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A study of Riemann-Liouville fractional nonlocal integral boundary value problems

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Abstract

In this paper, we discuss the existence and uniqueness of solutions for a Riemann-Liouville type fractional differential equation with nonlocal four-point Riemann-Liouville fractional-integral boundary conditions by means of classical fixed point theorems. An illustration of main results is also presented with the aid of some examples.

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Keywords: fractional differential equations; nonlocal fractional-integral boundary conditions; existence; fixed point

1 Introduction

In recent years, boundary value problems of nonlinear fractional differential equations with a variety of boundary conditions have been investigated by many researchers. Fractional differential equations appear naturally in various fields of science and engineering and constitute an important field of research [1–4]. As a matter of fact, fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. This is one of the characteristics of fractional-order differential operators that contributes to the popularity of the subject and has motivated many researchers and modelers to shift their focus from classical models to fractional order models. In consequence, there has been a significant progress in the theoretical analysis like periodicity, asymptotic behavior and numerical methods for fractional differential equations. Some recent work on the topic can be found in [5–20] and the references therein.

Fractional boundary conditions (FBC) involving fractional derivative D^α of order $\alpha \in (0,1)$ describe an intermediate boundary between the perfect electric conductor (PEC) and the perfect magnetic conductor (PMC), whereas $\alpha = 0$ and $\alpha = 1$ in FBC correspond to PEC and PMC, respectively. Fractional boundary conditions (FBC) are also matched with impedance boundary conditions (IBC) in the sense that the fractional order $\alpha = 0$ and $\alpha = 1$ in FBC correspond to the value of impedance $Z = 0$ and $Z = i\infty$. Recall that the value of the impedance Z varies from 0 for PEC to $i\infty$ for PMC. For more details, see [21].

In [22], the authors recently studied a problem of Riemann-Liouville fractional differential equations with fractional boundary conditions:

$$D^\alpha u(t) = f\big(t, u(t)\big), \quad t \in [0,T], \alpha \in (1,2],$$

$$D^{\alpha-2}u\big(0^+\big) = b_0 D^{\alpha-2}u\big(T^-\big), \qquad D^{\alpha-1}u\big(0^+\big) = b_1 D^{\alpha-1}u\big(T^-\big),$$

where D^α denotes the Riemann-Liouville fractional derivative of order α and $b_0 \neq 1$ and $b_1 \neq 1$.

In this paper, motivated by [22], we study a fully Riemann-Liouville fractional nonlocal integral boundary value problem given by

$$\begin{cases} D^\alpha u(t) = f(t, u(t)), \quad 1 < \alpha \leq 2, t \in [0, T], \\ D^{\alpha-1} u(0^+) - a D^{\alpha-1} u(T^-) = A I^\beta u(\xi), \\ D^{\alpha-2} u(0^+) - b D^{\alpha-2} u(T^-) = B I^\beta u(\eta), \quad \beta > 0; \xi, \eta \in (0, T), \end{cases} \tag{1.1}$$

where D^α denotes the Riemann-Liouville fractional derivative of order α, f is a given continuous function, I^β denotes the Riemann-Liouville integral of order β, and a, A, b, and B are real constants.

The paper is organized as follows. In Section 2, we establish an auxiliary lemma which is needed to define the solutions of the given problem. Section 3 contains main results. In Section 4, we discuss some examples for the illustration of the main results.

2 Preliminaries

Let us recall some basic definitions of fractional theory.

Definition 2.1 The Riemann-Liouville fractional integral of order α for a continuous function $g : [0, \infty) \to \mathbb{R}$ is defined as

$$I^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s)}{(t-s)^{1-\alpha}} \, ds, \quad \alpha > 0,$$

provided the integral exists.

Definition 2.2 For a continuous function $g : [0, \infty) \to \mathbb{R}$, the Riemann-Liouville derivative of fractional order $\alpha > 0$ is defined as

$$D^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} g(s) \, ds = \left(\frac{d}{dt}\right)^n I^{n-\alpha} g(t), \quad n-1 < \alpha \leq n,$$

$n = [\alpha] + 1$, where $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.1 *For* $1 < \alpha \leq 2$, *the solution of* $D^\alpha u(t) = \sigma(t)$, $t \in [0, T]$ *subject to the boundary conditions given by* (1.1) *is*

$$\begin{aligned} u(t) &= I^\alpha \sigma(t) + \left(\delta_1 t^{\alpha-1} - \delta_4 t^{\alpha-2}\right)\left[A I^{\alpha+\beta} \sigma(\xi) + a I^1 \sigma(T)\right] \\ &\quad + \left(\delta_2 t^{\alpha-1} + \delta_3 t^{\alpha-2}\right)\left[B I^{\alpha+\beta} \sigma(\eta) + b I^2 \sigma(T)\right], \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} &\delta_1 = \frac{\Gamma(\alpha-1)}{\delta}\left(1 - b - \frac{B\eta^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}\right); \qquad \delta_2 = \frac{A\Gamma(\alpha-1)\xi^{\alpha+\beta-2}}{\delta\Gamma(\alpha+\beta-1)}; \\ &\delta_3 = \frac{\Gamma(\alpha)}{\delta}\left(1 - a - \frac{A\xi^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\right); \qquad \delta_4 = \frac{\Gamma(\alpha)}{\delta}\left(bT - \frac{B\eta^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\right); \\ &\delta = \Gamma(\alpha)\Gamma(\alpha-1)\left(1 - a - \frac{A\xi^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\right)\left(1 - b - \frac{B\eta^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}\right) \\ &\quad - \frac{A\Gamma(\alpha)\Gamma(\alpha-1)\xi^{\alpha+\beta-2}}{\Gamma(\alpha+\beta)\Gamma(\alpha+\beta-1)}\left(bT\Gamma(\alpha+\beta) + B\eta^{\alpha+\beta-1}\right). \end{aligned} \tag{2.2}$$

Proof For arbitrary constants $c_1, c_2 \in \mathbb{R}$, it is well known that the general solution of the equation $D^\alpha u(t) = \sigma(t)$, $1 < \alpha \leq 2$, can be written as

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + I^\alpha \sigma(t). \tag{2.3}$$

From (2.3), we have

$$D^{\alpha-1}u(t) = c_1\Gamma(\alpha) + I^1\sigma(t), \qquad D^{\alpha-2}u(t) = c_1\Gamma(\alpha)t + c_2\Gamma(\alpha-1) + I^2\sigma(t), \tag{2.4}$$

$$I^\beta u(\varrho) = \frac{c_1\varrho^{\alpha+\beta-1}\Gamma(\alpha)}{\Gamma(\alpha+\beta)} + \frac{c_2\varrho^{\alpha+\beta-2}\Gamma(\alpha-1)}{\Gamma(\alpha+\beta-1)} + I^{\alpha+\beta}\sigma(\varrho), \tag{2.5}$$

where ϱ denotes ξ or η. Applying the given boundary conditions, we get

$$\begin{cases} (\Gamma(\alpha)(1-a) - \frac{\Gamma(\alpha)A\xi^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)})c_1 - \frac{A\Gamma(\alpha-1)\xi^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}c_2 = AI^{\alpha+\beta}\sigma(\xi) + aI^1\sigma(T), \\ -\Gamma(\alpha)(bT + \frac{B\eta^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)})c_1 + \Gamma(\alpha-1)(1-b-\frac{B\eta^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)})c_2 = BI^{\alpha+\beta}\sigma(\eta) + bI^2\sigma(T). \end{cases} \tag{2.6}$$

Solving the system of equations (2.6) for c_1, c_2, we find that

$$c_1 = \delta_1\big(AI^{\alpha+\beta}\sigma(\xi) + aI^1\sigma(T)\big) + \delta_2\big(BI^{\alpha+\beta}\sigma(\eta) + bI^2\sigma(T)\big),$$
$$c_2 = \delta_3\big(BI^{\alpha+\beta}\sigma(\eta) + bI^2\sigma(T)\big) - \delta_4\big(AI^{\alpha+\beta}\sigma(\xi) + aI^1\sigma(T)\big).$$

Substituting these values in (2.3), we get

$$\begin{aligned} u(t) &= I^\alpha\sigma(t) + A\big(\delta_1 t^{\alpha-1} - \delta_4 t^{\alpha-2}\big)I^{\alpha+\beta}\sigma(\xi) + B\big(\delta_2 t^{\alpha-1} + \delta_3 t^{\alpha-2}\big)I^{\alpha+\beta}\sigma(\eta) \\ &\quad + b\big(\delta_2 t^{\alpha-1} + \delta_3 t^{\alpha-2}\big)I^2\sigma(T) + a\big(\delta_1 t^{\alpha-1} - \delta_4 t^{\alpha-2}\big)I^1\sigma(T), \end{aligned} \tag{2.7}$$

where δ_1, δ_2, δ_3, δ_4 and δ are given by (2.2). This completes the proof. □

3 Existence results

Let $C[0,T]$ denote the Banach space of all continuous real-valued functions defined on $[0,T]$ with the norm $\|u\| = \sup\{|u(t)| : t \in [0,T]\}$. For $t \in [0,T]$, define $u_r(t) = t^r u(t)$, $r \geq 0$, and let $C_r[0,T]$ be the space of all functions u_r such that $u \in C[0,T]$ which turns out to be a Banach space when endowed with the norm $\|u\|_r = \sup\{t^r|u(t)| : t \in [0,T]\}$.

Let us define an operator $\mathcal{Q} : C_{2-\alpha}[0,T] \to C_{2-\alpha}[0,T]$ as

$$\begin{aligned} (\mathcal{Q}u)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(s,u(s)\big)\,ds \\ &\quad + \big(\delta_1 t^{\alpha-1} - \delta_4 t^{\alpha-2}\big)\bigg[A\int_0^\xi \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f\big(s,u(s)\big)\,ds + a\int_0^T f\big(s,u(s)\big)\,ds\bigg] \\ &\quad + \big(\delta_2 t^{\alpha-1} + \delta_3 t^{\alpha-2}\big)\bigg[B\int_0^\eta \frac{(\eta-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f\big(s,u(s)\big)\,ds \\ &\quad + b\int_0^T (T-s) f\big(s,u(s)\big)\,ds\bigg]. \end{aligned} \tag{3.1}$$

Observe that problem (1.1) has solutions only if the operator $\mathcal{Q}$ has fixed points.

To establish the first existence result, we need the following fixed point theorem.

Theorem 3.1 ([23]) *Let E be a Banach space. Let $T : E \to E$ be a completely continuous operator, and let the set $V = \{x \in E | x = \mu Tx, 0 < \mu < 1\}$ be bounded. Then the operator T has a fixed point in E.*

Theorem 3.2 *Assume that there exists a constant $M > 0$ such that $|f(t,u)| \leq M$, $\forall t \in [0,T]$, $u \in \mathbb{R}$. Then problem* (1.1) *has at least one solution in the space $C_{2-\alpha}[0,T]$.*

Proof As a first step, we show that the operator $\mathcal{Q}$ is completely continuous. The continuity of $\mathcal{Q}$ follows from the continuity of f. Let $\mathcal{H}$ be a bounded set in $C_{2-\alpha}[0,T]$. Hence $\mathcal{H}$ is bounded on $C[0,T]$. Then, $\forall u \in \mathcal{H}$, $t \in [0,T]$, we have

$$
\begin{aligned}
&t^{2-\alpha}\big|(\mathcal{Q}u)(t)\big| \\
&\quad \leq M\bigg|t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds + (\delta_1 t - \delta_4)\bigg|\bigg[A\int_0^\xi \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\,ds + a\int_0^T ds\bigg] \\
&\qquad + (\delta_2 t + \delta_3)\bigg[B\int_0^\eta \frac{(\eta-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\,ds + b\int_0^T (T-s)\,ds\bigg] \\
&\quad \leq M\bigg\{\frac{T^2}{\Gamma(\alpha+1)} + \big(|\delta_1| + |\delta_4|\big)\bigg(\frac{|A|\xi^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + |a|T\bigg) \\
&\qquad + \big(|\delta_2| + |\delta_3|\big)\bigg(\frac{|B|\eta^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|b|T^2}{2}\bigg)\bigg\} \\
&\quad = L,
\end{aligned}
$$

which implies that $\|(\mathcal{Q}u)\|_{2-\alpha} \leq L$. Hence $\mathcal{Q}\mathcal{H}$ is uniformly bounded. Also, for $t_1, t_2 \in [0,T]$, $u \in \mathcal{H}$, we have

$$
\begin{aligned}
&\big|t_1^{2-\alpha}(\mathcal{Q}u)(t_1) - t_2^{2-\alpha}(\mathcal{Q}u)(t_2)\big| \\
&\quad \leq M\bigg|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\big[t_1^{2-\alpha}(t_1-s)^{\alpha-1} - t_2^{2-\alpha}(t_2-s)^{\alpha-1}\big]\,ds - \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} t_2^{2-\alpha}(t_2-s)^{\alpha-1}\,ds \\
&\qquad + \delta_1(t_1-t_2)\bigg[A\int_0^\xi \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\,ds + a\int_0^T ds\bigg] \\
&\qquad + \delta_2(t_1-t_2)\bigg[B\int_0^\eta \frac{(\eta-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\,ds + b\int_0^T (T-s)\,ds\bigg]\bigg| \to 0 \quad \text{as } t_1 \to t_2.
\end{aligned}
$$

Thus $t^{2-\alpha}\mathcal{Q}\mathcal{H}$ and hence $\mathcal{Q}\mathcal{H}$ is equicontinuous. So, by the Arzela-Ascoli theorem, $\mathcal{Q}$ is completely continuous. Next, we consider the set

$$
V = \big\{t^{2-\alpha}u \in \mathbb{R} : t^{2-\alpha}u = \mu t^{2-\alpha}\mathcal{Q}u, 0 < \mu < 1\big\},
$$

and show that V is bounded. For $u \in V$, we have

$$
\big|t^{2-\alpha}u(t)\big| = \big|\mu t^{2-\alpha}(\mathcal{Q}u)(t)\big| \leq t^{2-\alpha}\big|\mathcal{Q}u(t)\big| \leq L.
$$

This implies that the set V is bounded independently of $\mu \in (0,1)$. Therefore, Theorem 3.1 applies and problem (1.1) has at least one solution on $[0,T]$. This completes the proof. □

Theorem 3.3 *Assume that there exists a constant $K > 0$ such that*

$$\left|f(t,u) - f(t,v)\right| \le K|u - v|, \quad \forall t \in [0,T], u, v \in \mathbb{R},$$

then problem (1.1) *has a unique solution in $C_{2-\alpha}[0,T]$ if $K\nu < 1$, where*

$$\nu = \left\{\frac{T^2}{\Gamma(\alpha+1)} + \frac{(|\delta_1| + |\delta_4|)(|A|\xi^{\alpha+\beta} + |a|T\Gamma(\alpha+\beta+1))}{\Gamma(\alpha+\beta+1)} + \frac{(|\delta_2| + |\delta_3|)(2|B|\eta^{\alpha+\beta} + |b|T^2\Gamma(\alpha+\beta+1))}{2\Gamma(\alpha+\beta+1)}\right\}. \tag{3.2}$$

Proof For every $t \in [0,T]$, $u, v \in \mathbb{R}$, we have

$$\begin{aligned}
&t^{2-\alpha}\left|(\mathcal{Q}u)(t) - (\mathcal{Q}v)(t)\right| \\
&\quad \le t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f(s,u(s)) - f(s,v(s))\right| ds \\
&\qquad + \left(|\delta_1 t - \delta_4|\right)\left(|A|\int_0^\xi \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left|f(s,u(s)) - f(s,v(s))\right| ds \right.\\
&\qquad \left. + |a|\int_0^T \left|f(s,u(s)) - f(s,v(s))\right| ds\right) \\
&\qquad + \left|(\delta_2 t + \delta_3)\right|\left(|B|\int_0^\eta \frac{(\eta-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left|f(s,u(s)) - f(s,v(s))\right| ds \right.\\
&\qquad \left. + |b|\int_0^T (T-s)\left|f(s,u(s)) - f(s,v(s))\right| ds\right) \\
&\quad \le K\left\{\frac{T^2}{\Gamma(\alpha+1)} + \frac{(|\delta_1| + |\delta_4|)(|A|\xi^{\alpha+\beta} + |a|T\Gamma(\alpha+\beta+1))}{\Gamma(\alpha+\beta+1)} \right.\\
&\qquad \left. + \frac{(|\delta_2| + |\delta_3|)(2|B|\eta^{\alpha+\beta} + |b|T^2\Gamma(\alpha+\beta+1))}{2\Gamma(\alpha+\beta+1)}\right\}|u - v|.
\end{aligned}$$

By the definition of $\|\cdot\|_{2-\alpha}$, we obtain

$$\left\|(\mathcal{Q}u)(t) - (\mathcal{Q}v)(t)\right\|_{2-\alpha} \le K\nu\|u - v\|_{2-\alpha} \le \|u - v\|_{2-\alpha}.$$

It follows that $\mathcal{Q}$ is a contraction. Hence, by the Banach contraction theorem, problem (1.1) has a unique solution in $C_{2-\alpha}[0,T]$. This completes the proof. □

Our next existence result is based on Leray-Schauder nonlinear alternative [24].

Lemma 3.1 (Leray-Schauder's nonlinear alternative type) *Let E be a Banach space, M be a closed, convex subset of E, U be an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact* (*that is, $F(U)$ is a relatively compact subset of C*) *map. Then either* (i) *F has a fixed point in $\overline{U}$ or* (ii) *there are $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda F(U)$.*

Theorem 3.4 *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Furthermore, assume that*:

(A_1) *There exist a function $p \in \mathcal{C}([0,T],\mathbb{R}^+)$ and a nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(t,u)| \le p(t)\psi(\|u\|)$, $\forall(t,u) \in [0,T] \times \mathbb{R}$;*

(A_2) *There exists a constant* $M > 0$ *such that*

$$\frac{M}{\psi(M)\nu\|p\|} > 1,$$

where ν *is given by* (3.2).

Then boundary value problem (1.1) *has at least one solution.*

Proof First we shall show that the operator $\mathcal{Q}$ defined by (3.1) maps bounded sets into bounded ones in $C_{2-\alpha}([0,T],\mathbb{R})$. For $r > 0$, let $\mathcal{H}_r = \{u \in C_{2-\alpha}[0,T] : \|u\|_{2-\alpha} \leq r\}$ be a bounded set in $C_{2-\alpha}([0,T],\mathbb{R})$. Then, for $u \in \mathcal{H}_r$, we have

$$\begin{aligned}
&t^{2-\alpha}\left|(\mathcal{Q}u)(t)\right| \\
&\quad\leq t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)\psi\left(\|u\|_{2-\alpha}\right)ds \\
&\qquad+\left(|\delta_1|+|\delta_4|\right)\left(|A|\int_0^\xi \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}p(s)\psi\left(\|u\|_{2-\alpha}\right)ds+|a|\int_0^T p(s)\psi\left(\|u\|_{2-\alpha}\right)ds\right) \\
&\qquad+\left(|\delta_2|+|\delta_3|\right)\left(|B|\int_0^\eta \frac{(\eta-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}p(s)\psi\left(\|u\|_{2-\alpha}\right)ds\right. \\
&\qquad\left.+|b|\int_0^T (T-s)p(s)\psi\left(\|u\|_{2-\alpha}\right)ds\right) \\
&\quad\leq \|p\|\psi(r)\left\{\frac{T^2}{\Gamma(\alpha+1)}+\left(|\delta_1|+|\delta_4|\right)\left(\frac{|A|\xi^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+|a|T\right)\right. \\
&\qquad\left.+\left(|\delta_2|+|\delta_3|\right)\left(\frac{|B|\eta^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{|b|T^2}{2}\right)\right\} \\
&\quad\leq \|p\|\psi(r)\nu,
\end{aligned}$$

where ν is given by (3.2).

Next, we shall show that the operator $\mathcal{Q}$ maps bounded sets into equicontinuous sets. Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ and $u \in \mathcal{H}_r$. Then we have

$$\begin{aligned}
&\left|t_1^{2-\alpha}(\mathcal{Q}u)(t_1)-t_2^{2-\alpha}(\mathcal{Q}u)(t_2)\right| \\
&\quad\leq \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[t_1^{2-\alpha}(t_1-s)^{\alpha-1}-t_2^{2-\alpha}(t_2-s)^{\alpha-1}\right]p(s)\psi(r)\,ds\right. \\
&\qquad-\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} t_2^{2-\alpha}(t_2-s)^{\alpha-1}p(s)\psi(r)\,ds \\
&\qquad+\delta_1(t_1-t_2)\left(A\int_0^\xi \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}p(s)\psi(r)\,ds+a\int_0^T p(s)\psi(r)\,ds\right) \\
&\qquad+\delta_2(t_1-t_2)\left(B\int_0^\eta \frac{(\eta-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}p(s)\psi(r)\,ds\right. \\
&\qquad\left.\left.+b\int_0^T (T-s)p(s)\psi(r)\,ds\right)\right| \\
&\quad\leq \|p\|\psi(r)\left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[t_1^{2-\alpha}(t_1-s)^{\alpha-1}-t_2^{2-\alpha}(t_2-s)^{\alpha-1}\right]ds\right.
\end{aligned}$$

$$- \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} t_2^{2-\alpha}(t_2 - s)^{\alpha-1}\,ds + \delta_1(t_1 - t_2)\biggl(A \int_0^{\xi} \frac{(\xi - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\,ds + a \int_0^T p(s)\psi(r)\,ds\biggr) + \delta_2(t_1 - t_2)\biggl(B \int_0^{\eta} \frac{(\eta - s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\,ds + b \int_0^T (T - s)\,ds\biggr)\biggr|,$$

which tends to zero independently of $u \in \mathcal{H}_r$ as $t_1 \to t_2$. Thus $\mathcal{Q}$ is completely continuous. Now let u be a solution of problem (1.1), then for $t \in [0, T]$ and $\lambda \in (0, 1)$, we have

$$t^{2-\alpha}\bigl|u(t)\bigr| \le \psi\bigl(\|u\|_{2-\alpha}\bigr)\|p\|\biggl\{\frac{T^2}{\Gamma(\alpha+1)} + \bigl(|\delta_1| + |\delta_4|\bigr)\biggl(\frac{|A|\xi^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + |a|T\biggr) + \bigl(|\delta_2| + |\delta_3|\bigr)\biggl(\frac{|B|\eta^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|b|T^2}{2}\biggr)\biggr\}, \tag{3.3}$$

which can be rewritten as

$$t^{2-\alpha}\bigl|u(t)\bigr|\biggl[\psi\bigl(\|u\|_{2-\alpha}\bigr)\|p\|\biggl\{\frac{T^2}{\Gamma(\alpha+1)} + \bigl(|\delta_1| + |\delta_4|\bigr)\biggl(\frac{|A|\xi^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + |a|T\biggr) + \bigl(|\delta_2| + |\delta_3|\bigr)\biggl(\frac{|B|\eta^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|b|T^2}{2}\biggr)\biggr\}\biggr]^{-1} \le 1. \tag{3.4}$$

By assumption (A_2), there exists M such that $t^{2-\alpha}|u(t)| \neq M$. Let us set

$$\mathcal{H}_M = \bigl\{u \in C_{2-\alpha}[0, T] : t^{2-\alpha}\bigl|u(t)\bigr| < M + 1\bigr\}.$$

Note that the operator $\mathcal{Q} : \overline{\mathcal{H}}_M \to C_{2-\alpha}[0, T]$ is completely continuous and by the definition of $\mathcal{H}_M$, there is no $u \in \partial\mathcal{H}_M$ such that $u = \lambda\mathcal{Q}(u)$ for some $\lambda \in (0, 1)$. In consequence, by Lemma 3.1, we conclude that $\mathcal{Q}$ has at least one fixed point $u \in \overline{\mathcal{H}}_M$, which is a solution of problem (1.1). □

4 Examples

Example 4.1 Consider the following fractional integral boundary value problem:

$$\begin{cases} D^{\alpha}u(t) = \frac{\sin u(t) + \cos u(t)}{5 + \cos^2 u(t)}, \quad t \in [0, 1], \\ D^{0.5}x(0^+) - D^{0.5}x(1^-) = I^{7/4}x(1/2), \\ D^{-0.5}x(0^+) - D^{-0.5}x(1^-) = I^{7/4}x(3/4). \end{cases} \tag{4.1}$$

Since

$$\bigl|f(t, u)\bigr| = \biggl|\frac{\sin u(t) + \cos u(t)}{5 + \cos^2 u(t)}\biggr| \le \frac{2}{5},$$

therefore, Theorem 3.2 applies and problem (4.1) has at least one solution on $[0, 1]$.

Example 4.2 Consider the problem

$$\begin{cases} D^{\frac{3}{2}}x(t) = K(\cos t + \tan^{-1} x(t)), \quad t \in [0, 1], \\ D^{0.5}x(0^+) - D^{0.5}x(1^-) = I^{7/4}x(1/2), \\ D^{-0.5}x(0^+) - D^{-0.5}x(1^-) = I^{7/4}x(3/4). \end{cases} \tag{4.2}$$

Here $\alpha = 3/2$, $T = 1$, $a = b = 1$, $\xi = 1/2$, $\eta = 3/4$, $\beta = 7/4$, $A = B = 1$. Clearly,

$$\left|f(t,u) - f(t,v)\right| \le K\left|\tan^{-1} u(t) - \tan^{-1} v(t)\right| \le K|u - v|,$$

$\nu \simeq 4.315066$ (ν is given by (3.2)) and in consequence, $K < 0.231746$. Thus, all the assumptions of Theorem 3.3 are satisfied. Therefore, by the conclusion of Theorem 3.3, there exists a unique solution for problem (4.2).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, BA, AA, AAS and RPA contributed to each part of this work equally and read and approved the final version of the manuscript.

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