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Positivity of the infimum eigenvalue for equations of $p(x)$-Laplace type in $\mathbb{R}^N$

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Abstract

We study the following elliptic equations with variable exponents

$$-\operatorname{div}(\phi(x,|\nabla u|)\nabla u) = \lambda f(x,u) \quad \text{in } \mathbb{R}^N.$$

Under suitable conditions on ϕ and f, we show the existence of positivity of the infimum of all eigenvalues for the problem above, and then give an example to demonstrate our main result.

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1 Introduction

The variable exponent problems appear in a lot of applications, for example, elastic mechanics, electro-rheological fluid dynamics and image processing, *etc.* The study of variable mathematical problems involving $p(x)$-growth conditions has attracted interest and attention in recent years. We refer the readers to [1–4] and references therein.

In this paper, we are concerned with the eigenvalue problem of a class of equations of $p(x)$-Laplacian type

$$-\operatorname{div}\big(\phi\big(x,|\nabla u|\big)\nabla u\big) = \lambda f(x,u) \quad \text{in } \mathbb{R}^N, \tag{E}$$

where the function $\phi(x,t)$ is of type $|t|^{p(x)-2}$ with continuous nonconstant function $p: \mathbb{R}^N \to (1,\infty)$ and $f: \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition. Recently, the authors in [5] obtained the positivity of the infimum of all eigenvalues for the $p(x)$-Laplacian type subject to the Dirichlet boundary condition. As far as the authors know, there are no results concerned with the eigenvalue problem for a more general $p(x)$-Laplacian type problem in the whole space $\mathbb{R}^N$.

When $\phi(x,t) = |t|^{p(x)-2}$, the operator involved in (E) is called the $p(x)$-Laplacian, *i.e.*, $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$. The studies for $p(x)$-Laplacian problems have been extensively performed by many researchers in various ways; see [5–11]. In particular, by using the Ljusternik-Schnirelmann critical point theory, Fan *et al.* [8] established the existence of the sequence of eigenvalues of the $p(x)$-Laplacian Dirichlet problem; see [12] for Neumann problems. Mihăilescu and Rădulescu in [13] obtained the existence of a continuous family of eigenvalues in a neighborhood of the origin under suitable conditions.

The $p(x)$-Laplacian is a natural generalization of the p-Laplacian, where $p > 1$ is a constant. There are a bunch of papers, for instance, [14–18] and references therein. But the $p(x)$-Laplace operator possesses more complicated nonlinearities than the p-Laplace operator, for example, it is nonhomogeneous, so a more complicated analysis has to be carefully carried out. Some properties of the p-Laplacian eigenvalue problems may not hold for a general $p(x)$-Laplacian. For example, under some conditions, the infimum of all eigenvalues for the $p(x)$-Laplacian might be zero; see [8]. The purpose of this paper is to give suitable conditions on ϕ and f to satisfy the positivity of the infimum of all eigenvalues for (E) still. This result generalizes Benouhiba's recent result in [6] in some sense.

This paper is organized as follows. In Section 2, we state some basic results for the variable exponent Lebesgue-Sobolev spaces, which are given in [19, 20]. In Section 3, we give sufficient conditions on ϕ and f to obtain the positivity of the infimum eigenvalue for the problem (E) above. Also, we present an example to illustrate our main result.

2 Preliminaries

In this section, we state some elementary properties for the variable exponent Lebesgue-Sobolev spaces, which will be used in the next section. The basic properties of the variable exponent Lebesgue-Sobolev spaces can be found from [19, 20].

To make a self-contained paper, we first recall some definitions and basic properties of the variable exponent Lebesgue spaces $L^{p(x)}(\mathbb{R}^N)$ and the variable exponent Lebesgue-Sobolev spaces $W^{1,p(x)}(\mathbb{R}^N)$.

Set

$$C_+(\mathbb{R}^N) = \Big\{h \in C(\mathbb{R}^N) : \inf_{x\in\mathbb{R}^N} h(x) > 1\Big\}.$$

For any $h \in C_+(\mathbb{R}^N)$, we define

$$h_+ = \sup_{x\in\mathbb{R}^N} h(x) \quad \text{and} \quad h_- = \inf_{x\in\mathbb{R}^N} h(x).$$

For any $p \in C_+(\mathbb{R}^N)$, we introduce the variable exponent Lebesgue space

$$L^{p(x)}(\mathbb{R}^N) := \Big\{u : u \text{ is a measurable real-valued function, } \int_{\mathbb{R}^N} \big|u(x)\big|^{p(x)}\,dx < \infty\Big\},$$

endowed with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\mathbb{R}^N)} = \inf\bigg\{\lambda > 0 : \int_{\mathbb{R}^N} \bigg|\frac{u(x)}{\lambda}\bigg|^{p(x)}\,dx \le 1\bigg\}.$$

The dual space of $L^{p(x)}(\mathbb{R}^N)$ is $L^{p'(x)}(\mathbb{R}^N)$, where $1/p(x) + 1/p'(x) = 1$. The variable exponent Lebesgue spaces are a special case of Orlicz-Musielak spaces treated by Musielak in [21].

The variable exponent Sobolev space $W^{1,p(x)}(\mathbb{R}^N)$ is defined by

$$W^{1,p(x)}(\mathbb{R}^N) = \big\{u \in L^{p(x)}(\mathbb{R}^N) : |\nabla u| \in L^{p(x)}(\mathbb{R}^N)\big\},$$

where the norm is

$$\|u\|_{W^{1,p(x)}(\mathbb{R}^N)} = \|u\|_{L^{p(x)}(\mathbb{R}^N)} + \|\nabla u\|_{L^{p(x)}(\mathbb{R}^N)}. \tag{2.1}$$

Definition 2.1 The exponent $p(\cdot)$ is said to be log-Hölder continuous if there is a constant C such that

$$\left|p(x)-p(y)\right| \leq \frac{C}{-\log|x-y|} \tag{2.2}$$

for every $x, y \in \mathbb{R}^N$ with $|x-y| \leq 1/2$.

Smooth functions are not dense in the variable exponent Sobolev spaces, without additional assumptions on the exponent $p(x)$. Zhikov [22] gave some examples of Lavrentiev's phenomenon for the problems with variable exponents. These examples show that smooth functions are not dense in variable exponent Sobolev spaces. However, when $p(x)$ satisfies the log-Hölder continuity condition, smooth functions are dense in variable exponent Sobolev spaces, and there is no confusion in defining the Sobolev space with zero boundary values, $W_0^{1,p(x)}(\mathbb{R}^N)$, as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|_{W^{1,p(x)}(\mathbb{R}^N)}$ (see [23, 24]).

Lemma 2.2 [19, 20] *The space $L^{p(x)}(\mathbb{R}^N)$ is a separable, uniformly convex Banach space, and its conjugate space is $L^{p'(x)}(\mathbb{R}^N)$, where $1/p(x)+1/p'(x)=1$. For any $u \in L^{p(x)}(\mathbb{R}^N)$ and $v \in L^{p'(x)}(\mathbb{R}^N)$, we have*

$$\left|\int_{\mathbb{R}^N} uv\,dx\right| \leq \left(\frac{1}{p_-}+\frac{1}{p'_-}\right)\|u\|_{L^{p(x)}(\mathbb{R}^N)}\|v\|_{L^{p'(x)}(\mathbb{R}^N)} \leq 2\|u\|_{L^{p(x)}(\mathbb{R}^N)}\|v\|_{L^{p'(x)}(\mathbb{R}^N)}.$$

Lemma 2.3 [19] *Denote*

$$\rho(u) = \int_{\mathbb{R}^N} |u|^{p(x)}\,dx \quad \text{for all } u \in L^{p(x)}\left(\mathbb{R}^N\right).$$

Then

(1) $\rho(u) > 1$ $(= 1; < 1)$ *if and only if* $\|u\|_{L^{p(x)}(\mathbb{R}^N)} > 1$ $(= 1; < 1)$, *respectively*;
(2) *if* $\|u\|_{L^{p(x)}(\mathbb{R}^N)} > 1$, *then* $\|u\|^{p_-}_{L^{p(x)}(\mathbb{R}^N)} \leq \rho(u) \leq \|u\|^{p_+}_{L^{p(x)}(\mathbb{R}^N)}$;
(3) *if* $\|u\|_{L^{p(x)}(\mathbb{R}^N)} < 1$, *then* $\|u\|^{p_+}_{L^{p(x)}(\mathbb{R}^N)} \leq \rho(u) \leq \|u\|^{p_-}_{L^{p(x)}(\mathbb{R}^N)}$.

Lemma 2.4 [11] *Let $q \in L^\infty(\mathbb{R}^N)$ be such that $1 \leq p(x)q(x) \leq \infty$ for almost all $x \in \mathbb{R}^N$. If $u \in L^{q(x)}(\mathbb{R}^N)$ with $u \neq 0$, then*

(1) *if* $\|u\|_{L^{p(x)q(x)}(\mathbb{R}^N)} > 1$, *then* $\|u\|^{q_-}_{L^{p(x)q(x)}(\mathbb{R}^N)} \leq \||u|^{q(x)}\|_{L^{p(x)}(\mathbb{R}^N)} \leq \|u\|^{q_+}_{L^{p(x)q(x)}(\mathbb{R}^N)}$;
(2) *if* $\|u\|_{L^{p(x)q(x)}(\mathbb{R}^N)} < 1$, *then* $\|u\|^{q_+}_{L^{p(x)q(x)}(\mathbb{R}^N)} \leq \||u|^{q(x)}\|_{L^{p(x)}(\mathbb{R}^N)} \leq \|u\|^{q_-}_{L^{p(x)q(x)}(\mathbb{R}^N)}$.

Lemma 2.5 [23] *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set with Lipschitz boundary, and let $p \in C_+(\overline{\Omega})$ with $1 < p_- \leq p_+ < N$ satisfy the log-Hölder continuity condition* (2.2). *If $q \in L^\infty(\Omega)$ with $q_- > 1$ satisfies*

$$q(x) \leq p^*(x) := \frac{Np(x)}{N-p(x)} \quad \text{for all } x \in \Omega,$$

then we have

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega),$$

and the imbedding is compact if $\inf_{x\in\Omega}(p^*(x)-q(x)) > 0$.

Lemma 2.6 [25] *Suppose that $p : \mathbb{R}^N \to \mathbb{R}^N$ is a Lipschitz function with $1 < p_- \le p_+ < N$. Let $q \in L^\infty(\mathbb{R}^N)$ and $p(x) \le q(x) \le p^*(x)$ for almost all $x \in \mathbb{R}^N$. Then there is a continuous embedding $W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{q(x)}(\mathbb{R}^N)$.*

3 Main result

In this section, we shall give the proof of the existence of the positive eigenvalue for the problem (E), by applying the basic properties of the spaces $L^{p(x)}(\mathbb{R}^N)$ and $W^{1,p(x)}(\mathbb{R}^N)$, which were given in the previous section.

Throughout this paper, let $p \in C_+(\mathbb{R}^N)$ satisfy the log-Hölder continuity condition (2.2) and $X := W_0^{1,p(x)}(\mathbb{R}^N)$ with the norm

$$\|u\|_X = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^N} \left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\},$$

which is equivalent to norm (2.1).

Definition 3.1 We say that $u \in X$ is a weak solution of the problem (E) if

$$\int_{\mathbb{R}^N} \phi\big(x, |\nabla u|\big) \nabla u(x) \cdot \nabla \varphi(x)\, dx = \lambda \int_{\mathbb{R}^N} f(x,u)\varphi(x)\, dx$$

for all $\varphi \in X$.

Denote

$$\Omega_1 = \left\{x \in \mathbb{R}^N : 1 < p(x) < 2\right\}, \qquad \Omega_2 = \left\{x \in \mathbb{R}^N : p(x) \ge 2\right\}$$

(we allow the case that one of these sets is empty). Then it is obvious that $\mathbb{R}^N = \Omega_1 \cup \Omega_2$. We assume that:

(H1) $p, q \in C_+(\mathbb{R}^N)$, $p(x) < N$, and $1 < p_- \le p_+ < q_- \le q_+ < p^*(x)$.

(HJ1) $\phi : \mathbb{R}^N \times [0,\infty) \to [0,\infty)$ satisfies the following conditions: $\phi(\cdot,\eta)$ is measurable on $\mathbb{R}^N$ for all $\eta \ge 0$ and $\phi(x,\cdot)$ is locally absolutely continuous on $[0,\infty)$ for almost all $x \in \mathbb{R}^N$.

(HJ2) There are a function $a \in L^{p'(x)}(\mathbb{R}^N)$ and a nonnegative constant b such that

$$\left|\phi\big(x,|v|\big)v\right| \le a(x) + b|v|^{p(x)-1}$$

for almost all $x \in \mathbb{R}^N$ and for all $v \in \mathbb{R}^N$.

(HJ3) There exists a positive constant c such that the following conditions are satisfied for almost all $x \in \mathbb{R}^N$:

$$\phi(x,\eta) \ge c\eta^{p(x)-2} \quad \text{and} \quad \eta \frac{\partial \phi}{\partial \eta}(x,\eta) + \phi(x,\eta) \ge c\eta^{p(x)-2} \tag{3.1}$$

for almost all $\eta \in (0,1)$. In case $x \in \Omega_2$, assume that condition (3.1) holds for almost all $\eta \in (1,\infty)$, and in case $x \in \Omega_1$, assume that for almost all $\eta \in (1,\infty)$ instead

$$\phi(x,\eta) \ge c \quad \text{and} \quad \eta \frac{\partial \phi}{\partial \eta}(x,\eta) + \phi(x,\eta) \ge c. \tag{3.2}$$

(HJ4) For all $x \in \mathbb{R}^N$ and all $\xi \in \mathbb{R}^N$, the estimate holds

$$0 \le a(x,\xi) \cdot \xi \le p_+ \Phi_0\big(x, |\xi|\big),$$

where $a(x,\xi) = \phi(x,|\xi|)\xi$.

Let us put

$$\Phi_0(x,t) = \int_0^t \phi(x,\eta)\eta \, d\eta$$

and define the functional $\Phi : X \to \mathbb{R}$ by

$$\Phi(u) = \int_{\mathbb{R}^N} \Phi_0\big(x, \big|\nabla u(x)\big|\big) \, dx.$$

Then $\Phi \in C^1(X,\mathbb{R})$ [5], and its Gateaux derivative is

$$\big\langle \Phi'(u), \varphi \big\rangle := \int_{\mathbb{R}^N} \phi\big(x, \big|\nabla u(x)\big|\big) \nabla u(x) \cdot \nabla \varphi(x) \, dx. \tag{3.3}$$

Let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a real-valued function. We assume that the function f satisfies the Carathéodory condition in the sense that $f(\cdot,t)$ is measurable for all $t \in \mathbb{R}$ and $f(x,\cdot)$ is continuous for almost all $x \in \mathbb{R}^N$. Denote

$$\gamma(x) = \frac{r(x)}{r(x) - q(x)} \quad \text{for almost all } x \in \mathbb{R}^N,$$

where q is given in (H1) and $q(x) < r(x) < p^*(x)$. We assume that

(F1) For all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, $f(x,t)t \ge 0$, and there is a nonnegative measurable function m with $m \in L^{\gamma(x)}(\mathbb{R}^N)$ such that

$$\big|f(x,t)\big| \le m(x)|t|^{q(x)-1}.$$

Denoting $F(x,t) = \int_0^t f(x,s) \, ds$, it follows from (F1) that

$$\text{(F1}'\text{)} \quad 0 \le F(x,t) \le \frac{m(x)}{q(x)} |t|^{q(x)} \quad \text{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Define the functional $\Psi, I_\lambda : X \to \mathbb{R}$ by

$$\Psi(u) = \int_{\mathbb{R}^N} F(x,u) \, dx \quad \text{and} \quad I_\lambda(u) = \Phi(u) - \lambda \Psi(u).$$

Then it is easy to check that $\Psi \in C^1(X,\mathbb{R})$, and its Gateaux derivative is

$$\big\langle \Psi'(u), \varphi \big\rangle = \int_{\mathbb{R}^N} f(x,u)\varphi(x) \, dx \tag{3.4}$$

for any $u, \varphi \in X$. Let us consider the following quantity:

$$\lambda^* = \inf_{u \in X \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \Phi_0(x, |\nabla u|) \, dx}{\int_{\mathbb{R}^N} F(x,u) \, dx}. \tag{3.5}$$

For the case of $\phi(x,|t|) = |t|^{p(x)-2}$ and $f(x,t) = m(x)|t|^{q(x)-2}t$, where $m(x)$ satisfies a suitable condition, Benouhiba [6] proved that $\lambda^* > 0$. In this section, we shall generalize the conditions on f and ϕ to satisfy $\lambda^* > 0$ still.

The following lemma plays a key role in obtaining the main result in this section.

Lemma 3.2 *Assume that assumptions* (HJ3)-(HJ4), (H1), *and* (F1) *hold and satisfy*

$$\text{(H2)} \quad q_+ - \frac{1}{2}p_- < q_-,$$

then the functionals Φ *and* Ψ *satisfy the following relations*:

$$\lim_{\|u\|_X \to 0} \frac{\Phi(u)}{\Psi(u)} = \infty \tag{3.6}$$

and

$$\lim_{\|u\|_X \to \infty} \frac{\Phi(u)}{\Psi(u)} = \infty. \tag{3.7}$$

Proof Applying Lemmas 2.2, 2.4 and 2.6, we get

$$\begin{aligned} \left|\Psi(u)\right| &= \left|\int_{\mathbb{R}^N} F(x,u)\,dx\right| \\ &\le \int_{\mathbb{R}^N} \left|\frac{m(x)}{q(x)}|u|^{q(x)}\right| dx \\ &\le \frac{2}{q_-}\|m\|_{L^{\gamma(x)}(\mathbb{R}^N)} \left\| |u|^{q(x)} \right\|_{L^{\frac{r(x)}{q(x)}}(\mathbb{R}^N)} \\ &\le \frac{2}{q_-}\|m\|_{L^{\gamma(x)}(\mathbb{R}^N)} \left(\|u\|^{q_+}_{L^{r(x)}(\mathbb{R}^N)} + \|u\|^{q_-}_{L^{r(x)}(\mathbb{R}^N)}\right) \\ &\le \frac{2C}{q_-}\|m\|_{L^{\gamma(x)}(\mathbb{R}^N)} \left(\|u\|_X^{q_+} + \|u\|_X^{q_-}\right) \end{aligned} \tag{3.8}$$

for some positive constant C. Let u in X with $\|u\|_X \le 1$. Then it follows from (HJ3), (HJ4), (3.8) and Lemma 2.3(3) that

$$\left|\frac{\Phi(u)}{\Psi(u)}\right| \ge \frac{\int_{\mathbb{R}^N} \Phi_0(x,|\nabla u|)\,dx}{\frac{4C}{q_-}\|m\|_{L^{\gamma(x)}(\mathbb{R}^N)}\|u\|_X^{q_-}} \ge \frac{\frac{c}{p_+}\|u\|_X^{p_+}}{\frac{4C}{q_-}\|m\|_{L^{\gamma(x)}(\mathbb{R}^N)}\|u\|_X^{q_-}}. \tag{3.9}$$

Since $q_- > p_+$, we conclude that

$$\frac{\Phi(u)}{\Psi(u)} \to \infty \quad \text{as } \|u\|_X \to 0.$$

Next, we show that relation (3.7) holds. From (H2), there exists a positive constant δ such that $q_+ - (1/2)p_- < \delta < q_-$, and thus we have

$$p_- > 2(q_+ - \delta) > 2(q_- - \delta). \tag{3.10}$$

Let $\ell(x)$ be a measurable function such that

$$\begin{aligned}&\max\left\{\frac{p(x)\gamma(x)}{p(x)+\delta\gamma(x)},\frac{p^*(x)}{p^*(x)+\delta-q(x)}\right\}\\&\quad\le\ell(x)\le\min\left\{\frac{p^*(x)\gamma(x)}{p^*(x)+\delta\gamma(x)},\frac{p(x)}{p(x)+\delta-q(x)}\right\}\end{aligned}\tag{3.11}$$

holds for almost all $x\in\mathbb{R}^N$ and

$$\delta\left(\frac{\ell_+}{\ell_-}+1\right)<q_-.\tag{3.12}$$

Then we have $\ell\in L^\infty(\mathbb{R}^N)$ and $1<\ell(x)<\gamma(x)$. Let $u\in X$ with $\|u\|_X>1$. Then it follows from (F1′) and Lemma 2.2 that

$$\begin{aligned}\left|\Psi(u)\right|&\le\frac{1}{q_-}\int_{\mathbb{R}^N}m(x)|u|^\delta|u|^{q(x)-\delta}\,dx\\&\le\frac{2}{q_-}\left\|m|u|^\delta\right\|_{L^{\ell(x)}(\mathbb{R}^N)}\left\||u|^{q(x)-\delta}\right\|_{L^{\ell'(x)}(\mathbb{R}^N)}.\end{aligned}$$

Therefore, without loss of generality, we may suppose that $\|m|u|^\delta\|_{L^{\ell(x)}(\mathbb{R}^N)}>1$. From the inequality above, by using Lemma 2.3, Lemma 2.2 and Lemma 2.4 in order, we have

$$\begin{aligned}\left|\Psi(u)\right|&\le\frac{2}{q_-}\left(\int_{\mathbb{R}^N}m^{\ell(x)}|u|^{\delta\ell(x)}\right)^{\frac{1}{\ell_-}}\left\||u|^{q(x)-\delta}\right\|_{L^{\ell'(x)}(\mathbb{R}^N)}\\&\le\frac{4}{q_-}\left\|m^{\ell(x)}\right\|^{\frac{1}{\ell_-}}_{L^{\frac{\gamma(x)}{\ell(x)}}(\mathbb{R}^N)}\left\||u|^{\delta\ell(x)}\right\|^{\frac{1}{\ell_-}}_{L^{(\frac{\gamma(x)}{\ell(x)})'}(\mathbb{R}^N)}\left\||u|^{q(x)-\delta}\right\|_{L^{\ell'(x)}(\mathbb{R}^N)}\\&\le\frac{4}{q_-}\|m\|^\alpha_{L^{\gamma(x)}(\mathbb{R}^N)}\left(\|u\|^{\delta\frac{\ell_+}{\ell_-}}_{L^{\delta\ell(x)(\frac{\gamma(x)}{\ell(x)})'}(\mathbb{R}^N)}+\|u\|^\delta_{L^{\delta\ell(x)(\frac{\gamma(x)}{\ell(x)})'}(\mathbb{R}^N)}\right)\\&\quad\times\left(\|u\|^{q_+-\delta}_{L^{(q(x)-\delta)\ell'(x)}(\mathbb{R}^N)}+\|u\|^{q_--\delta}_{L^{(q(x)-\delta)\ell'(x)}(\mathbb{R}^N)}\right),\end{aligned}$$

where $\alpha=\begin{cases}\ell_+/\ell_- & \text{if } \|m\|_{L^{\gamma(x)}(\mathbb{R}^N)}>1,\\ 1 & \text{if } \|m\|_{L^{\gamma(x)}(\mathbb{R}^N)}\le1.\end{cases}$

By Young's inequality, we get

$$\begin{aligned}\left|\Psi(u)\right|&\le\frac{4}{q_-}\|m\|^\alpha_{L^{\gamma(x)}(\mathbb{R}^N)}\left(\|u\|^{2\delta\frac{\ell_+}{\ell_-}}_{L^{\delta\ell(x)(\frac{\gamma(x)}{\ell(x)})'}(\mathbb{R}^N)}+\|u\|^{2\delta}_{L^{\delta\ell(x)(\frac{\gamma(x)}{\ell(x)})'}(\mathbb{R}^N)}\right.\\&\quad\left.+\|u\|^{2(q_+-\delta)}_{L^{(q(x)-\delta)\ell'(x)}(\mathbb{R}^N)}+\|u\|^{2(q_--\delta)}_{L^{(q(x)-\delta)\ell'(x)}(\mathbb{R}^N)}\right).\end{aligned}$$

Using (3.11), we get that

$$p(x)<\delta\ell(x)\left(\frac{\gamma(x)}{\ell(x)}\right)'\le p^*(x),\qquad p(x)<\bigl(q(x)-\delta\bigr)\ell'(x)\le p^*(x)$$

holds for almost all $x\in\mathbb{R}^N$. Hence it follows from Lemma 2.6 that

$$\left|\Psi(u)\right|\le\frac{4C}{q_-}\|m\|^\alpha_{L^{\gamma(x)}(\mathbb{R}^N)}\left(\|u\|_X^{2\delta\frac{\ell_+}{\ell_-}}+\|u\|_X^{2(q_+-\delta)}\right)\tag{3.13}$$

for some positive constant C. Therefore, we obtain that

$$\left|\frac{\Phi(u)}{\Psi(u)}\right| \geq \frac{\frac{c}{p_+}\|u\|_X^{p_-}}{\frac{4C}{q_-}\|m\|^{\alpha}_{L^{\gamma(x)}(\mathbb{R}^N)}(\|u\|_X^{2\delta\frac{\ell_+}{\ell_-}} + \|u\|_X^{2(q_+-\delta)})}.$$

From (3.10), with the inequality above, we conclude that relation (3.7) holds. □

Lemma 3.3 *Assume that* (HJ1)-(HJ3) *and* (H1) *hold. Then* Φ *is weakly lower semicontinuous, i.e.,* $u_n \rightharpoonup u$ *in* X *implies that* $\Phi(u) \leq \liminf_{n\to\infty}\Phi(u_n)$.

Proof Suppose that $u_n \rightharpoonup u$ in X as $n \to \infty$. Since (HJ3) implies that Φ' is strictly monotone on X, we have that Φ is convex, and so,

$$\Phi(u_n) \geq \Phi(u) + \langle \Phi'(u), u_n - u\rangle$$

for any n. Then we get that

$$\liminf_{n\to\infty}\Phi(u_n) \geq \Phi(u) + \liminf_{n\to\infty}\langle \Phi'(u), u_n - u\rangle = \Phi(u).$$

The proof is complete. □

Lemma 3.4 *Assume that* (H1) *and* (F1) *hold. For any* $K \in [0,\infty)$ *and all* $u \in X$, *the following estimate holds*:

$$\int_{|x|\geq K} F(x,u)\,dx \leq \frac{2C}{q_-}\left(\int_{|x|\geq K} m(x)\,dx\right)^{\frac{1}{\gamma_1}}\left(\|u\|_X^{q_+} + \|u\|_X^{q_-}\right), \tag{3.14}$$

where γ_1 *is either* γ_+ *or* γ_-.

Proof Applying Lemmas 2.2, 2.4 and 2.6, we get

$$\begin{aligned}
\int_{|x|\geq K} F(x,u)\,dx &\leq \int_{|x|\geq K}\frac{m(x)}{q(x)}|u|^{q(x)}\,dx \\
&\leq \frac{2}{q_-}\|m\|_{L^{\gamma(x)}(\{x\in\mathbb{R}^N:|x|\geq K\})}\left\||u|^{q(x)}\right\|_{L^{\frac{r(x)}{q(x)}}(\{x\in\mathbb{R}^N:|x|\geq K\})} \\
&\leq \frac{2}{q_-}\left(\int_{|x|\geq K} m(x)\,dx\right)^{\frac{1}{\gamma_1}}\left(\|u\|^{q_+}_{L^{r(x)}(\mathbb{R}^N)} + \|u\|^{q_-}_{L^{r(x)}(\mathbb{R}^N)}\right) \\
&\leq \frac{2C}{q_-}\left(\int_{|x|\geq K} m(x)\,dx\right)^{\frac{1}{\gamma_1}}\left(\|u\|_X^{q_+} + \|u\|_X^{q_-}\right)
\end{aligned}$$

for some positive constant C. □

Lemma 3.5 *Assume that* (H1) *and* (F1) *hold. For almost all* $x \in \mathbb{R}^N$ *and all* $t \in \mathbb{R}$, *the following estimate holds*:

$$F(x,t) \leq \frac{1}{q_-}\left(\frac{m(x)^{\gamma(x)}}{\gamma_-} + \frac{|t|^{r(x)}}{(\gamma_+)'}\right). \tag{3.15}$$

Proof Since $q(x)(\gamma(x))' = r(x)$, estimate (3.15) is obtained from (F1′) and Young's inequality. □

Lemma 3.6 *Assume that* (H1) *and* (F1) *hold. Then* Ψ *is weakly-strongly continuous, i.e.,* $u_n \rightharpoonup u$ *in* X *implies that* $\Psi(u_n) \to \Psi(u)$.

Proof Let $\{u_n\}$ be a sequence in X such that $u_n \rightharpoonup u$ in X. Then $\{u_n\}$ is bounded in X. By Lemma 3.4, for each $\varepsilon > 0$, there is a positive constant K_ε such that

$$\int_{|x|\geq K_\varepsilon} F(x,u_n)\,dx \leq \varepsilon \quad \text{and} \quad \int_{|x|\geq K_\varepsilon} F(x,u)\,dx \leq \varepsilon \tag{3.16}$$

holds for each $n \in \mathbb{N}$. It follows from Lemma 3.5 that the Nemytskij operator

$$u \mapsto F\bigl(x,u(x)\bigr)$$

is continuous from $L^{r(x)}(B_{K_\varepsilon}(0))$ into $L^1(B_{K_\varepsilon}(0))$; see Theorem 1.1 in [26]. This together with Lemma 2.5 yields that

$$\int_{|x|<K_\varepsilon} F(x,u_n)\,dx \to \int_{|x|<K_\varepsilon} F(x,u)\,dx. \tag{3.17}$$

Using (3.16) and (3.17), we deduce that $\Psi(u_n) \to \Psi(u)$ as $n \to \infty$. The proof is complete. □

We are in a position to state the main result about the existence of the positive eigenvalue for the problem (E).

Theorem 3.7 *Assume that* (HJ1)-(HJ4), (H1), (H2), *and* (F1) *hold. Then* λ^* *is a positive eigenvalue of the problem* (E). *Moreover, the problem* (E) *has a nontrivial weak solution for any* $\lambda \geq \lambda^*$.

Proof It is trivial by (3.5) that $\lambda^* \geq 0$. Suppose to the contrary that $\lambda^* = 0$. Let $\{u_n\}$ be a sequence in $X \setminus \{0\}$ such that

$$\lim_{n\to\infty} \frac{\Phi(u_n)}{\Psi(u_n)} = 0.$$

As in (3.9), we have

$$\left|\frac{\Phi(u_n)}{\Psi(u_n)}\right| \geq C\|u_n\|_X^{p_+-q_-}$$

for some positive constant C. Since $p_+ < q_-$, we obtain that $\|u_n\|_X \to \infty$ as $n \to \infty$. Hence it follows from Lemma 3.2 that

$$\lim_{n\to\infty} \frac{\Phi(u_n)}{\Psi(u_n)} = \infty,$$

which contradicts with the hypothesis. Hence we get $\lambda^* > 0$. The analogous argument as that in the proof of Theorem 4.5 in [5] proves that λ^* is an eigenvalue of the problem (E); see also Theorem 3.1 in [6].

Finally, we show that the problem (E) has a nontrivial weak solution for any $\lambda \geq \lambda^*$. Notice that u is a weak solution of (E) if and only if u is a critical point of I_λ. Assume that $\lambda > \lambda^*$ is fixed. Let $u \in X$ with $\|u\|_X > 1$. With the help of (HJ3) and (HJ4), it follows from proceeding as in the proof of relation (3.13) in Lemma 3.2 that

$$I_\lambda(u) \geq \frac{c}{p_+}\|u\|_X^{p_-} - \lambda\frac{4C}{q_-}\|m\|^{\alpha}_{L^{\gamma(x)}(\mathbb{R}^N)}\left(\|u\|_X^{2\delta\frac{\ell_+}{\ell_-}} + \|u\|_X^{2(q_+-\delta)}\right).$$

Since $p_- > 2(q_+ - \delta) > 2\delta(\ell_+/\ell_-)$, the inequality above implies that $I_\lambda(u) \to \infty$ as $\|u\|_X \to \infty$ for $\lambda > \lambda^*$, that is, I_λ is coercive. Also since the functional I_λ is weakly lower semicontinuous by Lemmas 3.3 and 3.6, we deduce that there exists a global minimizer u_0 of I_λ in X. Since $\lambda > \lambda^*$, we verify by definition (3.5) that there is an element ω in $X \setminus \{0\}$ such that $\Phi(\omega)/\Psi(\omega) < \lambda$. Then $I_\lambda(\omega) < 0$. So we obtain that

$$I_\lambda(u_0) = \inf_{v\in X\setminus\{0\}} I_\lambda(v) < 0.$$

Consequently, we conclude that $u_0 \not\equiv 0$. This completes the proof. □

Now, we consider an example to demonstrate our main result in this section.

Example 3.8 Let $p \in C(\mathbb{R}^N)$ with $2 \leq p(x) < N$ satisfy the log-Hölder continuity condition (2.2). Suppose that $a \in L^{2p'(x)}(\mathbb{R}^N)$, and there is a positive constant a_0 such that $a(x) \geq a_0$ for almost all $x \in \mathbb{R}^N$. Let us consider

$$-\operatorname{div}\left(\left(a(x) + |\nabla u|^2\right)^{\frac{p(x)-2}{2}}\nabla u\right) = \lambda m(x)|u|^{q(x)-2}u \quad \text{in } \mathbb{R}^N. \tag{E$_0$}$$

In this case, put

$$\phi\left(x,|v|\right) = \left(a(x) + |v|^2\right)^{\frac{p(x)-2}{2}} \quad \text{and} \quad \Phi_0\left(x,|v|\right) = \frac{1}{p(x)}\left(a(x) + |v|^2\right)^{\frac{p(x)}{2}}$$

for all $v \in \mathbb{R}^N$. Denote the quantities

$$\lambda^* = \inf_{u\in X\setminus\{0\}} \frac{\int_{\mathbb{R}^N}\frac{1}{p(x)}(a(x)+|\nabla u|^2)^{\frac{p(x)}{2}}\,dx}{\int_{\mathbb{R}^N}\frac{m(x)}{p(x)}|u|^{q(x)}\,dx} \quad \text{and} \quad \lambda_* = \inf_{u\in X\setminus\{0\}} \frac{\int_{\mathbb{R}^N}(a(x)+|\nabla u|^2)^{\frac{p(x)}{2}}\,dx}{\int_{\mathbb{R}^N} m(x)|u|^{q(x)}\,dx}.$$

If conditions (H1)-(H2) hold, then we have

(i) $0 < \lambda_* \leq \lambda^*$,
(ii) λ^* is a positive eigenvalue of the problem (E$_0$),
(iii) the problem (E$_0$) has a nontrivial weak solution for any $\lambda \geq \lambda^*$,
(iv) λ is not an eigenvalue of (E$_0$) for $\lambda < \lambda_*$.

Proof It is clear that conditions (HJ1)-(HJ4) and (F1) hold. From the definitions of λ_* and λ^*, we know that

$$\frac{q_-}{p_+}\lambda_* \leq \lambda^* \leq \frac{q_+}{p_-}\lambda_*,$$

and thus $\lambda_* \leq \lambda^*$. Also, from the same argument as that in Theorem 3.7, we have $\lambda^* > 0$, and thus $\lambda_* > 0$. Applying Theorem 3.7, the conclusions (ii) and (iii) hold. Let $\lambda < \lambda_*$. Suppose that λ is an eigenvalue of the problem (E_0). Then there is an element $v \in X \setminus \{0\}$ such that

$$\int_{\mathbb{R}^N} \bigl(a(x) + |\nabla v|^2\bigr)^{\frac{p(x)}{2}}\,dx - \lambda \int_{\mathbb{R}^N} m(x)|v|^{q(x)}\,dx = 0.$$

By the definition of λ_*, we get that

$$\begin{aligned}\lambda_* \int_{\mathbb{R}^N} m(x)|v|^{q(x)}\,dx &\leq \int_{\mathbb{R}^N} \bigl(a(x) + |\nabla v|^2\bigr)^{\frac{p(x)}{2}}\,dx \\ &= \lambda \int_{\mathbb{R}^N} m(x)|v|^{q(x)}\,dx < \lambda_* \int_{\mathbb{R}^N} m(x)|v|^{q(x)}\,dx,\end{aligned}$$

a contradiction. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to the manuscript and read and approved the final manuscript.

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