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A relaxed hybrid shrinking iteration approach to solving generalized mixed equilibrium problems for totally quasi-ϕ-asymptotically nonexpansive mappings

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Abstract

An optimal existing method for the approximation of common fixed points of countable families of nonlinear operators is introduced, by which a relaxed hybrid shrinking iterative algorithm is developed for the class of totally quasi-ϕ-asymptotically nonexpansive mappings, and a strong convergence theorem for solving generalized mixed equilibrium problems is established in the framework of Banach spaces. Since there is no need to impose the uniformity assumption on the involved countable family of mappings and no need to compute a complex series at each step in the iteration process, the result is more widely applicable than those of other authors with related interests.

MSC: 47J06; 47J25

Keywords: totally quasi-ϕ-asymptotically nonexpansive mappings; generalized mixed equilibrium problems; generalized projections

1 Introduction

Throughout this paper we assume that E is a real Banach space with its dual E^*, C is a nonempty closed convex subset of E and $J : E \to 2^{E^*}$ is the *normalized duality mapping* defined by

$$Jx = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}, \quad \forall x \in E.$$

In the sequel, we use $F(T)$ to denote the set of fixed points of a mapping T.

Definition 1.1 [1] (1) A mapping $T : C \to C$ is said to be *totally quasi-ϕ-asymptotically nonexpansive*, if $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\{v_n\}$, $\{\mu_n\}$ with $v_n, \mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ with $\zeta(0) = 0$ such that

$$\phi\left(p, T^n x\right) \leq \phi(p, x) + v_n \zeta\left(\phi(p, x)\right) + \mu_n, \quad \forall n \geq 1, x \in C, p \in F(T), \tag{1.1}$$

where $\phi : E \times E \to \mathbb{R}^+ \cup \{0\}$ denotes the *Lyapunov functional* defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \tag{1.2}$$

It is obvious from the definition of ϕ that

$$\big(\|x\| - \|y\|\big)^2 \le \phi(x,y) \le \big(\|x\| + \|y\|\big)^2, \quad \forall x,y \in E \tag{1.3}$$

and

$$\phi\big(x, J^{-1}(\lambda Jy + (1-\lambda)Jz)\big) \le \lambda \phi(x,y) + (1-\lambda)\phi(x,z), \quad \forall x,y \in E, \lambda \in [0,1]. \tag{1.4}$$

(2) A countable family of mappings $\{T_i\}_{i=1}^{\infty} : C \to C$ said to be *uniformly quasi-ϕ-asymptotically nonexpansive*, if $F := \bigcap_{i=1}^{\infty} F(T_i) \ne \emptyset$ and there exists a nonnegative real sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ (as $n \to \infty$) such that

$$\phi\big(p, T_i^n x\big) \le k_n \phi(p,x), \quad \forall n \ge 1, i \ge 1, x \in C, p \in F(T). \tag{1.5}$$

(3) A countable family of mappings $\{T_i\}_{i=1}^{\infty} : C \to C$ said to be *uniformly totally quasi-ϕ-asymptotically nonexpansive*, if $F := \bigcap_{i=1}^{\infty} F(T_i) \ne \emptyset$ and there exist nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n, \mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ with $\zeta(0) = 0$ such that

$$\phi\big(p, T_i^n x\big) \le \phi(p,x) + \nu_n \zeta\big(\phi(p,x)\big) + \mu_n, \quad \forall n \ge 1, i \ge 1, x \in C, p \in F(T). \tag{1.6}$$

(4) A mapping $T : C \to C$ is said to be *uniformly L-Lipschitz continuous*, if there exists a constant $L > 0$ such that

$$\big\|T^n x - T^n y\big\| \le L\|x - y\|, \quad \forall n \ge 1, x,y \in C. \tag{1.7}$$

Let $\theta : C \times C \to \mathbb{R}$ be a bifunction, $\psi : C \to \mathbb{R}$ a real valued function and $A : C \to E^*$ a nonlinear mapping. The so-called *generalized mixed equilibrium problem* (*GMEP*) is to find an $u \in C$ such that

$$\theta(u,y) + \langle Au, y - u \rangle + \psi(y) - \psi(u) \ge 0, \quad \forall y \in C, \tag{1.8}$$

whose set of solutions is denoted by Ω. The equilibrium problem is an unifying model for several problems arising in physics, engineering, science optimization, economics, transportation, network and structural analysis, Nash equilibrium problems in noncooperative games, and others. It has been shown that variational inequalities and mathematical programming problems can be viewed as a special realization of the abstract equilibrium problems. Many authors have proposed some useful methods to solve the EP (equilibrium problem), GEP (generalized equilibrium problem), MEP (mixed equilibrium problem), and GMEP. Concerning the weak and strong convergence of iterative sequences to a common element of the set of solutions for the GMEP, the set of solutions for variational inequality problems, and the set of common fixed points for relatively nonexpansive mappings, quasi-ϕ-nonexpansive mappings, quasi-ϕ-asymptotically nonexpansive mappings and total quasi-ϕ-asymptotically nonexpansive mappings have been studied by many authors in the setting of Hilbert or Banach spaces (see, for example, [2–17] and the references therein).

In 2010, Qin *et al.* [18] proposed the following shrinking projection method to find a common element of the set of solutions of an equilibrium problem and the set of common fixed points of a finite family of quasi-ϕ-nonexpansive mappings in the framework of Banach spaces:

$$
\begin{cases}
x_0 \in C; \quad C_0 = C, \\
y_n = J^{-1}[\alpha_{n,0} J x_n + \sum_{i=1}^{N} \alpha_{n,i} J T_i x_n], \\
u_n \in C \quad \text{such that } \forall y \in C, \\
\theta(u_n, y) + \frac{1}{r_n} \langle y - u_n, J u_n - J y_n \rangle \geq 0, \\
C_{n+1} = \{ v \in C_n : \phi(v, u_n) \leq \phi(v, x_n) \}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0,
\end{cases}
\tag{1.9}
$$

where $\Pi_{C_{n+1}}$ is the *generalized projection* (see (2.1)) of E onto C_{n+1}.

In 2011, Saewan and Kumam [19] introduced a modified new hybrid projection method to find a common element of the set of solutions of the generalized mixed equilibrium problems and the set of common fixed points of an infinite family of closed and *uniformly quasi-ϕ-asymptotically nonexpansive mappings* in an uniformly smooth and strictly convex Banach spaces E with the Kadec-Klee property:

$$
\begin{cases}
x_0 \in C; \quad C_0 = C, \\
y_n = J^{-1}[\alpha_n J x_n + (1 - \alpha_n) J z_n], \\
z_n = J^{-1}[\beta_{n,0} J x_n + \sum_{i=1}^{\infty} \beta_{n,i} J T_i^n x_n], \\
u_n \in C \quad \text{such that } u_n = K_{r_n} y_n, \\
C_{n+1} = \{ z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \xi_n \}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0,
\end{cases}
\tag{1.10}
$$

where $\xi_n := (k_n - 1) \sup_{p \in F} \zeta(\phi(p, x_n))$.

However, it is obviously a quite strong condition that the involved mappings are assumed to be a countable family of uniformly $(\{v_n\}, \{\mu_n\}, \zeta)$-quasi-$\phi$-asymptotically nonexpansive ones, which is a special case of totally quasi-ϕ-asymptotically nonexpansive mappings (see [10]). *In addition, the accurate computation of the series $\sum_{i=1}^{\infty} \beta_{n,i} J T_i^n y_n$ at each step of the iteration process is not easily attainable, which will lead to gradually increasing errors.*

Inspired and motivated by the studies mentioned above, by using a special way of choosing the indices, we propose a relaxed hybrid shrinking iteration scheme for approximating common fixed points of a countable family of totally quasi-ϕ-asymptotically nonexpansive mappings and obtain a strong convergence theorem for solving the generalized mixed equilibrium problems under suitable conditions, namely, *there is no need to assume uniformity for the totally quasi-ϕ-asymptotic property of the involved mappings, and no need to compute complex series in the iteration process.* The results extend and improve those of other authors with related interests.

2 Preliminaries

We say that a Banach space E is *strictly convex* if the following implication holds for $x, y \in E$:

$$
\|x\| = \|y\| = 1, \qquad x \neq y \quad \Rightarrow \quad \left\| \frac{x + y}{2} \right\| < 1.
\tag{2.1}
$$

It is also said to be *uniformly convex* if for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \qquad \|x - y\| \geq \epsilon \quad \Rightarrow \quad \left\|\frac{x + y}{2}\right\| \leq 1 - \delta. \tag{2.2}$$

It is well known that if E is a uniformly convex Banach space, then E is reflexive and strictly convex. A Banach space E is said to be *smooth* if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.3}$$

exists for each $x, y \in S(E) := \{x \in E : \|x\| = 1\}$. E is said to be *uniformly smooth* if the limit (2.3) is attained uniformly for $x, y \in S(E)$.

Following Alber [20], the *generalized projection* $\Pi_C : E \to C$ is defined by

$$\Pi_C = \arg \inf_{y \in C} \phi(y, x), \quad \forall x \in E. \tag{2.4}$$

Lemma 2.1 [20] *Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Then the following conclusions hold*:

(1) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y)$ *for all $x \in C$ and $y \in E$*;
(2) *If $x \in E$ and $z \in C$, then $z = \Pi_C x \Leftrightarrow \langle z - y, Jx - Jz \rangle \geq 0, \forall y \in C$*;
(3) *For $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$*.

Remark 2.2 The following basic properties for a Banach space E can be found in Cioranescu [21].

  (i) If E is uniformly smooth, then J is uniformly continuous on each bounded subset of E;
 (ii) If E is reflexive and strictly convex, then J^{-1} is norm-weak-continuous;
(iii) If E is reflexive smooth and strictly convex, then the normalized duality mapping J is single-valued, one-to-one and onto;
(iv) A Banach space E is uniformly smooth if and only if E^* is uniformly convex;
 (v) Each uniformly convex Banach space E has the *Kadec-Klee property*, *i.e.*, for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$ as $n \to \infty$.

Lemma 2.3 [22] *Let E be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, and C be a nonempty closed convex subset of E. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in C such that $x_n \to p$ and $\phi(x_n, y_n) \to 0$, where ϕ is the function defined by (1.2), then $y_n \to p$.*

Lemma 2.4 [22] *Let E and C be the same as in Lemma 2.3. Let $T : C \to C$ be a closed and totally quasi-ϕ-asymptotically nonexpansive mappings with nonnegative real sequences $\{v_n\}$, $\{\mu_n\}$ and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ such that $v_n, \mu_n \to 0$ and $\zeta(0) = 0$. If $\mu_1 = 0$, then the fixed point set $F(T)$ of T is a closed and convex subset of C.*

Lemma 2.5 [1] *Let E be a real uniformly convex Banach space and let $B_r(0)$ be the closed ball of E with center at the origin and radius $r > 0$. Then for any for any sequence $\{x_i\} \subset B_r(0)$ and for any sequence $\{\lambda_i\}$ of positive numbers with $\sum_{i=1}^{\infty} \lambda_i = 1$, there exists a*

continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ *with* $g(0) = 0$ *such that such that for any positive integer* $i \neq 1$, *the following hold*:

$$\left\| \sum_{i=1}^{\infty} \lambda_i x_i \right\|^2 \leq \sum_{i=1}^{\infty} \lambda_i \|x_i\|^2 - \lambda_1 \lambda_i g(\|x_1 - x_i\|), \tag{2.5}$$

and for all $x \in E$,

$$\phi\left(x, J^{-1}\left(\sum_{i=1}^{\infty} \lambda_i J x_i\right)\right) \leq \sum_{i=1}^{\infty} \lambda_i \phi(x, x_i) - \lambda_1 \lambda_i g(\|J x_1 - J x_i\|). \tag{2.6}$$

Assume that, to obtain the solution of GMEP, the function $\psi : C \to \mathbb{R}$ is convex and lower semicontinuous, the nonlinear mapping $A : C \to E^*$ is continuous and monotone, and the bifunction $\theta : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A₁) $\theta(x, x) = 0$;
(A₂) θ is monotone, *i.e.*, $\theta(x, y) + \theta(y, x) \leq 0$;
(A₃) $\limsup_{t \downarrow 0} \theta(x + t(z - x), y) \leq \theta(x, y)$;
(A₄) the mapping $y \mapsto \theta(x, y)$ is convex and lower semicontinuous.

Lemma 2.6 [15] *Let E be a smooth, strictly convex, and reflexive Banach space, and C be a nonempty closed convex subset of E. Let* $A : C \to E^*$ *be a continuous and monotone mapping,* $\psi : C \to \mathbb{R}$ *a lower semicontinuous and convex function, and* $\theta : C \times C \to \mathbb{R}$ *a bifunction satisfying the conditions* (A₁)-(A₄). *Let* $r > 0$ *and* $x \in E$. *Then, the following hold*:

(1) *There exists an* $u \in C$ *such that*

$$\theta(u, y) + \langle Au, y - u \rangle + \psi(y) - \psi(u) + \frac{1}{r}\langle y - u, Ju - Jx \rangle \geq 0, \quad \forall y \in C.$$

(2) *A mapping* $\kappa_r : C \to C$ *is defined by*

$$\kappa_r(x) = \left\{ u \in C : \theta(u, y) + \langle Au, y - u \rangle + \psi(y) - \psi(u) + \frac{1}{r}\langle y - u, Ju - Jx \rangle \geq 0, \forall y \in C \right\}.$$

Then, the mapping κ_r *has the following properties*:
  (i) κ_r *is single-valued*;
 (ii) κ_r *a firmly nonexpansive-type mapping, i.e.*,

$$\langle \kappa_r z - \kappa_r y, J \kappa_r z - J \kappa_r y \rangle \leq \langle \kappa_r z - \kappa_r y, Jz - Jy \rangle;$$

(iii) $F(\kappa_r) = \Omega = \tilde{F}(\kappa_r)$;
(iv) Ω *is a closed convex set of* C;
 (v) $\phi(p, \kappa_r z) + \phi(\kappa_r z, z) \leq \phi(p, z), \forall p \in F(\kappa_r), z \in E$,
where $\tilde{F}(\kappa_r)$ *denotes the set of asymptotic fixed points of* κ_r, *i.e.*,

$$\tilde{F}(\kappa_r) := \left\{ x \in C : \exists \{x_n\} \subset C, s.t., x_n \rightharpoonup x, \|x_n - \kappa_r x_n\| \to 0 \ (n \to \infty) \right\}.$$

Lemma 2.7 [23] *The unique solutions to the positive integer equation*

$$n = i + \frac{(m-1)m}{2}, \quad m \geq i, n = 1, 2, \ldots, \tag{2.7}$$

are

$$i = n - \frac{(m-1)m}{2}, \qquad m = -\left[\frac{1}{2} - \sqrt{2n + \frac{1}{4}}\right], \quad n = 1, 2, \ldots, \tag{2.8}$$

where $[x]$ *denotes the maximal integer that is not larger than* x.

3 Main results

Recall that a mapping T on a Banach space is closed if $x_n \to x$ and $Tx_n \to y$ as $n \to \infty$, then $Tx = y$.

Theorem 3.1 *Let E be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property and C a nonempty closed convex subset of E. Let $\theta : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions* (A$_1$)-(A$_4$), *$A : C \to E^*$ a continuous and monotone mapping, and $\psi : C \to \mathbb{R}$ a lower semicontinuous and convex function. Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be a countable family of closed and totally quasi-ϕ-asymptotically nonexpansive mappings with nonnegative real sequences $\{v_n^{(i)}\}$, $\{\mu_n^{(i)}\}$ satisfying $v_n^{(i)} \to 0$ and $\mu_n^{(i)} \to 0$ (as $n \to \infty$ and for each $i \geq 1$) and a sequence of strictly increasing and continuous functions $\{\zeta_i\} :$ $\mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ satisfying condition* (1.1). *Assume that each T_i is uniformly L_i-Lipschitz continuous with $\mu_1^{(i)} = 0$ for each $i \geq 1$. Let $\{\alpha_n\}$ be a sequence in $[0, \epsilon]$ for some $\epsilon \in (0, 1)$ and $\{\beta_i\}$ be a sequence in $(0, 1)$. Let $\{x_n\}$ be the sequence generated by*

$$
\begin{cases}
x_1 \in C; \quad C_1 = C, \\
y_n = J^{-1}[\alpha_n J x_n + (1 - \alpha_n) J z_n], \\
z_n = J^{-1}[\beta_{i_n} J x_n + (1 - \beta_{i_n}) J T_{i_n}^{m_n} x_n], \\
u_n \in C \quad \text{such that } \forall y \in C, \\
\theta(u_n, y) + \langle A u_n, y - u_n \rangle + \psi(y) - \psi(u_n) + \frac{1}{r_n} \langle y - u_n, J u_n - J y_n \rangle \geq 0, \\
C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n) + \xi_n\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n \in \mathbb{N},
\end{cases}
\tag{3.1}
$$

where $\xi_n := v_{m_n}^{(i_n)} \sup_{p \in F} \zeta_{i_n}(\phi(p, x_n)) + \mu_{m_n}^{(i_n)}$ and $\Pi_{C_{n+1}}$ is the generalized projection of E onto C_{n+1}; i_n and m_n are the solutions to the positive integer equation: $n = i + \frac{(m-1)m}{2}$ ($m \geq i$, $n = 1, 2, \ldots$), that is, for each $n \geq 1$, there exist unique i_n and m_n such that

$$i_1 = 1, \qquad i_2 = 1, \qquad i_3 = 2, \qquad i_4 = 1,$$

$$i_5 = 2, \qquad i_6 = 3, \qquad i_7 = 1, \qquad i_8 = 2, \ldots;$$

$$m_1 = 1, \qquad m_2 = 2, \qquad m_3 = 2, \qquad m_4 = 3,$$

$$m_5 = 3, \qquad m_6 = 3, \qquad m_7 = 4, \qquad m_8 = 4, \ldots.$$

If $G := F \cap \Omega \neq \emptyset$ and $F := \bigcap_{i=1}^{\infty} F(T_i)$ is bounded, then $\{x_n\}$ converges strongly to $\Pi_G x_1$.

Proof Two functions $\tau : C \times C \to \mathbb{R}$ and $\kappa_r : C \to C$ are defined by

$$\tau(x,y) = \theta(x,y) + \langle Ax, y - x \rangle + \psi(y) - \psi(x);$$

$$\kappa_r(x) = \left\{ u \in C : \tau(u,y) + \frac{1}{r}\langle y - u, Ju - Jx \rangle \geq 0, \forall y \in C \right\}.$$

By Lemma 2.6, we know that the function τ satisfies the conditions (A$_1$)-(A$_4$) and κ_r has the property (i)-(v). Therefore, (3.1) can be rewritten as

$$\begin{cases} x_1 \in C; \quad C_1 = C, \\ y_n = J^{-1}[\alpha_n J x_n + (1 - \alpha_n)J z_n], \\ z_n = J^{-1}[\beta_{i_n} J x_n + (1 - \beta_{i_n})J T_{i_n}^{m_n} x_n], \\ u_n \in C \quad \text{such that } \tau(u_n,y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1. \end{cases} \tag{3.2}$$

We divide the proof into several steps.

(I) F and C_n ($\forall n \geq 1$) both are closed and convex subsets in C.

In fact, it follows from Lemma 2.4 that each $F(T_i)$ is a closed and convex subset of C, so is F. In addition, with C_1 (= C) being closed and convex, we may assume that C_n is closed and convex for some $n \geq 2$. In view of the definition of ϕ we have

$$C_{n+1} = \{v \in C : \varphi(v) \leq a\} \cap C_n,$$

where $\varphi(v) = 2\langle v, Jx_n - Jy_n \rangle$ and $a = \|x_n\|^2 - \|y_n\|^2 + \xi_n$. This shows that C_{n+1} is closed and convex.

(II) G is a subset of $\bigcap_{n=1}^{\infty} C_n$.

It is obvious that $G \subset C_1$. Suppose that $G \subset C_n$ for some $n \geq 2$. Since $u_n = \kappa_{r_n} y_n$, by Lemma 2.6, it is easily shown that κ_{r_n} is quasi-ϕ-nonexpansive. Hence, for any $p \in G \subset C_n$, it follows from (1.4) that

$$\begin{aligned} \phi(p, u_n) &= \phi(p, \kappa_{r_n} y_n) \leq \phi(p, y_n) \\ &= \phi\big(p, J^{-1}[\alpha_n J x_n + (1 - \alpha_n)J z_n]\big) \\ &\leq \alpha_n \phi(p, x_n) + (1 - \alpha_n)\phi(p, z_n). \end{aligned} \tag{3.3}$$

Furthermore, it follows from Lemma 2.5 that for any $p \in G \subset C_n$,

$$\begin{aligned} \phi(p, z_n) &= \phi\big(p, J^{-1}[\beta_{i_n} J x_n + (1 - \beta_{i_n})J T_{i_n}^{m_n} x_n]\big) \\ &\leq \beta_{i_n}\phi(p, x_n) + (1 - \beta_{i_n})\phi\big(p, T_{i_n}^{m_n} x_n\big) - \beta_{i_n}(1 - \beta_{i_n})g\big(\|Jx_n - JT_{i_n}^{m_n} x_n\|\big) \\ &\leq \beta_{i_n}\phi(p, x_n) + (1 - \beta_{i_n})\big[\phi(p, x_n) + \nu_{m_n}^{(i_n)}\zeta_{i_n}\big(\phi(p, x_n)\big) + \mu_{m_n}^{(i_n)}\big] \\ &\quad - \beta_{i_n}(1 - \beta_{i_n})g\big(\|Jx_n - JT_{i_n}^{m_n} x_n\|\big) \\ &\leq \phi(p, x_n) + \nu_{m_n}^{(i_n)}\sup_{p \in F}\zeta_{i_n}\big(\phi(p, x_n)\big) + \mu_{m_n}^{(i_n)} - \beta_{i_n}(1 - \beta_{i_n})g\big(\|Jx_n - JT_{i_n}^{m_n} x_n\|\big) \\ &= \phi(p, x_n) + \xi_n - \beta_{i_n}(1 - \beta_{i_n})g\big(\|Jx_n - JT_{i_n}^{m_n} x_n\|\big). \end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.3) and simplifying it, we have

$$\phi(p, u_n) \leq \phi(p, y_n) \leq \phi(p, x_n) + (1 - \alpha_n)\xi_n - (1 - \alpha_n)\beta_{i_n}(1 - \beta_{i_n})g\left(\left\|Jx_n - JT_{i_n}^{m_n}x_n\right\|\right)$$

$$\leq \phi(p, x_n) + \xi_n - (1 - \alpha_n)\beta_{i_n}(1 - \beta_{i_n})g\left(\left\|Jx_n - JT_{i_n}^{m_n}x_n\right\|\right)$$

$$\leq \phi(p, x_n) + \xi_n. \tag{3.5}$$

This implies that $p \in C_{n+1}$, and so $G \subset C_{n+1}$.

(III) $x_n \to x^* \in C$ as $n \to \infty$.

In fact, since $x_n = \Pi_{C_n}x_1$, from Lemma 2.1(2) we have $\langle x_n - y, Jx_1 - Jx_n \rangle \geq 0$, $\forall y \in C_n$. Again since $F \subset \bigcap_{n=1}^{\infty} C_n$, we have $\langle x_n - p, Jx_1 - Jx_n \rangle \geq 0$, $\forall p \in F$. It follows from Lemma 2.1(1) that for each $p \in F$ and for each $n \geq 1$,

$$\phi(x_n, x_1) = \phi(\Pi_{C_n}x_1, x_1) \leq \phi(p, x_1) - \phi(p, x_n) \leq \phi(p, x_1),$$

which implies that $\{\phi(x_n, x_1)\}$ is bounded, so is $\{x_n\}$. Since for all $n \geq 1$, $x_n = \Pi_{C_n}x_1$ and $x_{n+1} = \Pi_{C_{n+1}}x_1 \in C_{n+1} \subset C_n$, we have $\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1)$. This implies that $\{\phi(x_n, x_1)\}$ is nondecreasing, hence the limit

$$\lim_{n \to \infty} \phi(x_n, x_1) \text{ exists.}$$

Since E is reflexive, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup x^* \in C$ as $i \to \infty$. Since C_n is closed and convex and $C_{n+1} \subset C_n$, this implies that C_n is weakly closed and $x^* \in C_n$ for each $n \geq 1$. In view of $x_{n_i} = \Pi_{C_{n_i}}x_1$, we have

$$\phi(x_{n_i}, x_1) \leq \phi(x^*, x_1), \quad \forall i \geq 1.$$

Since the norm $\|\cdot\|$ is weakly lower semicontinuous, we have

$$\liminf_{i \to \infty} \phi(x_{n_i}, x_1) = \liminf_{i \to \infty}\left(\|x_{n_i}\|^2 - 2\langle x_{n_i}, Jx_1 \rangle + \|x_1\|^2\right)$$

$$\geq \left\|x^*\right\|^2 - 2\langle x^*, Jx_1 \rangle + \|x_1\|^2$$

$$= \phi(x^*, x_1),$$

and so

$$\phi(x^*, x_1) \leq \liminf_{i \to \infty} \phi(x_{n_i}, x_1) \leq \limsup_{i \to \infty} \phi(x_{n_i}, x_1) \leq \phi(x^*, x_1).$$

This implies that $\lim_{i \to \infty} \phi(x_{n_i}, x_1) = \phi(x^*, x_1)$, and so $\|x_{n_i}\| \to \|x^*\|$ as $i \to \infty$. Since $x_{n_i} \rightharpoonup x^*$, by virtue of the *Kadec-Klee property* of E, we obtain

$$\lim_{i \to \infty} x_{n_i} = x^*.$$

Since $\{\phi(x_n, x_1)\}$ is convergent, this, together with $\lim_{i \to \infty} \phi(x_{n_i}, x_1) = \phi(x^*, x_1)$, shows that $\lim_{n \to \infty} \phi(x_n, x_1) = \phi(x^*, x_1)$. If there exists some subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to$

y as $j \to \infty$, then from Lemma 2.1(1) we have

$$
\begin{aligned}
\phi(x^*, y) &= \lim_{i,j \to \infty} \phi(x_{n_i}, x_{n_j}) = \lim_{i,j \to \infty} \phi(x_{n_i}, \Pi_{C_{n_j}} x_1) \\
&\leq \lim_{i,j \to \infty} \left(\phi(x_{n_i}, x_1) - \phi(\Pi_{C_{n_j}} x_1, x_1) \right) \\
&= \lim_{i,j \to \infty} \left(\phi(x_{n_i}, x_1) - \phi(x_{n_j}, x_1) \right) \\
&= \phi(x^*, x_1) - \phi(x^*, x_1) = 0,
\end{aligned}
$$

that is, $x^* = y$ and so

$$
\lim_{n \to \infty} x_n = x^*. \tag{3.6}
$$

(IV) x^* is a member of F.

Set $\mathcal{K}_i = \{k \in \mathbb{N} : k = i + \frac{(m-1)m}{2}, m \geq i, m \in \mathbb{N}\}$ for each $i \in \mathbb{N}$. Note that $v_{m_k}^{(i_k)} = v_{m_k}^{(i)}$, $\mu_{m_k}^{(i_k)} = \mu_{m_k}^{(i)}$ and $\zeta_{i_k} = \zeta_i$ whenever $k \in \mathcal{K}_i$ for each $i \geq 1$. For example, by Lemma 2.7 and the definition of $\mathcal{K}_1$, we have $\mathcal{K}_1 = \{1, 2, 4, 7, 11, 16, \ldots\}$ and $i_1 = i_2 = i_4 = i_7 = i_{11} = i_{16} = \cdots = 1$. Then we have

$$
\xi_k = v_{m_k}^{(i)} \sup_{p \in F} \zeta_i \left(\phi(p, x_k) \right) + \mu_{m_k}^{(i)}, \quad \forall k \in \mathcal{K}_i. \tag{3.7}
$$

Note that $\{m_k\}_{k \in \mathcal{K}_i} = \{i, i+1, i+2, \ldots\}$, i.e., $m_k \uparrow \infty$ as $\mathcal{K}_i \ni k \to \infty$. It follows from (3.6) and (3.7) that

$$
\lim_{\mathcal{K}_i \ni k \to \infty} \xi_k = 0, \quad \forall i \geq 1. \tag{3.8}
$$

Since $x_{n+1} \in C_{n+1}$, it follows from (3.1), (3.6), and (3.8) that

$$
\phi(x_{k+1}, y_k) \leq \phi(x_{k+1}, x_k) + \xi_k \to 0 \quad (\mathcal{K}_i \ni k \to \infty). \tag{3.9}
$$

Since $x_k \to x^*$ as $\mathcal{K}_i \ni k \to \infty$, it follows from (3.9) and Lemma 2.3 that

$$
\lim_{\mathcal{K}_i \ni k \to \infty} y_k = x^*, \quad \forall i \geq 1. \tag{3.10}
$$

Note that $T_{i_k}^{m_k} = T_i^{m_k}$ and $\beta_{i_k} = \beta_i$ whenever $k \in \mathcal{K}_i$ for each $i \geq 1$. From (3.5), for any $p \in F$, we have

$$
\phi(p, y_k) \leq \phi(p, x_k) + \xi_k - (1 - \alpha_k)\beta_i(1 - \beta_i)g\left(\|Jx_k - JT_i^{m_k} x_k\| \right), \quad \forall k \in \mathcal{K}_i,
$$

that is,

$$
(1 - \alpha_k)\beta_i(1 - \beta_i)g\left(\|Jx_k - JT_i^{m_k} x_k\| \right) \leq \phi(p, x_k) + \xi_k - \phi(p, y_k) \to 0 \quad (\mathcal{K}_i \ni k \to \infty).
$$

This shows that $\lim_{\mathcal{K}_i \ni k \to \infty} g(\|Jx_k - JT_i^{m_k} x_k\|) = 0$. In view of the property of g, we have

$$
\lim_{\mathcal{K}_i \ni k \to \infty} \|Jx_k - JT_i^{m_k} x_k\| = 0, \quad \forall i \geq 1.
$$

In addition, $Jx_k \to Jx^*$ ($\mathcal{K}_i \ni k \to \infty$) implies that $\lim_{\mathcal{K}_i \ni k \to \infty} JT_i^{m_k} x_k = Jx^*$. From Remark 2.2(ii) it yields that, as $\mathcal{K}_i \ni k \to \infty$,

$$T_i^{m_k} x_k \rightharpoonup x^*, \quad \forall i \geq 1. \tag{3.11}$$

Again since for each $i \geq 1$, as $\mathcal{K}_i \ni k \to \infty$,

$$\left| \left\| T_i^{m_k} x_k \right\| - \left\| x^* \right\| \right| = \left| \left\| JT_i^{m_k} x_k \right\| - \left\| Jx^* \right\| \right| \leq \left\| JT_i^{m_k} x_k - Jx^* \right\| \to 0,$$

this, together with (3.11) and the *Kadec-Klee property* of E, shows that

$$\lim_{\mathcal{K}_i \ni k \to \infty} T_i^{m_k} x_k = x^*, \quad \forall i \geq 1. \tag{3.12}$$

We use the assumptions that for each $i \geq 1$, T_i is uniformly L_i-Lipschitz continuous. Noting again that $\{m_k\}_{k \in \mathcal{K}_i} = \{i, i+1, i+2, \ldots\}$, i.e., $m_{k+1} - 1 = m_k$ for all $k \in \mathcal{K}_i$, we then have

$$\begin{aligned}
\left\| T_i^{m_{k+1}} x_k - T_i^{m_k} x_k \right\| &\leq \left\| T_i^{m_{k+1}} x_k - T_i^{m_{k+1}} x_{k+1} \right\| + \left\| T_i^{m_{k+1}} x_{k+1} - x_{k+1} \right\| \\
&\quad + \left\| x_{k+1} - x_k \right\| + \left\| x_k - T_i^{m_k} x_k \right\| \\
&\leq (L_i + 1) \| x_{k+1} - x_k \| + \left\| T_i^{m_{k+1}} x_{k+1} - x_{k+1} \right\| \\
&\quad + \left\| x_k - T_i^{m_k} x_k \right\|.
\end{aligned} \tag{3.13}$$

From (3.12) and $x_k \to x^*$ as $\mathcal{K}_i \ni k \to \infty$ we have $\lim_{\mathcal{K}_i \ni k \to \infty} \| T_i^{m_{k+1}} x_k - T_i^{m_k} x_k \| = 0$ and $\lim_{\mathcal{K}_i \ni k \to \infty} T_i^{m_{k+1}} x_k = x^*$, i.e., $\lim_{\mathcal{K}_i \ni k \to \infty} T_i(T_i^{m_{k+1}-1} x_k) = x^*$. It then follows that, for each $i \geq 1$,

$$\lim_{\mathcal{K}_i \ni k \to \infty} T_i\left(T_i^{m_k} x_k\right) = x^*, \quad \forall i \geq 1. \tag{3.14}$$

In view of the closedness of T_i, it follows from (3.12) that $T_i x^* = x^*$, i.e., for each $i \geq 1$, $x^* \in F(T_i)$ and hence $x^* \in F$.

(V) x^* is also a member of G.

Since $x_{n+1} = \Pi_{C_{n+1}} x_1$, it follows from (3.1) and (3.6) that

$$\phi(x_{k+1}, u_k) \leq \phi(x_{k+1}, x_k) + \xi_k \to 0 \quad (\mathcal{K}_i \ni k \to \infty).$$

Since $x_k \to x^*$ as $\mathcal{K}_i \ni k \to \infty$, by virtue of Lemma 2.1 we have

$$\lim_{\mathcal{K}_i \ni k \to \infty} u_k = x^*, \quad \forall i \geq 1. \tag{3.15}$$

This, together with (3.10), shows that $\lim_{\mathcal{K}_i \ni k \to \infty} \| u_k - y_k \| = 0$ and $\lim_{\mathcal{K}_i \ni k \to \infty} \| Ju_k - Jy_k \| = 0$. By the assumption that $\{r_k\}_{k \in \mathcal{K}_i} \subset [a, \infty)$ for some $a > 0$, we have

$$\lim_{\mathcal{K}_i \ni k \to \infty} \frac{\| Ju_k - Jy_k \|}{r_k} = 0, \quad \forall i \geq 1. \tag{3.16}$$

Since $\tau(u_k, y) + \frac{1}{r_k}\langle y - u_k, Ju_k - Jy_k\rangle \geq 0$, $\forall y \in C$, by condition (A$_1$), we have

$$\frac{1}{r_k}\langle y - u_k, Ju_k - Jy_k\rangle \geq -\tau(u_k, y) \geq \tau(y, u_k), \quad \forall y \in C. \tag{3.17}$$

By the assumption that the mapping $y \mapsto \tau(x, y)$ is convex and lower semicontinuous, letting $\mathcal{K}_i \ni k \to \infty$ in (3.17), from (3.15) and (3.16), we have $\tau(y, x^*) \leq 0$, $\forall y \in C$. For any $t \in (0, 1]$ and any $y \in C$, set $y_t = ty + (1 - t)x^*$. Then $\tau(y_t, x^*) \leq 0$ since $y_t \in C$. By conditions (A$_1$) and (A$_4$), we have

$$0 = \tau(y_t, y_t) \leq t\tau(y_t, y) + (1 - t)\tau(y_t, x^*) \leq t\tau(y_t, y).$$

Dividing both sides of the above equation by t, we have $\tau(y_t, y) \geq 0$, $\forall y \in C$. Letting $t \downarrow 0$, from condition (A$_3$), we have $\tau(x^*, y) \geq 0$, $\forall y \in C$, i.e., $x^* \in \Omega$ and so $x^* \in G$.

(VI) $x^* = \Pi_G x_1$, and so $x_n \to \Pi_G x_1$ as $n \to \infty$.

Put $u = \Pi_G x_1$. Since $u \in G \subset C_n$ and $x_n = \Pi_{C_n} x_1$, we have $\phi(x_n, x_1) \leq \phi(u, x_1)$, $\forall n \geq 1$. Then

$$\phi(x^*, x_1) = \lim_{n \to \infty}\phi(x_n, x_1) \leq \phi(u, x_1), \tag{3.18}$$

which implies that $x^* = u$ since $u = \Pi_{Gx_1}$, and hence $x_n \to x^* = \Pi_F x_1$ as $n \to \infty$. This completes the proof. □

We now provide a nontrivial family of mappings satisfying the conditions of Theorem 3.1.

Example 3.2 Let $E = \mathbb{R}^1$ with the standard norm $\|\cdot\| = |\cdot|$ and $C = [-1, 1]$. Let $\{T_i\}: C \to C$ be a countable family of mappings defined by

$$T_i x = \begin{cases} \frac{-x}{i^2 + 1}, & x \in (0, 1], \\ x, & x \in [-1, 0]. \end{cases}$$

We first show that $\{T_i\}$ is uniformly L-Lipschitzian. If $x \in (0, 1]$ and $y \in [-1, 0]$, then

$$\left|T_i^n x - T_i^n y\right| = \left|\frac{(-1)^n}{(i^2 + 1)^n}x - y\right| \leq |x| + |y| \leq 2|x - y|.$$

The rest is trivial. Second, we claim that $\{T_i\}$ is a family of closed and totally quasi-ϕ-asymptotically nonexpansive mappings. In fact, for any $x \in (0, 1]$ and $p \in \bigcap_{i=1}^{\infty} F(T_i) = [-1, 0]$, we have, for all $n \geq 2$,

$$\begin{aligned}
\phi(p, T_i^n x) = \left|T_i^n x - p\right|^2 &= \left|\frac{(-1)^n}{(i^2 + 1)^n}x - p\right|^2 \\
&\leq \left[|p| + \frac{1}{(i^2 + 1)^n}|x|\right]^2 \\
&\leq \left[|x - p| + \frac{1}{(i^2 + 1)^n}\right]^2 \\
&= \phi(x, p) + \nu_n^{(i)}\zeta(\phi(x, p)) + \mu_n^{(i)},
\end{aligned}$$

where $\phi(x,p) = |x - p|^2$, $\nu_n^{(i)} = \frac{2}{(i^2+1)^n}$, $\mu_n^{(i)} = \frac{1}{(i^2+1)^{2n}}$ and $\zeta(x) = \sqrt{x}$. Note that $|T_i x - p|^2 \leq |x - p|^2$, that is, $\mu_1^{(i)} = 0$ for each $i \geq 1$.

Next, we define a bifunction $\theta : C \times C \to \mathbb{R}$ satisfying the conditions (A$_1$)-(A$_4$) by

$$\theta(x,y) = y^2 - x^2.$$

Let $A = 0$ and $\psi = 0$. Then the set of solutions Ω to the generalized mixed equilibrium problem for θ, A and ψ is obviously $\{0\}$. Since $G := \Omega \cap F \neq \emptyset$ and F is bounded, it follows from Theorem 3.1 that the sequence $\{x_n\}$ defined by (3.1) converges strongly to $\Pi_G x_1$.

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