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A new method for the research of best proximity point theorems of nonlinear mappings

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Abstract

The purpose of this paper is to present a new method for the research of best proximity point theorems of nonlinear mappings in metric spaces. In this paper, the *P*-operator technique, which changes non-self-mapping to self-mapping, provides a new and simple method of proof. Best proximity point theorems for weakly contractive and weakly Kannan mappings, generalized best proximity point theorems for generalized contractions, and best proximity points for proximal cyclic contraction mappings have been proved by using this new method. Meanwhile, many recent results in this area have been improved.

MSC: 47H05; 47H09; 47H10

Keywords: best proximity point; weak *P*-property; weakly contractive mapping; generalized best proximity point; proximal cyclic contraction

1 Introduction and preliminaries

Several problems can be changed to equations of the form $Tx = x$, where T is a given self-mapping defined on a subset of a metric space, a normed linear space, a topological vector space or some suitable space. However, if T is a non-self-mapping from A to B, then the aforementioned equation does not necessarily admit a solution. In this case, one would contemplate finding an approximate solution x in A such that the error $d(x, Tx)$ is minimum, where d is the distance function. In view of the fact that $d(x, Tx)$ is at least $d(A,B)$, a best proximity point theorem (for short BPPT) guarantees the global minimization of $d(x, Tx)$ by the requirement that an approximate solution x satisfies the condition $d(x, Tx) = d(A,B)$. Such optimal approximate solutions are called best proximity points of the mapping T. Interestingly, best proximity point theorems also serve as a natural generalization of fixed point theorems, for a best proximity point becomes a fixed point if the mapping under consideration is a self-mapping. Research on the best proximity point is an important topic in the nonlinear functional analysis and applications (see [1–18]).

Let A, B be two nonempty subsets of a complete metric space and consider a mapping $T : A \to B$. The best proximity point problem is whether we can find an element $x_0 \in A$ such that $d(x_0, Tx_0) = \min\{d(x, Tx) : x \in A\}$. Since $d(x, Tx) \geq d(A,B)$ for any $x \in A$, in fact, the optimal solution to this problem is the one for which the value $d(A,B)$ attained.

Let A, B be two nonempty subsets of a metric space (X,d). We denote by A_0 and B_0 the following sets:

$$A_0 = \{x \in A : d(x,y) = d(A,B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : d(x,y) = d(A,B) \text{ for some } x \in A\},$$

where $d(A,B) = \inf\{d(x,y) : x \in A \text{ and } y \in B\}$.

It is interesting to notice that A_0 and B_0 are contained in the boundaries of A and B, respectively, provided A and B are closed subsets of a normed linear space such that $d(A,B) > 0$ (see [1]).

2 BPPT for weakly contractive and weakly Kannan mappings

Let A and B be nonempty subsets of a metric space (X,d). An operator $T : A \to B$ is said to be contractive if there exists $k \in [0,1)$ such that $d(Tx,Ty) \le kd(x,y)$ for any $x,y \in A$. The well-known Banach contraction principle says: Let (X,d) be a complete metric space, and $T : X \to X$ be a contraction of X into itself. Then T has a unique fixed point in X.

In the last 50 years, the Banach contraction principle has been extensively studied and generalized in many settings. One of the generalizations is the weakly contractive mapping.

Definition 2.1 [3] Let (X,d) be a metric space. A mapping $f : X \to X$ is said to be *weakly contractive* provided that

$$d\big(f(x),f(y)\big) \le \bar{\alpha}(x,y)d(x,y)$$

for all $x,y \in X$, where the function $\bar{\alpha} : X \times X \to [0,1)$, holds, for every $0 < a < b$, that

$$\theta(a,b) = \sup\{\bar{\alpha}(x,y) : a \le d(x,y) \le b\} < 1.$$

The fixed point theorem for weakly contractive mapping was presented in [3].

Theorem 2.2 *Let (X,d) be a complete metric space. If $f : X \to X$ is a weakly contractive mapping, then f has a unique fixed point x^* and the Picard sequence of iterates $\{f^n(x)\}_{n\in N}$ converges, for every $x \in X$, to x^*.*

One type of contraction which is different from the Banach contraction is Kannan mappings. In [11], Kannan obtained the following fixed point theorem.

Theorem 2.3 [11] *Let (X,d) be a complete metric space and let $f : X \to X$ be a mapping such that*

$$d\big(f(x),f(y)\big) \le \frac{\alpha}{2}\big[d\big(x,f(x)\big) + d\big(y,f(y)\big)\big]$$

for all $x,y \in X$ and some $\alpha \in [0,1)$, then f has a unique fixed point $x^* \in X$. Moreover, the Picard sequence of iterates $\{f^n(x)\}_{n\in N}$ converges, for every $x \in X$, to x^*.

In [12], the authors introduce a more general weakly Kannan mapping and obtain its fixed point theorem.

Definition 2.4 [12] Let (X, d) be a metric space. A mapping $f : X \to X$ is said to be *weakly Kannan* if there exists $\bar{\alpha} : X \times X \to [0, 1)$, which satisfies for every $0 < a \leq b$ and for all $x, y \in X$

$$\theta(a, b) = \sup\{\bar{\alpha}(x, y) : a \leq d(x, y) \leq b\} < 1$$

and

$$d(f(x), f(y)) \leq \frac{\bar{\alpha}(x, y)}{2}\left[d(x, f(x)) + d(y, f(y))\right].$$

Theorem 2.5 [12] *Let (X, d) be a complete metric space. If $f : X \to X$ is a weakly Kannan mapping, then f has a unique fixed point x^* and the Picard sequence of iterates $\{f^n(x)\}_{n \in N}$ converges, for every $x \in X$, to x^*.*

In this section, we first obtain best proximity point theorems for weakly contractive mapping and weakly Kannan mapping in metric spaces. Further, we extend the results to partial metric spaces. The P-operator technique, which changes non-self-mapping to self-mapping, provides a new and simple proof. Many recent results in this area have been improved.

Before giving the main results, we need the following notations and basic facts.

Definition 2.6 [13] Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the *P-property* if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$\begin{cases} d(x_1, y_1) = d(A, B), \\ d(x_2, y_2) = d(A, B) \end{cases} \Rightarrow \quad d(x_1, x_2) = d(y_1, y_2).$$

In [13], the author proves that any pair (A, B) of nonempty closed convex subsets of a real Hilbert space H satisfies the P-property.

In [4], the P-property has been weakened to the weak P-property. An example that satisfies the P-property but not the weak P-property can be found there.

Definition 2.7 [4] Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the *weak P-property* if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$\begin{cases} d(x_1, y_1) = d(A, B), \\ d(x_2, y_2) = d(A, B) \end{cases} \Rightarrow \quad d(x_1, x_2) \leq d(y_1, y_2).$$

Example [4] Consider (R^2, d), where d is the Euclidean distance and the subsets $A = \{(0, 0)\}$ and $B = \{y = 1 + \sqrt{1 - x^2}\}$.

Obviously, $A_0 = \{(0, 0)\}$, $B_0 = \{(-1, 1), (1, 1)\}$ and $d(A, B) = \sqrt{2}$. Furthermore,

$$d((0, 0), (-1, 1)) = d((0, 0), (1, 1)) = \sqrt{2},$$

however,

$$0 = d\big((0,0),(0,0)\big) < d\big((-1,1),(1,1)\big) = 2.$$

We can see that the pair (A,B) satisfies the weak P-property but not the P-property.

Firstly, we present the following definitions.

Definition 2.8 Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d). A mapping $f: A \to B$ is said to be *weakly contractive* provided that

$$d\big(f(x),f(y)\big) \le \bar{\alpha}(x,y)d(x,y)$$

for all $x,y \in A$, where the function $\bar{\alpha}: X \times X \to [0,1)$ holds, for every $0 < a < b$, and

$$\theta(a,b) = \sup\big\{\bar{\alpha}(x,y) : a \le d(x,y) \le b\big\} < 1.$$

Definition 2.9 Let (A,B) be a pair of nonempty closed subsets of a complete metric space. A mapping $f: A \to B$ is said to be *weakly Kannan* if there exists $\bar{\alpha}: X \times X \to [0,1)$ which satisfies for every $0 < a \le b$ and for all $x,y \in X$

$$\theta(a,b) = \sup\big\{\bar{\alpha}(x,y) : a \le d(x,y) \le b\big\} < 1$$

and

$$d\big(f(x),f(y)\big) \le \frac{\bar{\alpha}(x,y)}{2}\big[d\big(x,f(x)\big) + d\big(y,f(y)\big) - 2d(A,B)\big].$$

Next we prove the best proximity point theorems for weakly contractive and weakly Kannan mappings in metric spaces.

Theorem 2.10 *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that $A_0 \neq \emptyset$. Let $T: A \to B$ be a weakly contractive mapping defined as Definition* 2.8. *Suppose that $T(A_0) \subseteq B_0$ and the pair (A,B) has the weak P-property. Then T has a unique best proximity point $x^* \in A_0$ and the iteration sequence $\{x_{2k}\}_{n=0}^{\infty}$ defined by*

$$x_{2k+1} = Tx_{2k}, \quad d(x_{2k+2},x_{2k+1}) = d(A,B), \quad k = 0,1,2,\ldots$$

converges, for every $x_0 \in A_0$, to x^*.

Proof We first prove that B_0 is closed. Let $\{y_n\} \subseteq B_0$ be a sequence such that $y_n \to q \in B$. It follows from the weak P-property that

$$d(y_n,y_m) \to 0 \quad \Rightarrow \quad d(x_n,x_m) \to 0,$$

as $n,m \to \infty$, where $x_n,x_m \in A_0$ and $d(x_n,y_n) = d(A,B)$, $d(x_m,y_m) = d(A,B)$. Then $\{x_n\}$ is a Cauchy sequence so that $\{x_n\}$ converges strongly to a point $p \in A$. By the continuity of metric d we have $d(p,q) = d(A,B)$, that is, $q \in B_0$ and hence B_0 is closed.

Let $\overline{A}_0$ be the closure of A_0; we claim that $T(\overline{A}_0) \subseteq B_0$. In fact, if $x \in \overline{A}_0 \setminus A_0$, then there exists a sequence $\{x_n\} \subseteq A_0$ such that $x_n \to x$. By the continuity of T and the closedness of B_0 we have $Tx = \lim_{n\to\infty} Tx_n \in B_0$. That is $T(\overline{A}_0) \subseteq B_0$.

Define an operator $P_{A_0} : T(\overline{A}_0) \to A_0$, by $P_{A_0}y = \{x \in A_0 : d(x,y) = d(A,B)\}$. Since the pair (A,B) has the weak P-property, we have

$$d(P_{A_0}Tx_1, P_{A_0}Tx_2) \leq d(Tx_1, Tx_2) \leq \bar{\alpha}(x_1,x_2)d(x_1,x_2)$$

for any $x_1, x_2 \in \overline{A}_0$. This shows that $P_{A_0}T : \overline{A}_0 \to \overline{A}_0$ is a weak contraction from complete metric subspace $\overline{A}_0$ into itself. Using Theorem 2.2, we can see that $P_{A_0}T$ has a unique fixed point x^*. That is, $P_{A_0}Tx^* = x^* \in A_0$, which implies that

$$d\big(x^*, Tx^*\big) = d(A,B).$$

Therefore, x^* is the unique one in A_0 such that $d(x^*, Tx^*) = d(A,B)$. It is easy to see that x^* is also the unique one in A such that $d(x^*, Tx^*) = d(A,B)$. The Picard iteration sequence

$$x_{n+1} = P_{A_0}Tx_n, \quad n = 0,1,2,\ldots$$

converges, for every $x_0 \in A_0$, to x^*. The iteration sequence $\{x_{2k}\}_{n=0}^{\infty}$ defined by

$$x_{2k+1} = Tx_{2k}, \quad d(x_{2k+2}, x_{2k+1}) = d(A,B), \quad k = 0,1,2,\ldots,$$

is exactly the subsequence of $\{x_n\}$, so that it converges, for every $x_0 \in A_0$, to x^*. This completes the proof. □

Theorem 2.11 *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that $A_0 \neq \emptyset$. Let $T : A \to B$ be a continuous weakly Kannan mapping defined as Definition* 2.9. *Suppose that $T(A_0) \subseteq B_0$ and the pair (A,B) has the weak P-property. Then T has a unique best proximity point $x^* \in A_0$ and the iteration sequence $\{x_{2k}\}_{n=0}^{\infty}$ defined by*

$$x_{2k+1} = Tx_{2k}, \quad d(x_{2k+2}, x_{2k+1}) = d(A,B), \quad k = 0,1,2,\ldots$$

converges, for every $x_0 \in A_0$, to x^*.

Proof The closedness of B_0 and $T(\overline{A}_0) \subseteq B_0$ have been proved in Theorem 2.10. Now we define an operator $P_{A_0} : T(\overline{A}_0) \to A_0$, by $P_{A_0}y = \{x \in A_0 : d(x,y) = d(A,B)\}$. Since the pair (A,B) has weak P-property, we have

$$\begin{aligned}
d(P_{A_0}Tx_1, P_{A_0}Tx_2) &\leq d(Tx_1, Tx_2) \\
&\leq \frac{\bar{\alpha}(x,y)}{2}\big[d(x_1, Tx_1) + d(x_2, Tx_2) - 2d(A,B)\big] \\
&\leq \frac{\bar{\alpha}(x,y)}{2}\big[d(x_1, P_{A_0}Tx_1) + d(P_{A_0}Tx_1, Tx_1) \\
&\quad + d(x_2, P_{A_0}Tx_2) + d(P_{A_0}Tx_2, Tx_2) - 2d(A,B)\big] \\
&= \frac{\bar{\alpha}(x,y)}{2}\big[d(x_1, P_{A_0}Tx_1) + d(x_2, P_{A_0}Tx_2)\big]
\end{aligned}$$

for any $x_1, x_2 \in \overline{A}_0$. This shows that $P_{A_0}T : \overline{A}_0 \to \overline{A}_0$ is a weakly Kannan mapping from complete metric subspace $\overline{A}_0$ into itself. Using Theorem 2.5, we can see that $P_{A_0}T$ a unique fixed point x^*. That is, $P_{A_0}Tx^* = x^* \in A_0$, which implies that

$$d\left(x^*, Tx^*\right) = d(A, B).$$

Therefore, x^* is the unique one in A_0 such that $d(x^*, Tx^*) = d(A, B)$. It is easy to see that x^* is also the unique one in A such that $d(x^*, Tx^*) = d(A, B)$. The Picard iteration sequence

$$x_{n+1} = P_{A_0}Tx_n, \quad n = 0, 1, 2, \ldots$$

converges, for every $x_0 \in A_0$, to x^*. Since the iteration sequence $\{x_{2k}\}_{n=0}^{\infty}$ defined by

$$x_{2k+1} = Tx_{2k}, \quad d(x_{2k+2}, x_{2k+1}) = d(A, B), \quad k = 0, 1, 2, \ldots,$$

is exactly the subsequence of $\{x_n\}$, it converges, for every $x_0 \in A_0$, to x^*. This completes the proof. □

Example 2.12 Let $X = R^2$, $A = \{(1, y) : y \geq 0\}$, $B = \{(0, y) : y \geq 0\}$, and define $f : A \to B$ as follows:

$$f(1, y) = \left(0, \frac{y^2}{y+1}\right).$$

We have $A_0 = A \neq \emptyset$, $B_0 = B$, $f(A_0) \subseteq B_0$. It is obvious that (A, B) satisfy the P-property so it must satisfy the weakly P-property. Meanwhile

$$\begin{aligned}
d\left(f(1, y_1), f(1, y_2)\right) &= \left|\frac{y_1^2}{y_1 + 1} - \frac{y_2^2}{y_2 + 1}\right| \\
&= \frac{|y_1^2(y_2 + 1) - y_2^2(y_1 + 1)|}{(y_1 + 1)(y_2 + 1)} \\
&= \frac{|(y_1 y_2 + y_1 + y_2)(y_2 - y_1)|}{(y_1 + 1)(y_2 + 1)} \\
&= \frac{y_1 y_2 + y_1 + y_2}{(y_1 + 1)(y_2 + 1)}|y_1 - y_2| \\
&= \frac{y_1 y_2 + y_1 + y_2}{y_1 y_2 + y_1 + y_2 + 1}|y_1 - y_2| \\
&= \frac{y_1 y_2 + y_1 + y_2}{y_1 y_2 + y_1 + y_2 + 1} d\left((1, y_1), (1, y_2)\right) \\
&= \bar{\alpha}\left((1, y_1), (1, y_2)\right) d\left((1, y_1), (1, y_2)\right),
\end{aligned}$$

where

$$\bar{\alpha}\left((1, y_1), (1, y_2)\right) = \frac{y_1 y_2 + y_1 + y_2}{y_1 y_2 + y_1 + y_2 + 1}.$$

That is, f is a weakly contractive mapping. All conditions of Theorem 2.10 hold, the conclusion of Theorem 2.10 is also correct, that is, f has a unique best proximity point

$z^* = (1,0) \in A_0$ such that $d(z^*, f(z^*)) = d((1,0),(0,0)) = 1 = d(A,B)$. On the other hand, it is obvious that the iteration sequence $\{z_{2k}\}_{n=0}^{\infty}$ defined by

$$z_{2k+1} = f(z_{2k}), \quad d(z_{2k+2}, z_{2k+1}) = d(A,B) = 1, \quad k = 0,1,2,\ldots,$$

converges, for every $z_0 \in A_0$, to z^*, since

$$z_{2(k+1)} = (1, y_{2(k+1)}) = \left(1, \frac{y_{2k}^2}{y_{2k}+1}\right) \to (1,0).$$

In fact, from $y_{2(k+1)} = \frac{y_{2k}^2}{y_{2k}+1}$ we know that $y_{2(k+1)} \le y_{2k}$, so there exists a number y^* such that $y_{2k} \to y^*$. Furthermore, $y^* = \frac{(y^*)^2}{y^*+1}$ and hence $y^* = 0$.

Example 2.13 Let $X = R^2$, $A = \{(1,y), y \ge 0\}$, $B = \{(0,y), y \ge 0\}$. For $y \ge 0$, $z \ge 0$, we have the following equivalence relations:

$$\begin{aligned}
&3z = \sqrt{(y-z)^2+1} - 1 \\
&\quad\Leftrightarrow\quad 3z+1 = \sqrt{(y-z)^2+1} \\
&\quad\Leftrightarrow\quad (3z+1)^2 = (y-z)^2 + 1 = y^2 - 2yz + z^2 + 1 \\
&\quad\Leftrightarrow\quad 9z^2 + 6z + 1 = y^2 - 2yz + z^2 + 1 \\
&\quad\Leftrightarrow\quad 8z^2 + (6+2y)z - y^2 = 0 \\
&\quad\Leftrightarrow\quad z = \frac{\sqrt{(6+2y)^2 + 32y^2} - (6+2y)}{16}.
\end{aligned}$$

We define a function $f : [0,+\infty) \to [0,+\infty)$ as follows:

$$z = f(y) = \frac{\sqrt{(6+2y)^2 + 32y^2} - (6+2y)}{16}.$$

From the above equivalence relations, we get

$$3f(y) = \sqrt{\big(y - f(y)\big)^2 + 1} - 1.$$

Therefore, we define a mapping $T : A \to B$ as follows:

$$T : (1,y) \mapsto \big(0, f(y)\big) = \left(0, \frac{\sqrt{(6+2y)^2 + 32y^2} - (6+2y)}{16}\right).$$

We have $A_0 = A \neq \emptyset$, $B_0 = B$, $T(A_0) \subseteq B_0$. It is obvious that (A,B) satisfy the P-property and so must satisfy the weakly P-property. Meanwhile the following inequality holds:

$$\begin{aligned}
d\big(T(1,y) - T(1,h)\big) &= \big|f(y) - f(h)\big| = \frac{1}{3}\big|3f(y) - 3f(h)\big| \\
&= \frac{1}{3}\Big|\Big(\sqrt{\big(y-f(y)\big)^2+1} - 1\Big) - \Big(\sqrt{\big(h-f(h)\big)^2+1} - 1\Big)\Big| \\
&= \frac{1}{3}\Big|\sqrt{\big(y-f(y)\big)^2+1} + \sqrt{\big(h-f(h)\big)^2+1} - 2\sqrt{\big(h-f(h)\big)^2+1}\Big|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{3}\left|\sqrt{\left(y-f(y)\right)^2+1}+\sqrt{\left(h-f(h)\right)^2+1}-2\right| \\
&= \frac{1}{3}\left|d\left((1,y),T(1,y)\right)+d\left((1,h),T(1,h)\right)-2d(A,B)\right| \\
&\leq \frac{y+h+2}{2(y+h+3)}\left|d\left((1,y),T(1,y)\right)+d\left((1,h),T(1,h)\right)-2d(A,B)\right| \\
&= \frac{\bar{\alpha}(y,h)}{2}\left|d\left((1,y),T(1,y)\right)+d\left((1,h),T(1,h)\right)-2d(A,B)\right|,
\end{aligned}$$

where $\bar{\alpha}(y,h) = \frac{y+h+2}{(y+h+3)}$. That is, T is a continuous weakly Kannan mapping. All conditions of Theorem 2.11 hold, the conclusion of Theorem 2.11 is also correct, that is, T has a unique best proximity point $z^* = (1,0) \in A_0$ such that $d(z^*, T(z^*)) = d((1,0),(0,0)) = 1 = d(A,B)$. On the other hand, it is obvious that the iteration sequence $\{z_{2k}\}_{n=0}^{\infty}$ defined by

$$z_{2k+1} = T(z_{2k}), \quad d(z_{2k+2}, z_{2k+1}) = d(A,B) = 1, \quad k = 0,1,2,\ldots,$$

converges, for every $z_0 \in A_0$, to z^*, since $z_{2(k+1)} = (1, \frac{1}{3}\sqrt{(y_{2k}-f(y_{2k}))^2+1}-1) \to (1,0)$.

3 Generalized BPPT for generalized contractions

Definition 3.1 [3] A mapping $T: A \to B$ is said to be a proximal contraction of the first kind if there exists a non-negative number $\alpha < 1$ such that

$$\begin{cases} d(u_1, Tx_1) = d(A,B), \\ d(u_2, Tx_2) = d(A,B) \end{cases} \quad \Rightarrow \quad d(u_1,u_2) \leq \alpha d(x_1,x_2)$$

for all $u_1, u_2, x_1, x_2 \in A$.

Recently in [9], Amini-Harandi *et al.* introduced the following new class of proximal contractions and proved the following result.

Definition 3.2 [9] A mapping $T: A \to B$ is said to be a (φ, g)-contraction if

$$\begin{cases} d(u_1, Tx_1) = d(A,B), \\ d(u_2, Tx_2) = d(A,B) \end{cases} \quad \Rightarrow \quad d(u_1,u_2) \leq \varphi\left(d\left(g(x_1),g(x_2)\right)\right)$$

for all $u_1, u_2, x_1, x_2 \in A$, where $\varphi: [0,\infty) \to [0,\infty)$ is a function obeying the following conditions:

$$\varphi(0) = 0, \qquad \varphi(t) < t, \qquad \limsup_{s \to t^+} \varphi(s) < t, \quad \forall t > 0,$$

and $g: A \to A$ is a mapping. If g is the identity operator, $T: A \to B$ is said to be a φ-contraction.

Definition 3.3 An element x in A is said to be a best proximity point of the mapping $T: A \to B$ if it satisfies the condition that $d(x, Tx) = d(A,B)$.

Theorem 3.4 [9] *Let A and B be nonempty closed subsets of a complete metric space (X,d) such that B is approximately compact with respect to A. Moreover, assume that A_0 and B_0 are nonempty. Let $T: A \to B$ and $g: A \to A$ satisfy the following conditions.*

(a) *T is a (φ, g)-proximal contraction,*
(b) $T(A_0) \subseteq B_0$,
(c) *g is a one-to-one continuous map such that $g^{-1} : g(A) \to A$ is uniformly continuous,*
(d) $A_0 \subseteq g(A_0)$.

Then there exists a unique element $x \in A$ such that $d(g(x), Tx) = d(A, B)$. Further, for any fixed element $x \in A_0$, the sequence $\{x_n\}$ defined by $d(g(x_{n+}), Tx_n) = d(A, B)$ converges to x.

The purpose of this section is to improve the result of Amini-Harandi *et al.* by using a new simple method of proof without the hypothesis of approximate compactness to B.

The following lemma is important for our results, which is actually a generalized Banach's fixed point theorem.

Lemma 3.5 *Let A be a subset of a complete metric space (X, d), and let $T : X \to X$ a continuous mapping with conditions*

$$d(Tx, Ty) \leq \varphi\big(d(x, y)\big), \quad \forall x, y \in A,$$

and $T(A) \subseteq A$, where $\varphi : [0, \infty) \to [0, \infty)$ is a function obeying the following conditions:

$$\varphi(0) = 0, \qquad \varphi(t) < t, \qquad \limsup_{s \to t^+} \varphi(s) < t, \quad \forall t > 0.$$

Then for any fixed element $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ converges to a point $x \in \overline{A}$. Further, x is a fixed point of T.

Proof We claim that $\{x_n\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there exist $\varepsilon_0 > 0$ and two subsequences of integers $\{n_k\}$, $\{m_k\}$ such that

$$d(x_{n_k}, x_{m_k}) \geq \varepsilon_0, \quad k = 1, 2, 3, \ldots. \tag{3.1}$$

Since $d(x_n, x_{n+1}) \to 0$ $(n \to \infty)$ is obvious, we may also assume

$$d(x_{n_k}, x_{m_k-1}) < \varepsilon_0 \tag{3.2}$$

by choosing m_k to be the smallest number exceeding n_k for which (3.1) holds. From (3.1) and (3.2) we have

$$\varepsilon_0 \leq d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k}) < \varepsilon_0 + d(x_{m_k-1}, x_{m_k}).$$

Taking the limit as $k \to \infty$, we get

$$\lim_{k \to \infty} d(x_{n_k}, x_{m_k}) = \varepsilon_0.$$

By the triangle inequality

$$\begin{aligned} d(x_{n_k}, x_{m_k}) &\leq d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, x_{m_k+1}) + d(x_{m_k+1}, x_{m_k}) \\ &\leq d(x_{n_k}, x_{n_k+1}) + \varphi\big(d(x_{n_k}, x_{m_k})\big) + d(x_{m_k+1}, x_{m_k}). \end{aligned}$$

Taking the sup-limit as $k \to \infty$, we get

$$\varepsilon_0 \leq \limsup_{k\to\infty} \varphi\big(d(x_{n_k}, x_{m_k})\big) < \varepsilon_0,$$

a contradiction. Therefore $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x \in X$ such that $x_n \to x$. It is obvious from the continuity of T and $d(x_n, Tx_n) \to 0$ that x is a fixed point of T. This completes the proof. □

Now, we are ready to state our main result in this section.

Theorem 3.6 *Let A and B be nonempty closed subsets of a complete metric space such that A_0 and B_0 are nonempty. Let $T : A \to B$ and $g : A \to A$ satisfy the following conditions.*

(a) *g is a one-to-one continuous map such that $g^{-1} : g(A) \to A$ is uniformly continuous;*

(b) *T is a (φ, g)-contraction with $T(A_0) \subset B_0$.*

Then there exists a unique element $x^* \in A$ such that $d(g(x^*), Tx^*) = d(A, B)$. Further, for any fixed element $x_0 \in A_0$, the sequence defined by $d(g(x_{n+1}), Tx_n) = d(A, B)$, converges to x^*.

Proof Let

$$D(x, y) = d\big(g(x), g(y)\big), \quad \forall x, y \in A.$$

It is obvious that $D(x, y)$ is a metric on the A. Now we prove (A, D) is a complete metric space. Let $\{x_n\} \subseteq (A, D)$ be a Cauchy sequence, we have

$$\lim_{n,m\to\infty} D(x_n, x_m) = \lim_{n,m\to\infty} d\big(g(x_n), g(x_m)\big) = 0.$$

Since $g^{-1} : g(A) \to A$ is uniformly continuous, we have

$$\lim_{n,m\to\infty} d(x_n, x_m) = \lim_{n,m\to\infty} d\big(g^{-1}g(x_n), g^{-1}g(x_m)\big) = 0$$

and hence $\{x_n\} \subseteq (A, d)$ is a Cauchy sequence. Since (A, d) is a complete metric space, there exists an element $x \in A$ such that $d(x_n, x) \to 0$ as $n \to \infty$. Since g is continuous, we have $D(x_n, x) \to 0$ as $n \to \infty$. This completes the proof of the completeness of (A, D).

For any $x \in A_0$, from (b) we know $Tx \in B_0$. Since T is a (φ, g)-contraction, there exists a unique $z \in A_0$ such that $d(z, Tx) = d(A, B)$. We denote $z = T_0 Tx$. Then $T_0 : S(A_0) \to A_0$ is a mapping. Further, we define a composite mapping $u = T_0 Tx$ from A_0 into itself. Since T is a (φ, g)-contraction, we have

$$d(T_0 Tx_1, T_0 Tx_2) \leq \varphi\big(d\big(g(x_1), g(x_2)\big)\big)$$

for any $x_1, x_2 \in A_0$,

$$d\big(gg^{-1}(T_0 Tx_1), gg^{-1}(T_0 Tx_2)\big) \leq \varphi\big(d\big(g(x_1), g(x_2)\big)\big)$$

for any $x_1, x_2 \in A_0$, and

$$D\big(g^{-1}T_0 Tx_1, g^{-1}T_0 Tx_2\big) \leq \varphi\big(D(x_1, x_2)\big)$$

for any $x_1, x_2 \in A_0$. From the above inequality, we also know that the mapping $u = g^{-1}T_0Tx$ is continuous on the A_0, so we can expand the definition of $u = g^{-1}T_0Tx$ onto $\overline{A}_0$ such that it is still continuous on the $\overline{A}_0$. By using Lemma 3.5, we know for any fixed element $x_0 \in A_0$, that the sequence $\{x_n\}$ defined by

$$x_{n+1} = g^{-1}T_0Tx_n,$$

which is equivalent to $d(g(x_{n+1}), Tx_n) = d(A,B)$, converges to a point $x^* \in \overline{A}$. Further, x^* is a fixed point of $g^{-1}T_0T$. That is, $x^* = g^{-1}T_0Tx^*$ which is equivalent to $d(g(x^*), Tx^*) = d(A,B)$. Since T is a (φ, g)-contraction, this x^* is unique. This completes the proof. □

Corollary 3.7 *Let A and B be nonempty closed subsets of a complete metric space such that A_0 and B_0 are nonempty. Let $T : A \to B$ and $g : A \to A$ satisfy the following conditions.*

(a) *g is a one-to-one continuous map such that $g^{-1} : g(A) \to A$ is uniformly continuous;*

(b) *T is a proximal contraction of the first kind with $T(A_0) \subset B_0$.*

Then there exists a unique element $x^* \in A$ such that $d(g(x^*), Tx^*) = d(A,B)$. Further, for any fixed element $x_0 \in A_0$, the sequence defined by $d(g(x_{n+1}), Tx_n) = d(A,B)$ converges to x^*.

Corollary 3.8 *Let A and B be nonempty closed subsets of a complete metric space such that A_0 and B_0 are nonempty. Let $T : A \to B$ be is a φ-contraction with $T(A_0) \subset B_0$. Then there exists a unique best proximity point $x^* \in A$ of T. Further, for any fixed element $x_0 \in A_0$, the sequence defined by $d(x_{n+1}, Tx_n) = d(A,B)$ converges to x^*.*

Remark 3.9 In Theorem 3.6, we do not need the hypothesis of approximate compactness to B. Therefore, Theorem 3.6 improved substantially the results of Theorem 3.4. On the other hand, the method of proof is also different.

4 BPPT for proximal cyclic contraction mappings

Definition 4.1 [1] Given non-self-mappings $S : A \to B$ and $T : B \to A$, the pair (S,T) is said to form a proximal cyclic contraction if there exists a non-negative number $\alpha < 1$ such that

$$\begin{cases} d(u, Sx) = d(A,B), \\ d(v, Ty) = d(A,B) \end{cases} \quad \Rightarrow \quad d(u,v) \le \alpha d(x,y) + (1-\alpha)d(A,B)$$

for all $u, x \in A$ and $v, y \in B$.

Definition 4.2 [1] A mapping $S : A \to B$ is said to be a proximal contraction of the first kind if there exists a non-negative number $\alpha < 1$ such that

$$\begin{cases} d(u_1, Sx_1) = d(A,B), \\ d(u_2, Sx_2) = d(A,B) \end{cases} \quad \Rightarrow \quad d(u_1,u_2) \le \alpha d(x_1,x_2)$$

for all $u_1, u_2, x_1, x_2 \in A$.

Definition 4.3 An element x in A is said to be a best proximity point of the mapping $S : A \to B$ if it satisfies the condition that $d(x, Sx) = d(A,B)$.

In [1], the author proved the following result, a generalized best proximity point theorem for non-self-proximal contractions of the first kind.

Theorem 4.4 [1] *Let A and B be nonempty closed subsets of a complete metric space such that A_0 and B_0 are nonempty. Let $S : A \to B$, $T : B \to A$ and $g : A \cup B \to A \cup B$ satisfy the following conditions.*

(a) *S and T are proximal contractions of the first kind.*
(b) *$S(A_0) \subset B_0$ and $T(B_0) \subset A_0$.*
(c) *The pair (S, T) forms a proximal cyclic contraction.*
(d) *g is an isometry.*
(e) *$A_0 \subset g(A_0)$ and $B_0 \subset g(B_0)$.*

Then there exist a unique element x in A and a unique element y in B satisfying the conditions that

$$d(x, y) = d(gx, Sx) = d(gy, Ty) = d(A, B).$$

Further, for any fixed element x_0 in A_0, the sequence $\{x_n\}$, defined by

$$d(gx_{n+1}, Sx_n) = d(A, B),$$

converges to the element x. For any fixed element y_0 in B_0, the sequence $\{y_n\}$, defined by

$$d(gy_{n+1}, Ty_n) = d(A, B),$$

converges to the element y.

On the other hand, a sequence $\{u_n\}$ of elements in A converges to x if there is a sequence $\{\varepsilon_n\}$ of positive numbers for which $\lim_{n\to\infty} \varepsilon_n = 0$, $d(u_{n+1}, z_{n+1}) \leq \varepsilon_n$, where $z_{n+1} \in A$ satisfies the condition that $d(z_{n+1}, Su_n) = d(A, B)$.

In 1973, Geraghty introduced the Geraghty-contraction and obtained Theorem 4.6.

Definition 4.5 [14] Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a *Geraghty-contraction* if there exists $\beta \in \Gamma$ such that for any $x, y \in X$

$$d(Tx, Ty) \leq \beta\big(d(x, y)\big) \cdot d(x, y),$$

where the class Γ denotes those functions $\beta : [0, \infty) \to [0, 1)$ satisfying the following condition:

$$\beta(t_n) \to 1 \quad \Rightarrow \quad t_n \to 0.$$

Theorem 4.6 [14] *Let (X, d) be a complete metric space and $T : X \to X$ be a Geraghty-contraction. Then T has a unique fixed point x^* and, for any $x_0 \in X$, the iterative sequence $x_{n+1} = Tx_n$ converges to x^*.*

Definition 4.7 [2] A mapping $S : A \to B$ is called Geraghty's proximal contraction of the first kind if there exists $\beta \in \Gamma$ such that

$$\begin{cases} d(u_1, Sx_1) = d(A, B), \\ d(u_2, Sx_2) = d(A, B) \end{cases} \quad \Rightarrow \quad d(u_1, u_2) \leq \beta\big(d(x_1, x_2)\big) d(x_1, x_2)$$

for all $u_1, u_2, x_1, x_2 \in A$.

In [2], the authors proved the following result.

Theorem 4.8 [2] *Let A and B be nonempty closed subsets of a complete metric space such that A_0 and B_0 are nonempty. Let $S : A \to B$, $T : B \to A$, and $g : A \cup B \to A \cup B$ satisfy the following conditions.*

(a) *S, T are Geraghty's proximal contractions of the first kind.*
(b) *$S(A_0) \subset B_0$ and $T(B_0) \subset A_0$.*
(c) *The pair (S, T) forms a proximal cyclic contraction.*
(d) *g is an isometry.*
(e) *$A_0 \subset g(A_0)$ and $B_0 \subset g(B_0)$.*

Then there exist a unique element x^* in A and a unique element y^* in B satisfying the conditions that

$$d\bigl(x^*, y^*\bigr) = d\bigl(gx^*, Sx^*\bigr) = d\bigl(gy^*, Ty^*\bigr) = d(A, B).$$

Further, for any fixed element x_0 in A_0, the sequence $\{x_n\}$, defined by

$$d(gx_{n+1}, Sx_n) = d(A, B),$$

converges to the element x^*. For any fixed element y_0 in B_0, the sequence $\{y_n\}$, defined by

$$d(gy_{n+1}, Ty_n) = d(A, B),$$

converges to the element y^*.

On the other hand, a sequence $\{u_n\}$ of elements in A converges to x if there is a sequence $\{\varepsilon_n\}$ of positive numbers for which $\lim_{n\to\infty} \varepsilon_n = 0$, $d(u_{n+1}, z_{n+1}) \le \varepsilon_n$, where $z_{n+1} \in A$ satisfies the condition that $d(z_{n+1}, Su_n) = d(A, B)$.

The purpose of this section is to prove best proximity point theorems for proximal cyclic contractions and weakly proximal contractions by using the new method of proof. Our results improve and extend the recent results of some others. Meanwhile, we point out a mistake in Theorem 4.8.

Theorem 4.9 *Let A and B be nonempty closed subsets of a complete metric space such that A_0 and B_0 are nonempty. Let $S : A \to B$, $T : B \to A$, and $g : A \cup B \to A \cup B$ satisfy the following conditions.*

(a) *S, T are Geraghty's proximal contractions of the first kind.*
(b) *$S(A_0) \subset B_0$ and $T(B_0) \subset A_0$.*
(c) *The pair (S, T) forms a proximal cyclic contraction.*
(d) *g is an isometry.*
(e) *$A_0 \subset g(A_0)$ and $B_0 \subset g(B_0)$.*

Then there exist a unique element x^* in A and a unique element y^* in B satisfying the conditions that

$$d\bigl(x^*, y^*\bigr) = d\bigl(gx^*, Sx^*\bigr) = d\bigl(gy^*, Ty^*\bigr) = d(A, B).$$

Further, for any fixed element x_0 in A_0, the sequence $\{x_n\}$, defined by

$$d(gx_{n+1}, Sx_n) = d(A, B),$$

converges to the element x^*. For any fixed element y_0 in B_0, the sequence $\{y_n\}$, defined by

$$d(gy_{n+1}, Ty_n) = d(A, B),$$

converges to the element y^*.

On the other hand, assume $\beta(t) \le \alpha < 1$. Then a sequence $\{u_n\}$ of elements in A converges to x^* if there is a sequence $\{\varepsilon_n\}$ of positive numbers for which $\lim_{n\to\infty} \varepsilon_n = 0$, $d(u_{n+1}, z_{n+1}) \le \varepsilon_n$, where $z_{n+1} \in A$ satisfies the condition that $d(gz_{n+1}, Su_n) = d(A, B)$.

Proof For any $x \in A_0$, from (b) we know $Sx \in B_0$. Since S is a Geraghty-contraction, there exists a unique $z \in A_0$ such that $d(z, Sx) = d(A, B)$. We denote $z = S_0Sx$. Then $S_0 : S(A_0) \to A_0$ is a mapping. Further, we define a composite mapping $u = g^{-1}S_0Sx$ from A_0 into itself. Since S is a Geraghty-contraction, we have

$$d\big(g^{-1}S_0Sx_1, g^{-1}S_0Sx_2\big) = d(S_0Sx_1, S_0Sx_2) \le \beta\big(d(x_1, x_2)\big)\big(d(x_1, x_2)\big) \tag{4.1}$$

for any $x_1, x_2 \in A_0$. From the above inequality, we also know that the mapping $u = g^{-1}S_0Sx$ is continuous, so we can expand the definition of $u = g^{-1}S_0Sx$ onto $\overline{A}_0$. Because we do not need the continuity of function $\beta(t)$, we define another function $\bar{\beta}(t) : [0, \infty) \to [0, 1)$ as follows:

$$\bar{\beta}(t) = \begin{cases} \beta(0), & t = 0, \\ \max\{\limsup_{r_n\to t} \beta(r_n), \beta(t)\}, & t > 0. \end{cases}$$

It is easy to see $\bar{\beta}(t) \in \Gamma$. From (4.1) we get

$$d\big(g^{-1}S_0Sx_1, g^{-1}S_0Sx_2\big) = d(S_0Sx_1, S_0Sx_2) \le \bar{\beta}\big(d(x_1, x_2)\big)\big(d(x_1, x_2)\big) \tag{4.2}$$

for any $x_1, x_2 \in \overline{A}_0$. From (4.2) we know $g^{-1}S_0S : \overline{A}_0 \to \overline{A}_0$ is a Geraghty-contraction. By using Theorem 4.6, we claim $g^{-1}S_0S$ has a unique fixed point x^* in $\overline{A}_0$, that is, $x^* = g^{-1}S_0Sx^*$, which implies $gx^* = S_0Sx^*$ and hence $d(gx^*, Sx^*) = d(A, B)$. By using the same method, we can prove that there exists a unique element y^* in $\overline{B}_0$ such that $d(gy^*, Ty^*) = d(A, B)$. On the other hand, from (c) we have

$$d\big(x^*, y^*\big) = d\big(gx^*, gy^*\big) \le \alpha d\big(x^*, y^*\big) + (1 - \alpha)d(A, B),$$

which implies $d(x^*, y^*) \le d(A, B)$ and hence $d(x^*, y^*) = d(A, B)$.

Since $g^{-1}S_0S$ is a Geraghty-contraction, for any fixed element x_0 in A_0, the sequence $\{x_n\}$, defined by $x_{n+1} = g^{-1}S_0Sx_n$ converges to the element x^*. This sequence $\{x_n\}$ also is defined by $d(gx_{n+1}, Sx_n) = d(A, B)$. For the same reason, for any fixed element y_0 in B_0, the sequence $\{y_n\}$, defined by $y_{n+1} = g^{-1}T_0Tx_n$, converges to the element y^*. This sequence $\{y_n\}$ also is defined by $d(gy_{n+1}, Ty_n) = d(A, B)$.

Finally, $d(gz_{n+1}, Su_n) = d(A,B) \Leftrightarrow z_{n+1} = g^{-1}S_0Su_n$, which gives us

$$\begin{aligned} d(x_{n+1}, u_{n+1}) &\le d(x_{n+1}, z_{n+1}) + d(z_{n+1}, u_{n+1}) \\ &\le d\bigl(g^{-1}S_0Sx_n, g^{-1}S_0Su_n\bigr) + d(z_{n+1}, u_{n+1}) \\ &\le \beta\bigl(d(x_n, u_n)\bigr)d(x_n, u_n) + d(z_{n+1}, u_{n+1}) \\ &\le \alpha d(x_n, u_n) + d(z_{n+1}, u_{n+1}). \end{aligned}$$

It is easy to prove $d(x_{n+1}, u_{n+1}) \to 0$ which implies $u_n \to x^*$. This completes the proof. □

Remark 4.10 If $\beta(t) \equiv \alpha < 1$, then Theorem 4.9 yields Theorem 4.4.

Remark 4.11 In the reference [2], from line 15 to line 20 on page 7, the following argument is wrong, so the final conclusion of Theorem 1.8 is not correct.

The wrong argument For any $\varepsilon > 0$, choose a positive integer N such that $\varepsilon_n \le \varepsilon$ for all $n > N$. Observe that

$$\begin{aligned} d(x_{n+1}, u_{n+1}) &\le d(x_{n+1}, z_{n+1}) + d(z_{n+1}, u_{n+1}) \\ &\le \beta\bigl(d(x_n, u_n)\bigr)d(x_n, u_n) + \varepsilon_n \\ &\le d(x_n, u_n) + \varepsilon. \end{aligned}$$

Since ε is arbitrary, we can conclude that for all $n \ge N$ the sequence $\{d(x_n, u_n)\}$ is nonincreasing and bounded below and hence converges to some non-negative real number r.

Counter-example Let

$$d_n = \sum_{k=1}^{n} \frac{1}{k}, \quad n = 1, 2, 3, \ldots,$$

then

$$d_{n+1} \le d_n + \frac{1}{n+1},$$

and, for any $\varepsilon > 0$, we can choose a positive integer N such that $\frac{1}{n+1} \le \varepsilon$ for all $n > N$, and hence

$$d_{n+1} \le d_n + \varepsilon.$$

But d_n is not nonincreasing and $d_n \to \infty$ as $n \to \infty$.

Next we prove the best proximal point theorem for a weakly proximal contractive mapping.

Definition 4.12 Let A and B be nonempty subsets of a complete metric space. A mapping $S : A \to B$ is called weak proximal contraction if

$$\begin{cases} d(u_1, Sx_1) = d(A,B), \\ d(u_2, Sx_2) = d(A,B) \end{cases} \quad \Rightarrow \quad d(u_1, u_2) \le \bar{\alpha}(x_1, x_2)d(x_1, x_2)$$

for all $u_1, u_2, x_1, x_2 \in A$, where for the function $\bar{\alpha} : X \times X \to [0,1)$ we have, for every $0 < a < b$,

$$\theta(a,b) = \sup\{\bar{\alpha}(x,y) : a \leq d(x,y) \leq b\} < 1.$$

Theorem 4.13 *Let A and B be nonempty closed subsets of a complete metric space such that A_0 and B_0 are nonempty. Let $S : A \to B$, $T : B \to A$ and $g : A \cup B \to A \cup B$ satisfy the following conditions.*

(a) *S, T are weakly proximal contractions.*
(b) *$S(A_0) \subset B_0$ and $T(B_0) \subset A_0$.*
(c) *The pair (S,T) forms a proximal cyclic contraction.*
(d) *g is an isometry.*
(e) *$A_0 \subset g(A_0)$ and $B_0 \subset g(B_0)$.*

Then there exist a unique element x^* in A and a unique element y^* in B satisfying the conditions that

$$d(x^*, y^*) = d(gx^*, Sx^*) = d(gy^*, Ty^*) = d(A,B).$$

Further, for any fixed element x_0 in A_0, the sequence $\{x_n\}$, defined by

$$d(gx_{n+1}, Sx_n) = d(A,B),$$

converges to the element x^*. For any fixed element y_0 in B_0, the sequence $\{y_n\}$, defined by

$$d(gy_{n+1}, Ty_n) = d(A,B),$$

converges to the element y^*.

On the other hand, assume $\bar{\alpha}(x,y) \leq \alpha < 1$. Then a sequence $\{u_n\}$ of elements in A converges to x^* if there is a sequence $\{\varepsilon_n\}$ of positive numbers for which $\lim_{n\to\infty} \varepsilon_n = 0$, $d(u_{n+1}, z_{n+1}) \leq \varepsilon_n$, where $z_{n+1} \in A$ satisfies the condition that $d(gz_{n+1}, Su_n) = d(A,B)$.

Proof For any $x \in A_0$, from (b) we know $Sx \in B_0$. Since S is a weakly contractive mapping, there exists a unique $z \in A_0$ such that $d(z, Sx) = d(A,B)$. We denote $z = S_0Sx$. Then $S_0 : S(A_0) \to A_0$ is a mapping. Further, we define a composite mapping $u = g^{-1}S_0Sx$ from A_0 into itself. Since S is a weakly contractive mapping, then we have

$$d(g^{-1}S_0Sx_1, g^{-1}S_0Sx_2) = d(S_0Sx_1, S_0Sx_2) \leq \bar{\alpha}(x_1,x_2)d(x_1,x_2)$$

for any $x_1, x_2 \in A_0$. From above inequality, we also know the mapping $u = g^{-1}S_0Sx$ is continuous, so we can expand the definition of $u = g^{-1}S_0Sx$ onto $\overline{A}_0$. From the above inequality we know that $g^{-1}S_0S : \overline{A}_0 \to \overline{A}_0$ is a weak contractive mapping. By using Theorem 2.2, we claim that $g^{-1}S_0S$ has a unique fixed point x^* in $\overline{A}_0$, that is, $x^* = g^{-1}S_0Sx^*$, which implies $gx^* = S_0Sx^*$ and hence $d(gx^*, Sx^*) = d(A,B)$. By using the same method, we can prove that there exists a unique element y^* in $\overline{B}_0$ such that $d(gy^*, Ty^*) = d(A,B)$. On the other hand, from (c) we have

$$d(x^*, y^*) = d(gx^*, gy^*) \leq \alpha d(x^*, y^*) + (1-\alpha)d(A,B),$$

which implies $d(x^*, y^*) \leq d(A,B)$ and hence $d(x^*, y^*) = d(A,B)$.

Since $g^{-1}S_0S$ is a weak contractive mapping, for any fixed element x_0 in A_0, the sequence $\{x_n\}$, defined by $x_{n+1} = g^{-1}S_0Sx_n$ converges to the element x^*. This sequence $\{x_n\}$ also is defined by $d(gx_{n+1}, Sx_n) = d(A,B)$. By the same reason, for any fixed element y_0 in B_0, the sequence $\{y_n\}$, defined by $y_{n+1} = g^{-1}T_0Tx_n$ converges to the element y^*. This sequence $\{y_n\}$ also is defined by $d(gy_{n+1}, Ty_n) = d(A,B)$.

Finally, $d(gz_{n+1}, Su_n) = d(A,B) \Leftrightarrow z_{n+1} = g^{-1}S_0Su_n$, which gives us

$$\begin{aligned} d(x_{n+1}, u_{n+1}) &\le d(x_{n+1}, z_{n+1}) + d(z_{n+1}, u_{n+1}) \\ &\le d\bigl(g^{-1}S_0Sx_n, g^{-1}S_0Su_n\bigr) + d(z_{n+1}, u_{n+1}) \\ &\le \bar{\alpha}(x_n, u_n)d(x_n, u_n) + d(z_{n+1}, u_{n+1}) \\ &\le \alpha d(x_n, u_n) + d(z_{n+1}, u_{n+1}). \end{aligned}$$

It is easy to prove $d(x_{n+1}, u_{n+1}) \to 0$, which implies $u_n \to x^*$, This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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