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General complementarities on complete partial orders

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Abstract

This paper proves the existence of a Nash equilibrium for *extended* (*semi-*) *uniform* *g-modular* games, *i.e.*, non-cooperative games where the strategy space is a complete partially ordered set, and the best reply correspondence satisfies certain monotonicity requirements.

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1 Introduction

Games with strategic complementarities (see Amir [1], Vives [2], among others) are based on two fundamental properties: (i) the ability to order the elements of the players' strategy sets; and (ii) the strategic complementarity which implies upward sloping best replies (see Topkis [3], Cooper [4]). Contrary to the topologically oriented approaches that generally require quasiconcavity of each utility function in own action, the properties of games with strategic complementarities release the reliance on mixed strategies to ensure the existence of a Nash equilibrium.

Games with strategic complementarities in which the joint best reply correspondence is Veinott increasing (Veinott [5], Calciano [6]) rely on Zhou's [7] extension of Tarski fixed point theorem (Tarski [8]) to set-valued maps. Earlier attempts that aim to generalize games with strategic complementarities have mainly concentrated on the increasingness notions (see Antoniadou [9], Calciano [6]). In particular, recently, using the notions of upper and lower increasingness (Smithson [10]) which are substantially weaker than Veinott increasingness, Calciano [11] has presented games with general complementarities and g-modular games that retain the main properties of supermodular games. On the other hand, among those very few attempts to extend the set of strategy sets, d'Orey [12] has employed quasilattices while requiring even a stronger notion of increasingness than that of Veinott [5]. A notion of quasilattices has also been introduced by Calciano [11], and this notion has been linked to that of lattices by means of specific theorems, contrary to the notion of d'Orey [12]. Specifically, Calciano [11] has provided the conditions under which quasilattices and sublattices coincide after showing that the set of maximizers of a g-modular function on a lattice is a quasilattice.[a]

The purpose of this paper is to further extend the theory and the scope of application of games with strategic complementarities. Extending the set of games with (weak) general complementarities (see Calciano [11]) to include games with complete partially ordered (CPO) strategy sets, our aim is to provide a weaker structure on the set of strategy profiles. To do so, we show that the fixed point set of an order-preserving set-valued map on a CPO is itself a nonempty CPO. Note that the nonemptiness of the fixed point set already follows from Abian and Brown [13] and from Smithson [10], whereas the existence of the least fixed point follows from Stouti [14]. However, we present short proofs in the way that Echenique [15] has proved Tarski [8] and Zhou [7] fixed point theorems by means of transfinite recursion. Beside the existence result, the approach may provide a convenient way of construction towards the least Nash equilibrium. For this approach, we assume that the correspondence satisfies a certain notion of increasingness (Calciano [16]) weaker than that of Veinott [5], hence than that of Stouti [14]. Also, we require that the correspondence has a bottom element for each member of its domain. As for the result on the order structure, we additionally prove that the chains in the fixed point set have a supremum.

The contribution of this paper consists in showing that the theory of games with general complementarities developed by Calciano [11] can be shown to hold in the context of CPOs as well. In particular, the paper analyzes the existence and the order structure of Nash equilibria for *extended* (*semi-*) *uniform g-modular* games in which the strategy space is a CPO, and the joint best reply correspondence satisfies monotonicity requirements weaker than that of Veinott.

As the strategy set of each player is no longer required to be a complete lattice, our results prove to be crucial in providing the existence of equilibrium for the games in which at least one of the players has a multidimensional strategy set and faces a form of budget constraint, or capacity constraint, or law regulation that makes some of her strategies infeasible or unavailable. For instance, if such constraints are introduced into multi-stage R&D models (Amir [1]), or into Bertrand competition with pricing and advertising (Vives [2], Calciano [17]), or into generalized contest games (Acemoglu and Jensen [18]), the strategy set would no longer be a lattice, but a CPO. In particular, in a generalized contest game, the players make two types of costly effort, each corresponding to a separate contest. One contest can correspond to an educational competition while the other can represent competition in sports.[b] Since the total amount of effort that can be made by players is bounded from above, the maximum amount cannot be exerted for both type of contests. Thus, the strategy set ceases to be a lattice. For such cases, the existence of equilibrium cannot be verified by the existing results in the context of games with strategic complementarities. However, utilizing games with general complementarities, we show that the set of equilibrium is indeed a nonempty CPO.

The article is organized as follows. The next section introduces the related definitions, and it shows the existence of fixed points on CPOs. In Section 3, extended (semi-) uniform g-modular games are presented, and the Nash equilibrium set is characterized. Finally, Section 4 concludes.

2 Preliminaries

Let X be a nonempty set on which a reflexive, antisymmetric, and transitive binary relation, denoted by $\leq$, is defined. $(X, \leq)$ turns out to be a complete partially ordered set (CPO) if (i) X has a bottom element, $\perp$; and (ii) for each directed subset D of X, the supremum exists. An equivalent definition writes as (ii′) each chain in X has a supremum.[c]

Furthermore, only for notational purposes, we let Y to be some complete lattice satisfying $X \subset Y$. By the topological characterization of completeness (see Birkhoff [19]), Y is compact in its interval topology which is the topology generated by taking the close intervals, $[y, z] = \{x \in Y : y \leq x \leq z\}$ with $y, z \in Y$, as a subbasis of closed sets. Moreover, let 1 denote the greatest element of Y.[d]

In the definitions we use throughout the paper, upper contour set of every $x \in X$, formally $U_x \equiv \{y \in X \mid x \leq y\}$, is essential. On a CPO, this set can naturally be defined as follows. U_x is the union of all directed subsets of X including x as a bottom element. Though, in this study, we obtain this set by a different approach: Take an element outside X which is greater than x. The greatest element, 1, of the complete lattice Y is utilized only at this stage. It is straightforward that $U_x = [x, 1] \cap X$ for every $x \in X$. We prefer this approach, because the following proofs become more tractable. Also, our results become comparable with those of Calciano [11] which this paper builds on. Finally, since X is a CPO, $\vee$ ($\wedge$) may not be well defined on X; if so, we set $\vee$ ($\wedge$) as $\vee_Y$ ($\wedge_Y$).

A map $f : X \to X$ is said to be order preserving if for any $x, y \in X$ with $x < y$, we have $f(x) \leq f(y)$. For the following definitions, consider a correspondence $F : X \to X$. The correspondence F is upper increasing if for every $x, y \in X$, $x \leq y$ implies that for every $a \in F(x)$, there is some $b \in F(y)$ such that $a \leq b$. It is lower increasing if for every $x, y \in X$, $x \leq y$ implies that for every $b \in F(y)$, there is some $a \in F(x)$ such that $a \leq b$. These definitions are attributed to Smithson [10]. Moreover, the notions of strong upper/lower increasingness introduced by Calciano [16] are defined as follows. The correspondence F is strongly upper increasing if for every $x, y \in X$ with $x \leq y$, every $a \in F(x)$, and every $b \in F(y)$, there is some $p \in F(y)$ such that $p \in [a, a \vee b] \cap X$. It is strongly lower increasing if for every $x, y \in X$ with $x \leq y$, every $a \in F(x)$, and every $b \in F(y)$, there is some $q \in F(x)$ such that $q \in [a \wedge b, b] \cap X$.

In this paper, we utilize uniform g-modularity defined by Calciano [11]. We first state a regularity condition, and then we slightly change the definition in order to guarantee that our definition is applicable to a CPO-setting for any CPO.

Condition 1 Let Y be a complete lattice, $X \subset Y$ be a CPO. We say that $(a, b) \in X \times X$ satisfies Condition 1 if $b \nleq a$ implies that $[a \wedge b, b) \cap X \neq \varnothing$ and $a \nleq b$ implies that $[a \wedge b, a) \cap X \neq \varnothing$.

Definition 1 Let Y be a complete lattice, $X \subset Y$ be a CPO, and T be a poset. A function $u : X \times T \to R$ is semi-uniform g-modular in (x, t) on $X \times T$ if for every $(a, b) \in X \times X$ satisfying Condition 1:

(i) $b \nleq a$ implies that there are $p \in [a \wedge b, b)$ and $q \in X$ such that:
 (i.a) $\forall t \in T$: $u(a, t) + u(b, t) \leq u(p, t) + u(q, t)$,
 (i.b) and furthermore, $\forall t', t'' \in T$ with $t' < t''$:

$$u(q, t') - u(b, t') \leq u(q, t'') - u(b, t'');$$

(ii) $a \nleq b$ implies that there are $p' \in [a \wedge b, a)$ and $q' \in X$ such that:
 (ii.a) $\forall t \in T$: $u(a, t) + u(b, t) \leq u(p', t) + u(q', t)$,
 (ii.b) and furthermore, $\forall t', t'' \in T$ with $t' < t''$:

$$u(q', t') - u(b, t') \leq u(q', t'') - u(b, t'').$$

The above definition of semi-uniform g-modularity does not require q and q' to be included in specific sets. For our results, we also need a version in which the sets including q and q' are restricted.

Condition 2 Let Y be a complete lattice, $X \subset Y$ be a CPO. We say that $(a,b) \in X \times X$ satisfies Condition 2 if $b \not\leq a$ implies that $(a, a \vee b] \cap X \neq \varnothing$ and $a \not\leq b$ implies that $(b, a \vee b] \cap X \neq \varnothing$.

Definition 2 Let Y be a complete lattice, $X \subset Y$ be a CPO, and T be a poset. A function $u : X \times T \to R$ is uniform g-modular on CPO in (x,t) on $X \times T$ if for every $(a,b) \in X \times X$ satisfying Conditions 1 and 2:

(i) $b \not\leq a$ implies that there are $p \in [a \wedge b, b) \cap X$ and $q \in (a, a \vee b] \cap X$ such that:
  (i.a) $\forall t \in T$: $u(a,t) + u(b,t) \leq u(p,t) + u(q,t)$,
  (i.b) and furthermore, $\forall t', t'' \in T$ with $t' < t''$:

$$u(q,t') - u(b,t') \leq u(q,t'') - u(b,t'');$$

(ii) $a \not\leq b$ implies that there are $p' \in [a \wedge b, a) \cap X$ and $q' \in (b, a \vee b] \cap X$ such that:
  (ii.a) $\forall t \in T$: $u(a,t) + u(b,t) \leq u(p',t) + u(q',t)$,
  (ii.b) and furthermore, $\forall t', t'' \in T$ with $t' < t''$:

$$u(q',t') - u(b,t') \leq u(q',t'') - u(b,t'').$$

Finally, the set of fixed points of f relative to X, denoted by $\varepsilon(f)$, is defined as follows:[e]

$$\varepsilon(f) = \{x \in X : x = f(x)\}.$$

Synonymously, the set of fixed points of F relative to X is then given by

$$\varepsilon(F) = \{x \in X : x \in F(x)\}.$$

2.1 The existence of fixed points on CPOs

The first theorem is the existence of fixed points of an order-preserving self-map on a CPO. In fact, the nonemptiness of the fixed point set already follows from Abian and Brown [13]; however, our proof is in line with Echenique's [15] arguments. Echenique's [15] proof is constructive in the sense that it gives a procedure for finding a fixed point. Yet, since the proof utilizes ordinal numbers, there are notions of constructiveness that the proof would not satisfy.

Theorem 1 (Abian and Brown [13]) *Let X be a CPO, and $f : X \to X$ be an order-preserving self-map. Then the set of fixed points of f is a nonempty CPO.*

Proof See the Appendix. □

Moving to correspondences on CPOs, we know that Smithson [10] has proved the nonemptiness of the fixed point set, and that Stouti [14] has shown the existence of the least fixed point. The following theorem indicates that a weaker notion of increasingness is sufficient for the existence of the least fixed point. The proof relies on the approach given in the proof of Theorem 1.

Theorem 2 *Let X be a CPO, and $F : X \to X$ be a correspondence such that for every $x \in X$, $F(x)$ has a bottom element. If F is lower increasing, then F has a least fixed point.*

Proof See the Appendix. □

2.2 The order structure of the set of fixed points

In this section, we provide the conditions under which the set of fixed points turns out to be a CPO. To our knowledge, this paper is the first to present such a result. At this point, we need further notations to state the necessary assumptions for the proof. Let

$$A = \{x \in X : \exists y \in F(x) : x \leq y\}$$

be the set of elements of X at which F weakly jumps the diagonal. For a fixed $h \in A$, define $F_h : [h, 1] \cap X \to [h, 1] \cap X$ as

$$F_h(x) = F(x) \cap [h, 1].$$

The set A and the correspondence F_h are borrowed from Calciano [11], and they are well defined in a CPO setting. They are essentially utilized in Lemma 1 and Theorem 3 in a similar way to Calciano's [11].

Lemma 1 *Let $F : X \to X$ be a correspondence. If F is upper increasing, then F_h is nonempty-valued.*

Proof See Calciano [11]. □

Theorem 3 *Let Y be a complete lattice, $X \subset Y$ be a CPO, and $F : X \to X$ be a correspondence. If* (i) *F is strongly upper increasing and strongly lower increasing*; (ii) $\forall x \in X$: *$F(x)$ is a CPO in X; and* (iii) $\forall h \in A$ *and* $\forall x \in X$: *$F_h(x)$ has a bottom element whenever nonempty, then the fixed point set of F is a CPO.*

Proof Since strong lower increasingness implies lower increasingness, we say that $\varepsilon(F)$ has a bottom element by Theorem 2. Now, take any directed nonempty subset E of $\varepsilon(F)$. We have to show that $\bigvee_{\varepsilon(F)} E$ exists. If E is finite, then the result is trivial. Then assume otherwise, and let $h = \bigvee E$. Noting that E is a directed set, let (x_n) be an increasing sequence consisting of all elements of E. As h is the least upper bound of E, the infinite intersection $\bigcap\{x \in X \mid x_i \leq x < h\} = \varnothing$.

Assume that $\exists a \in F(h)$ such that $a \geq h$. Then $h \in A$. If not, there are two cases: (i) $\exists a \in F(h)$ such that $a < h$; and (ii) $\forall a \in F(h)$, a and h are unordered. Case (i) has two subcases: (i.a) $\exists x_k \in E$ such that $x_k \geq a$; and (i.b) $\forall x_i \in E$, $x_i \ngeq a$. Under case (i.a), by strong upper increasingness, there exists $a_i \in F(h)$ such that $a_i \in [x_i, x_i \vee a]$ for every $i \geq k$. That is $a_i = x_i$ for every $i \geq k$. We then have a directed subset E' of $F(h)$. It is obvious that $\bigvee E' = h$. Since $F(h)$ is a CPO, h is included in $F(h)$. Under cases (i.b) and (ii), we can construct a directed subset of $F(h)$, denoted by E'', in such a way that for every $x_i \in E$, there exists

$b_i \in E''$ such that $b_i \geq x_i$.[f] Since $F(h)$ is a CPO, the supremum $\bigvee E''$ is included in $F(h)$. Since the supremum of E is unique, by construction, we have $\bigvee E'' \geq h$. That is to say, the correspondence F weakly jumps the diagonal at h.

In all of the above cases, $h \in A$. For every $x \in [h, 1] \cap X$, Lemma 1 verifies that $F_h(x)$ is nonempty. Thus, $F_h(x)$ has a bottom element by assumption. Take any $z, z' \in [h, 1] \cap X$ such that $z \leq z'$, and take any $y \in F_h(z)$ and $y' \in F_h(z')$. If $y \leq y'$, then F_h is lower increasing. If otherwise, there are two cases. Either $y' < y$, or y and y' are unordered. For the former case, by strong lower increasingness of F, there exists $t \in F(z)$ such that $h \leq y' = y \wedge y' \leq t \leq y'$, concluding that $y' \in F_h(z)$. For the latter case, by strong lower increasingness of F, there exists $t \in F(z)$ with $y \wedge y' \leq t \leq y'$. As $h \leq y \wedge y' \leq t$, we have $t \in F_h(z)$. By applying Theorem 2 for F_h, we find that $\varepsilon(F_h)$ is nonempty and has a least element. Let $e^* \in \varepsilon(F_h)$ be the least fixed point of F_h. Check that $e^* \in \varepsilon(F)$ by the definition of a fixed point. If $\bar{e} \in \varepsilon(F)$ is an upper bound on E, then $\bar{e} \geq h$. This implies that $\bar{e} \in \varepsilon(F_h)$, *i.e.*, $e^* \leq \bar{e}$. We conclude that $e^* = \bigvee_{\varepsilon(F)} E$ which completes the proof that the supremum of E exists. Thus, $\varepsilon(F)$ is a CPO. □

3 General complementarities on CPOs

Let $\Gamma = (N, (X_i)_{i\in N}, (u_i)_{i\in N})$ be a normal form game in which N is the player set, X_i is the set of strategies for player $i \in N$, and $u_i : X \to \mathbb{R}$ is the utility function for player $i \in N$ where $X \equiv \prod_{i\in N} X_i$ denotes the set of strategy profiles.

For a normal form game, a strategy profile $x^* \in X$ is a Nash equilibrium if for every $i \in N$, and every $x_i \in X_i$,

$$u_i(x^*) \geq u_i(x_i, x^*_{-i}).$$

Accordingly, for a given $x_{-i} \in X_{-i} \equiv \prod_{j\in N\setminus\{i\}} X_j$, the best response of player $i \in N$ is defined as *the set of maximizers*. In particular, the best response correspondence of $i \in N$, denoted by $B_i : X_{-i} \to X_i$, is

$$B_i(x_{-i}) = \{x^*_i \mid \forall x'_i \in X_i : u_i(x^*_i, x_{-i}) \geq u_i(x'_i, x_{-i})\}.$$

Then the joint best response correspondence, denoted by $F : X \to X$, is defined as

$$F(x) = \prod_{i\in N} B_i(x_{-i}).$$

This, in turn, implies that a strategy profile $x^* \in X$ is a Nash equilibrium if $x^* \in F(x^*)$; *i.e.*, if it is a fixed point of the joint best response correspondence.

3.1 Extended semi-uniform *g*-modular games and the existence of Nash equilibria

In this section, we first show that under semi-uniform g-modularity, the set of maximizers is strongly lower increasing which provides us more than we need to apply Theorem 2.

Theorem 4 *Let Y be a complete lattice, $X \subset Y$ be a compact CPO in its interval topology, and T be a poset. Assume that $u : X \times T \to R$ is upper semi-continuous in x on X, for every t in T, and is semi-uniform g-modular in (x, t) on $X \times T$. Let $B(t)$ denote the set of* *maximizers for a given $t \in T$. If every $(a,b) \in B(t) \times B(t')$ satisfies Condition 1 for every $t,t' \in T$ with $t \leq t'$, then B is strongly lower increasing.*

Proof This result can, in fact, be attributed to Calciano [11] since an extension to CPOs does not affect the monotonicity of the set of maximizers. See the Appendix for a detailed proof. □

Next two definitions extend the domain of games with weak general complementarities (see Calciano [11]) to CPOs.

Definition 3 A game Γ has extended weak general complementarities on CPOs (hence, is a GEWGC) if (i) each strategy set X_i (a subset of a complete lattice Y) is a CPO; (ii) $\forall x \in X$: the joint best reply correspondence F is nonempty and has a least element; and (iii) F is lower increasing.

Definition 4 A game Γ is extended semi-uniform g-modular if (i) each individual strategy space X_i (a subset of a complete lattice Y) is a compact CPO; (ii) every utility function u_i is upper semi-continuous in own strategies x_i for every strategy profile of the other players x_{-i}; (iii) every u_i is semi-uniform g-modular in (x_i, x_{-i}); and (iv) $\forall x, x' \in X$ with $x \leq x'$: every $(a,b) \in F(x) \times F(x')$ satisfies Condition 1 where F is the corresponding joint best reply correspondence.

Theorem 5 *Let Γ be an extended semi-uniform g-modular game. Then Γ is a GEWGC, and the least Nash equilibrium of Γ exists.*

Proof We need to show that all three properties of a GEWGC are satisfied: (i) is trivial and (iii) follows from Theorem 4. For (ii), we refer to Calciano [11], and we provide a detailed proof in the Appendix. Then, by Theorem 2, there exists the least fixed point of the joint best reply correspondence, which is the least Nash equilibrium of the game. □

So far, we have provided the conditions for games with CPO strategy sets to have a Nash equilibrium. The following result is the comparative statics property due to Topkis [3], a well-appreciated property of extremal equilibria in games with strategic complementarities.

Theorem 6 *Let T be a partially ordered set, and $(\Gamma_t)_{t\in T}$ be a collection of games with extended weak general complementarities such that the joint best reply correspondence $F(\cdot,t)$ is lower increasing in (x,t) on $X \times T$ and has a bottom element for every $(x,t) \in X \times T$. Then the least Nash equilibrium is increasing in t.*

Proof Let $F(\cdot,t)$ be the joint best reply correspondence of the game Γ_t. Noting that a bottom element exists in every $F(x,t)$, define $f_t(x) = \bigwedge F(x,t)$ for every $(x,t) \in X \times T$. The function f_t is order preserving since $F(\cdot,t)$ is lower increasing. Pick any $t' < t''$ in T. Let Ω be the set of all ordinal numbers, and $\succ$ be the linear order on ordinal numbers. For every $t \in \{t', t''\}$, define $g_t : \Omega \to X$ by transfinite recursion as $g_t(0) = \bot$, and

$$g_t(\beta) = \bigvee \{ f_t(g_t(\alpha)) : \beta \succ \alpha \}.$$

Recalling the arguments from Theorems 1 and 2, we know that the smallest fixed points $e^*_{t'}$ and $e^*_{t''}$ of the games $\Gamma_{t'}$ and $\Gamma_{t''}$ can be obtained by using $g_{t'}$ and $g_{t''}$ respectively. Also, lower increasingness of $F(\cdot,t)$ in (x,t) implies that g_t is order preserving. Noting that $g_{t'}(\beta) \le g_{t''}(\beta)$ for every $\beta \in \Omega$, one has $e^*_{t'} \le e^*_{t''}$. □

The following section is devoted to the order structure of Nash equilibria.

3.2 Extended uniform *g*-modular games and the order structure of Nash equilibria

In this section, we present additional assumptions under which the set of Nash equilibrium is a CPO. The theorem below indicates that under uniform g-modularity, the set of maximizers is strongly upper increasing and strongly lower increasing.

Theorem 7 *Let Y be a complete lattice, $X \subset Y$ be a compact CPO in its interval topology, and T be a poset. Assume that $u : X \times T \to R$ is upper semi-continuous in x on X, for every t in T, and is uniform g-modular on CPO in (x,t) on $X \times T$. Let $B(t)$ denote the set of maximizers for a given $t \in T$. If every $(a,b) \in B(t) \times B(t')$ satisfies Conditions* 1 *and* 2 *for every $t, t' \in T$ with $t \le t'$, then B is strongly upper increasing and strongly lower increasing.*

Proof The result follows from Calciano [11]. See the Appendix for a detailed proof. □

Then we define two classes of games, and show that the set of Nash equilibrium is a CPO for these games.

Definition 5 A game Γ has extended general complementarities on CPOs (hence, is a GEGC) if (i) each strategy set X_i (a subset of a complete lattice Y) is a CPO; (ii) the joint best reply correspondence F is strongly upper increasing; (iii) F is strongly lower increasing; (iv) $\forall x \in X$: $F(x)$ is a CPO in X; and (v) $\forall h \in A$ and $\forall x \in X$: the correspondence $F_h(x)$ has a bottom element whenever nonempty.

Definition 6 A game Γ is extended uniform g-modular if (i) each strategy set X_i (a subset of a complete lattice Y) is a compact CPO such that the lower contour set of each $x \in X$ is closed;[g] (ii) every utility function u_i is upper semi-continuous in own strategies x_i for every strategy profile of the other players x_{-i}; (iii) every u_i is uniform g-modular on CPO in (x_i, x_{-i}); and (iv) $\forall x, x' \in X$ with $x \le x'$: every $(a,b) \in F(x) \times F(x')$ satisfies Conditions 1 and 2 where F is the corresponding joint best reply correspondence.

Theorem 8 *Let Γ be an extended uniform g-modular game. Then Γ is a GEGC, and the Nash equilibrium set of Γ is a CPO.*

Proof We need to show that all four properties of a GEGC are satisfied: (i) is trivial, (ii) and (iii) follow from Theorem 7, and (v) follows from an approach similar to the one used in the proof of Theorem 5. For (iv), we need to show that $F(x)$ is a CPO for every $x \in X$. Note that since $F_\perp$ coincides with F, it is true that F has a bottom element. Then take any $x \in X$ and any directed set $E \subset F(x)$. We need to show that $\bigvee_{F(x)} E$ exists. If E is finite, then the result trivially follows. Assume otherwise, and set $k = \bigvee E$. Now, we aim to construct a sequence in E converging to k. We start by taking $e_1 \in E$. Then there must be an element $e_2 \in E$

such that $e_1 < e_2 < k$. This follows because E is infinite and directed. Since X is compact, and the lower contour sets are closed, this construction returns a sequence $(e_n) \in E$ such that $(e_n) \to k$. Since each u_i is upper semi-continuous, $k \in E \subset F(x)$. Hence, $\bigvee_{F(x)} E = k$ so that $\bigvee_{F(x)} E$ exists. Then, by Theorem 3, the set of fixed points of the joint best reply correspondence, that is, the Nash equilibrium set is a CPO. □

4 Conclusion

In this study, we first provide a short proof for the existence of fixed points of monotone correspondences defined on CPOs, in a constructive way that Echenique [15] has proved Tarski [8] and Zhou [7] fixed point theorems. We also give conditions under which the set of fixed points turns out to be a nonempty CPO. Thereafter, we prove the nonemptiness of the set of Nash equilibria for extended semi-uniform g-modular games, and we show that the set of Nash equilibria is a nonempty CPO for extended uniform g-modular games. Finally, we provide a monotone comparative statics result on the equilibrium set.

Appendix

Proof of Theorem 1[h] Let Ω be the set of all ordinal numbers, and $\succ$ be the linear order on ordinal numbers. Define $g : \Omega \to X$ by transfinite recursion as $g(0) = \bot$, and

$$g(\beta) = \bigvee \{ f(g(\alpha)) : \beta \succ \alpha \}.$$

The function g is order preserving by definition. For each $\alpha \in \Omega$, it follows that $g(\alpha + 1) = f(g(\alpha))$. By *the axiom of replacement*, there must exist $\gamma \in \Omega$ such that $g(\gamma) = g(\gamma + 1)$. Let γ^* be the smallest of such γ's.[i] Finally, let $e^* = g(\gamma^*)$. Then $e^* = f(e^*)$, so that e^* is a fixed point of f. In fact, it is the smallest fixed point of f. To see this, take any $e \in \varepsilon(f)$. If $e = \bot$, then $e^* = \bot$ is the smallest element of $\varepsilon(f)$. If $e > \bot$, there exists α such that $g(\alpha) < e$. Since f is an order-preserving map, $g(\alpha + 1) = f(g(\alpha)) \leq f(e) = e$. By transfinite induction, we have $e^* \leq e$, concluding that e^* is the smallest fixed point.

Then we need to show that each directed nonempty subset E of $\varepsilon(f)$ has a supremum; *i.e.*, $\bigvee_{\varepsilon(f)} E$ exists. Let $x = \bigvee_{\varepsilon(f)} E$, and let $U_E = \{y \in X : x \leq y\}$ be the set of upper bounds of E. Note that $f(U_E)$ is a subset of U_E, because for every $y \in U_E$ and every $e \in E$, we have $e \leq f(y)$ since $e = f(e) \leq f(y)$. Let $\varphi = f|_{U_E}$. Then φ maps U_E into U_E, and φ is order preserving as well. Hence, $\varepsilon(\varphi)$ has a smallest element. By definition of φ, this smallest element is $\bigvee_{\varepsilon(f)} E$. Thus, $\varepsilon(f)$ is a CPO. □

Proof of Theorem 2 Define $f : X \to X$ such that $f(x) = \bigwedge F(x)$ for every $x \in X$. Note that $f(x) \in F(x)$ and f is order preserving by construction. By Theorem 1, there is a smallest fixed point; say $e^* \in \varepsilon(f)$. We also have $e^* = f(e^*) \in F(e^*)$ and, thus, $e^* \in \varepsilon(F)$. We have shown that $\varepsilon(F)$ is nonempty. Take any $e \in \varepsilon(F)$. As in the proof of Theorem 1, one can then easily show $e^* \leq e$, so that e^* is the smallest fixed point. □

Proof of Theorem 3 Here, we construct the directed subset $E'' \subset F(h)$ in such a way that for every $x_i \in E$, there exists $b_i \in E''$ such that $b_i \geq x_i$. Consider the bottom element of $F(h)$, say $\underline{a}$. For every $x_i \in E$, by strong upper increasingness, $\exists a_i \in F(h)$ such that $a_i \in [x_i, x_i \vee \underline{a}]$. Since $\underline{a}$ is the bottom element, we also have $\underline{a} \leq a_i$. That is to say, $a_i \in [\underline{a}, x_i \vee \underline{a}]$.

Consequently, $a_i \in [x_i, x_i \vee \underline{a}] \cap [\underline{a}, x_i \vee \underline{a}]$. This implies that $a_i = x_i \vee \underline{a}$. Recalling that (x_n) is an increasing sequence, the set of all a_i's is the desired set E''. □

Proof of Theorem 4 Since u is upper semi-continuous, and X is compact, $B(t)$ is closed and nonempty. Take $a \in B(t')$ and $b \in B(t'')$ with $t' \leq t''$. If $a \leq b$, then the result trivially follows. Consider the case that they are unordered. Since $b \nleq a$, there exist $p \in [a \wedge b, b)$ and $q \in X$ such that

$$0 \leq u(a,t') - u(p,t') \leq u(q,t') - u(b,t') \leq u(q,t'') - u(b,t'') \leq 0,$$

where the second inequality follows from Definition 1(i.a), the third follows from Definition 1(i.b), and the rest follow from the optimality of a and b. Thus, $p \in B(t')$ which we need to show.

The only remaining case is $b < a$. Now, we have to show that $b \in B(t')$ to prove strong lower increasingness. Note that there exist $p_1' \in [a \wedge b, a)$ and $q' \in X$ such that

$$0 \leq u(a,t') - u(p_1',t') \leq u(q',t') - u(b,t') \leq u(q',t'') - u(b,t'') \leq 0,$$

implying that $p_1' \in B(t')$. Since $[a \wedge b, a) \equiv [b, a)$, we have $p_1' \geq b$. If $p_1' = b$, we are done. If not, then it must be the case that $p_1' > b$. By the same argument we can find some $p_2' \in [p_1', a)$ such that $p_2' \in B(t')$. If $p_2' = b$, we are done. If not, we can find p_3' by repeating the argument. We either have some $p_n' \in B(t')$ with $p_n' = b$, or we find a sequence (p_n') which is strictly decreasing and bounded from below by b. The former completes the proof, and the latter implies that the sequence converges to some $m_1 \in B(t')$ with $m_1 \geq b$ by the fact that $B(t')$ is closed in the interval topology of X. Now, if $m_1 = b$, we are done. If not, then $m_1 > b$. By the same process, we can construct a convergent sequence between m_1 and b, which converges to m_2 which is not less than b. At the end, either we have some $m_n = b$, or we find a sequence $(m_n) \in B(t')$ converging to b. Since $B(t')$ is closed, we have $b \in B(t')$, concluding that $B(t)$ is strongly lower increasing. □

Proof of Theorem 5 Here, we only prove (ii) which states that $F(x)$ has a bottom element. Take any $x \in X$. If $F(x)$ is a chain, we are done. If not, take any unordered $y, y' \in F(x)$. Consider $y_i, y_i' \in B_i(x_{-i})$. We need to show that $y_i \wedge y_i' \in B_i(x_{-i})$. By semi-uniform g-modularity,

$$u_i(y_i, x_{-i}) + u_i(y_i', x_{-i}) \leq u_i(p_1, x_{-i}) + u_i(q_1, x_{-i})$$

for some $p_1 \in [y_i \wedge y_i', y_i)$ and $q_1 \in X_i$, implying that $p_1, q_1 \in B_i(x_{-i})$. By a sequence construction similar to the one in the proof of Theorem 4, one can show that $y_i \wedge y_i' \in B_i(x_{-i})$. Repeating similar arguments for each $i \in N$, we have $y \wedge y' \in F(x)$. Hence, $F(x)$ has a bottom element. □

Proof of Theorem 7 Since u is upper semi-continuous, and X is compact, $B(t)$ is closed and nonempty. Take $a \in B(t')$ and $b \in B(t'')$ with $t' \leq t''$. If $a \leq b$, then the result trivially follows. Consider the case that they are unordered. Since $b \nleq a$, by uniform g-modularity, there exist $p \in [a \wedge b, b) \cap X$ and $q \in (a, a \vee b] \cap X$ such that

$$0 \leq u(a,t') - u(p,t') \leq u(q,t') - u(b,t') \leq u(q,t'') - u(b,t'') \leq 0,$$

where the second inequality follows from Definition 2(i.a), the third follows from Definition 2(i.b), and the rest follow from the optimality of a and b. Thus, $p \in B(t')$ and $q \in B(t'')$. The former implies strong lower increasingness, and the latter implies strong upper increasingness. The only remaining case is $b < a$. Now, showing that $a \in B(t'')$ and $b \in B(t')$ is enough to prove strong upper increasingness and strong lower increasingness, respectively. Since we have $a \nleq b$, by uniform g-modularity, there exist $p' \in [a \wedge b, a) \cap X$ and $q' \in (b, a \vee b] \cap X$ such that

$$0 \leq u(a,t') - u(p',t') \leq u(q',t') - u(b,t') \leq u(q',t'') - u(b,t'') \leq 0,$$

implying that $p' \in B(t')$ and $q' \in B(t'')$. Since $(b, a \vee b] \equiv (b,a]$ and $[a \wedge b, a) \equiv [b,a)$, we have $q' \leq a$ and $p' \geq b$. Let us consider strong lower increasingness first: If $p' = b$, we are done. If not, then it must be the case that $p' > b$. By a sequence construction similar to the one in the proof of Theorem 4, we can find a decreasing sequence bounded by b which converges to $m_1 \in B(t')$ with $m_1 \geq b$. We continue constructing such sequences until we reach some $m_n = b$, or the sequence (m_n) converges to $b \in B(t')$, concluding that B is strongly lower increasing. For strong upper increasingness, a similar method applies to find increasing convergent sequences bounded by a from above. That completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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Endnotes
[a] Moreover, Calciano [20] has extended the monotone comparative statics results of Milgrom and Shannon [21] to general binary algebras.
[b] See Acemoglu and Jensen [18] for the details in an aggregate large game context.
[c] This definition is provided by Davey and Priestly [22]. In some studies, such a set is also referred to as CPO with bottom element or pointed CPO.
[d] Note that the set Y is arbitrary, and so is its greatest element 1. In fact, we only need a set including an element which is greater than every $x \in X$.
[e] It is also referred to as the fixed point set of f.
[f] The construction of this directed set is relegated to the Appendix.
[g] Lower contour set of x on X is defined as $\{y \in X \mid y \leq x\}$. Note that *closedness* is a topological property, and recall that we use the interval topology.
[h] We do not claim the originality of the proof as it significantly borrows from Echenique [15]. We give this proof to show that Echenique's [15] proof can be generalized to hold on a CPO; a result that may not seem trivial to the reader.
[i] Note that γ^* is well defined as any set of ordinal numbers has a smallest element.

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