = (u_1, u_2, \dots, u_N)$, $m_{kj} \in \{0,1\}$, α_{ki}, β_{ki} are complex numbers, a_k are complex-valued functions, G, G_{k0}, G_{kb} are defined as in (3.1)-(3.2), and

$$\theta_{kj} = \frac{m_{kj} + \frac{1}{p}}{2}, \qquad s_{kj} = s(1 - \theta_{kj}), \quad s > 0, \qquad B_{kj} = l_q^{s_{kj}}, \quad j = 1, 2,$$

$$B_{0p} = \prod_{k,j} B_{kj}, \qquad \alpha_{km_{k1}} \neq 0, \qquad \beta_{km_{k2}} \neq 0, \quad k = 1, 2, \dots, n.$$

Let A be the operator in $l_q(N)$ defined by

$$D(A) = l_q^s(N), \qquad A = \left[d_{mj}(x)\right], \qquad d_{mj}(x) = g_m(x) 2^{sj}, \quad m, j = 1, 2, \dots, N,$$

where

$$l_q(N) = \left\{ u = \{u_j\}, j = 1, 2, \dots N, \|u\|_{l_q(N)} = \left(\sum_{j=1}^{N} |u_j|^q \right)^{\frac{1}{q}} < \infty \right\},$$

$$l_q(A) = \left\{ u \in l_q(N), \|u\|_{l_q(A)} = \|Au\|_{l_q(N)} = \left(\sum_{j=1}^{N} \left|2^{sj} u_j\right|^q \right)^{\frac{1}{q}} < \infty \right\},$$

$$x \in G, 1 < q < \infty, N = 1, 2, \dots, \infty.$$

Let $b_{kj}(x) = M_{kj}(x)2^{\sigma j}$ and

$$B = L\big(L_p\big(G; l_q(N)\big)\big).$$

From Theorem 6.1 we obtain the following result.

Theorem 7.1 *Let the following condition hold*:

(1) a_k *are continuous functions on* $\bar{G}$, $a_k(x) \in S(\varphi_1) \cap \mathbb{C}/R_+$;

(2) $s \geq \frac{2np(2-q)}{q(p-1)}$, $0 < \sigma < s_0$, $s_0 = \frac{s(p-1)}{2p}$;

(3) $g_j \in C(\bar{G})$, $N_{kj} \in C(\bar{G})$; *the eigenvalues of the matrix* $[d_{mi}(x)]$ *and* $d_{ii}(x)$ *are positive for all* $x \in \bar{G}$, $m, i = 1, 2, \dots, N$; *there is a positive constant* C *such that*

$$\sum_{k=1}^{n} \sum_{j=1}^{N} M_{kj}^{q_1}(x) \leq C \sum_{j=1}^{N} g_j^{q_1}(x) < \infty, \quad x \in G, \frac{1}{q} + \frac{1}{q_1} = 1;$$

(4) *the function* $F(\cdot, \upsilon) = (F_1(\cdot, \upsilon), \dots, F_N(\cdot, \upsilon))$ *is measurable for each* $\upsilon \in B_{0p}$ *and the function* $F(x, \cdot)$ *for a.a.* $x \in G$ *is continuous and* $f(\cdot) = F(\cdot, 0) \in L_p(G; l_q)$; *for each* $R > 0$ *there is a function* $\Psi_R \in L_\infty(G)$ *such that*

$$\left\|F(x, U) - F(x, \bar{U})\right\|_{l_q} \leq \Psi_R(x)\|U - \bar{U}\|_{l_q(A)}$$

a.a. $x \in G$ *and*

$$U, \bar{U} \in B_{0p}, \qquad \|U\|_{B_{0p}} \leq R, \qquad \|U\|_{B_{0p}} \leq R,$$

$$U = \{u_{kj}\}, \qquad \bar{U} = \{\bar{u}_{kj}\}, \qquad u_{kj}, \bar{u}_{kj} \in B_{0p}.$$

Then problem (7.1)-(7.3) *has a unique solution* $u = \{u_m(x)\}_1^N$ *that belongs to the space* $W_p^{1,2}(G_T, l_q(A), l_q)$.

Proof By virtue of [26] the space $l_q(N)$ is a UMD-space. It is easy to see that the operator A is R-positive in $l_q(N)$. Then by using conditions (1)-(3) we see that condition (5) of Theorem 6.1 holds. So in view of Theorem 6.1 we obtain the conclusion. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All abstract results belong to VS; the application part belongs to AS.

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