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Cyclic contractions via auxiliary functions on *G*-metric spaces

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Abstract

In this paper, we prove the existence and uniqueness of fixed points of certain cyclic mappings via auxiliary functions in the context of *G*-metric spaces, which were introduced by Zead and Sims. In particular, we extend, improve and generalize some earlier results in the literature on this topic.

MSC: 47H10; 54H25

Keywords: fixed point; *G*-metric space; cyclic maps; cyclic contractions

1 Introduction and preliminaries

It is well established that fixed point theory, which mainly concerns the existence and uniqueness of fixed points, is today's one of the most investigated research areas as a major subfield of nonlinear functional analysis. Historically, the first outstanding result in this field that guaranteed the existence and uniqueness of fixed points was given by Banach [1]. This result, known as the Banach mapping contraction principle, simply states that every contraction mapping has a unique fixed point in a complete metric space. Since the first appearance of the Banach principle, the ever increasing application potential of the fixed point theory in various research fields, such as physics, chemistry, certain engineering branches, economics and many areas of mathematics, has made this topic more crucial than ever. Consequently, after the Banach celebrated principle, many authors have searched for further fixed point results and reported successfully new fixed point theorems conceived by the use of two very effective techniques, combined or separately.

The first one of these techniques is to 'replace' the notion of a metric space with a more general space. Quasi-metric spaces, partial metric spaces, rectangular metric spaces, fuzzy metric space, *b*-metric spaces, *D*-metric spaces, *G*-metric spaces are generalizations of metric spaces and can be considered as examples of 'replacements'. Amongst all of these spaces, *G*-metric spaces, introduced by Zead and Sims [2], are ones of the interesting. Therefore, in the last decade, the notion of a *G*-metric space has attracted considerable attention from researchers, especially from fixed point theorists [3–25].

The second one of these techniques is to modify the conditions on the operator(s). In other words, it entails the examination of certain conditions under which the contraction mapping yields a fixed point. One of the attractive results produced using this approach was given by Kirk *et al.* [26] in 2003 through the introduction of the concepts of cyclic mappings and best proximity points. After this work, best proximity theorems and, in particular, the fixed point theorems in the context of cyclic mappings have been studied extensively (see, *e.g.*, [27–43]).

The two upper mentioned topics, cyclic mappings and G-metric spaces, have been combined by Aydi in [22] and Karapınar *et al.* in [36]. In these papers, the existence and uniqueness of fixed points of cyclic mappings are investigated in the framework of G-metric spaces. In this paper, we aim to improve on certain statements proved on these two topics. For the sake of completeness, we will include basic definitions and crucial results that we need in the rest of this work.

Mustafa and Sims [2] defined the concept of G-metric spaces as follows.

Definition 1.1 (See [2]) Let X be a nonempty set, $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

(G1) $G(x,y,z) = 0$ if $x = y = z$,
(G2) $0 < G(x,x,y)$ for all $x,y \in X$ with $x \neq y$,
(G3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$,
(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (symmetry in all three variables),
(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ (rectangle inequality) for all $x,y,z,a \in X$.

Then the function G is called a generalized metric or, more specifically, a G-metric on X, and the pair (X,G) is called a G-metric space.

Note that every G-metric on X induces a metric d_G on X defined by

$$d_G(x,y) = G(x,y,y) + G(y,x,x) \quad \text{for all } x,y \in X. \tag{1}$$

For a better understanding of the subject, we give the following examples of G-metrics.

Example 1.1 Let (X,d) be a metric space. The function $G : X \times X \times X \to [0,+\infty)$, defined by

$$G(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\}$$

for all $x,y,z \in X$, is a G-metric on X.

Example 1.2 (See, *e.g.*, [2]) Let $X = [0,\infty)$. The function $G : X \times X \times X \to [0,+\infty)$, defined by

$$G(x,y,z) = |x-y| + |y-z| + |z-x|$$

for all $x,y,z \in X$, is a G-metric on X.

In their initial paper, Mustafa and Sims [2] also defined the basic topological concepts in G-metric spaces as follows.

Definition 1.2 (See [2]) Let (X,G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X. We say that $\{x_n\}$ is G-convergent to $x \in X$ if

$$\lim_{n,m\to+\infty} G(x,x_n,x_m) = 0,$$

that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$ for all $n, m \geq N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim_{n\to+\infty} x_n = x$.

Proposition 1.1 (See [2]) *Let (X, G) be a G-metric space. The following are equivalent*:

(1) *$\{x_n\}$ is G-convergent to x,*
(2) *$G(x_n, x_n, x) \to 0$ as $n \to +\infty$,*
(3) *$G(x_n, x, x) \to 0$ as $n \to +\infty$,*
(4) *$G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$.*

Definition 1.3 (See [2]) Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called a G-Cauchy sequence if, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $m, n, l \geq N$, that is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

Proposition 1.2 (See [2]) *Let (X, G) be a G-metric space. Then the following are equivalent*:

(1) *the sequence $\{x_n\}$ is G-Cauchy,*
(2) *for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$ for all $m, n \geq N$.*

Definition 1.4 (See [2]) A G-metric space (X, G) is called G-complete if every G-Cauchy sequence is G-convergent in (X, G).

Definition 1.5 Let (X, G) be a G-metric space. A mapping $F : X \times X \times X \to X$ is said to be continuous if for any three G-convergent sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converging to x, y and z respectively, $\{F(x_n, y_n, z_n)\}$ is G-convergent to $F(x, y, z)$.

Note that each G-metric on X generates a topology τ_G on X whose base is a family of open G-balls $\{B_G(x, \varepsilon), x \in X, \varepsilon > 0\}$, where $B_G(x, \varepsilon) = \{y \in X, G(x, y, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. A nonempty set $A \subset X$ is G-closed in the G-metric space (X, G) if $\overline{A} = A$. Observe that

$$x \in \overline{A} \iff B_G(x, \varepsilon) \cap A \neq \emptyset$$

for all $\varepsilon > 0$. We recall also the following proposition.

Proposition 1.3 (See, *e.g.*, [36]) *Let (X, G) be a G-metric space and A be a nonempty subset of X. The set A is G-closed if for any G-convergent sequence $\{x_n\}$ in A with limit x, we have $x \in A$.*

Mustafa [5] extended the well-known Banach contraction principle mapping in the framework of G-metric spaces as follows.

Theorem 1.1 (See [5]) *Let (X, G) be a complete G-metric space and $T : X \to X$ be a mapping satisfying the following condition for all $x, y, z \in X$*:

$$G(Tx, Ty, Tz) \leq kG(x, y, z), \tag{2}$$

where $k \in [0, 1)$. Then T has a unique fixed point.

Theorem 1.2 (See [5]) *Let (X, G) be a complete G-metric space and $T : X \to X$ be a mapping satisfying the following condition for all $x, y \in X$:*

$$G(Tx, Ty, Ty) \leq kG(x, y, y), \tag{3}$$

where $k \in [0, 1)$. Then T has a unique fixed point.

Remark 1.1 We notice that the condition (2) implies the condition (3). The converse is true only if $k \in [0, \frac{1}{2})$. For details, see [5].

Lemma 1.1 ([5]) *By the rectangle inequality* (G5) *together with the symmetry* (G4), *we have*

$$G(x, y, y) = G(y, y, x) \leq G(y, x, x) + G(x, y, x) = 2G(y, x, x). \tag{4}$$

A map $T : X \to X$ on a metric space (X, d) is called a weak ϕ-contraction if there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$d(Tx, Ty) \leq d(x, y) - \phi\bigl(d(x, y)\bigr),$$

for all $x, y \in X$. We notice that these types of contractions have also been a subject of extensive research (see, *e.g.*, [44–49]). In what follows, we recall the notion of cyclic weak ψ-contractions on G-metric spaces. Let Ψ be the set of continuous functions $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ and $\phi(t) > 0$ for $t > 0$. In [36], the authors concentrated on two types of cyclic contractions: cyclic-type Banach contractions and cyclic weak ϕ-contractions.

Theorem 1.3 *Let (X, G) be a G-complete G-metric space and $\{A_j\}_{j=1}^m$ be a family of nonempty G-closed subsets of X with $Y = \bigcup_{j=1}^m A_j$. Let $T : Y \to Y$ be a map satisfying*

$$T(A_j) \subseteq A_{j+1}, \quad j = 1, \dots, m, \text{ where } A_{m+1} = A_1. \tag{5}$$

Suppose that there exists a function $\phi \in \Psi$ such that the map T satisfies the inequality

$$G(Tx, Ty, Tz) \leq M(x, y, z) - \phi\bigl(M(x, y, z)\bigr) \tag{6}$$

for all $x \in A_j$ and $y, z \in A_{j+1}$, $j = 1, \dots, m$, where

$$M(x, y, z) = \max\bigl\{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\bigr\}. \tag{7}$$

Then T has a unique fixed point in $\bigcap_{j=1}^m A_j$.

The following result, which can be considered as a corollary of Theorem 1.3, is stated in [36].

Theorem 1.4 (See [36]) *Let (X, G) be a G-complete G-metric space and $\{A_j\}_{j=1}^m$ be a family of nonempty G-closed subsets of X. Let $Y = \bigcup_{j=1}^m A_j$ and $T : Y \to Y$ be a map satisfying*

$$T(A_j) \subseteq A_{j+1}, \quad j = 1, \dots, m, \text{ where } A_{m+1} = A_1. \tag{8}$$

If there exists $k \in (0,1)$ *such that*

$$G(Tx, Ty, Tz) \leq kG(x, y, z) \tag{9}$$

holds for all $x \in A_j$ *and* $y, z \in A_{j+1}$, $j = 1, \dots, m$, *then* T *has a unique fixed point in* $\bigcap_{j=1}^{m} A_j$.

In this paper, we extend, generalize and enrich the results on the topic in the literature.

2 Main results

We start this section by defining some sets of auxiliary functions. Let $\mathcal{F}$ denote all functions $f : [0,\infty) \to [0,\infty)$ such that $f(t) = 0$ if and only if $t = 0$. Let Ψ and Φ be the subsets of $\mathcal{F}$ such that

$$\Psi = \{\psi \in \mathcal{F} : \psi \text{ is continuous and nondecreasing}\},$$
$$\Phi = \{\phi \in \mathcal{F} : \phi \text{ is lower semi-continuous}\}.$$

Lemma 2.1 *Let* (X, G) *be a G-complete G-metric space and* $\{x_n\}$ *be a sequence in* X *such that* $G(x_n, x_{n+1}, x_{n+1})$ *is nonincreasing,*

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{10}$$

If $\{x_n\}$ *is not a Cauchy sequence, then there exist* $\varepsilon > 0$ *and two sequences* $\{n_k\}$ *and* $\{\ell_k\}$ *of positive integers such that the following sequences tend to* ε *when* $k \to \infty$:

$$\begin{aligned} &G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}), \qquad G(x_{\ell(k)}, x_{n(k)+1}, x_{n(k)+1}), \qquad G(x_{\ell(k)-1}, x_{n(k)}, x_{n(k)}), \\ &G(x_{\ell(k)-1}, x_{n(k)+1}, x_{n(k)+1}), \qquad G(x_{n(k)}, x_{\ell(k)}, x_{\ell(k)+1}). \end{aligned} \tag{11}$$

Proof Due to Lemma 1.1, we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq 2G(x_n, x_{n+1}, x_{n+1}).$$

Letting $n \to \infty$ regarding the assumption of the lemma, we derive that

$$\lim_{n\to\infty} G(x_n, x_n, x_{n+1}) = 0. \tag{12}$$

If $\{x_n\}$ is not G-Cauchy, then, due to Proposition 1.2, there exist $\varepsilon > 0$ and corresponding subsequences $\{n(k)\}$ and $\{\ell(k)\}$ of $\mathbb{N}$ satisfying $n(k) > \ell(k) > k$ for which

$$G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}) \geq \varepsilon, \tag{13}$$

where $n(k)$ is chosen as the smallest integer satisfying (13), that is,

$$G(x_{\ell(k)}, x_{n(k)-1}, x_{n(k)-1}) < \varepsilon. \tag{14}$$

By (13), (14) and the rectangle inequality (G5), we easily derive that

$$\begin{aligned}\varepsilon &\leq G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}) \leq G(x_{\ell(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) \\ &< \varepsilon + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}).\end{aligned} \tag{15}$$

Letting $k \to \infty$ in (15) and using (10), we get

$$\lim_{k\to\infty} G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}) = \varepsilon. \tag{16}$$

Further,

$$G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}) \leq G(x_{\ell(k)}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)+1}, x_{n(k)}, x_{n(k)}), \tag{17}$$

and

$$G(x_{\ell(k)}, x_{n(k)+1}, x_{n(k)+1}) \leq G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}). \tag{18}$$

Passing to the limit when $k \to \infty$ and using (10) and (16), we obtain that

$$\lim_{k\to\infty} G(x_{\ell(k)}, x_{n(k)+1}, x_{n(k)+1}) = \varepsilon. \tag{19}$$

In a similar way,

$$G(x_{\ell(k)-1}, x_{n(k)}, x_{n(k)}) \leq G(x_{\ell(k)-1}, x_{\ell(k)}, x_{\ell(k)}) + G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}), \tag{20}$$

and

$$G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}) \leq G(x_{\ell(k)}, x_{\ell(k)-1}, x_{\ell(k)-1}) + G(x_{\ell(k)-1}, x_{n(k)}, x_{n(k)}). \tag{21}$$

Passing to the limit when $k \to \infty$ and using (10) and (16), we obtain that

$$\lim_{k\to\infty} G(x_{\ell(k)-1}, x_{n(k)}, x_{n(k)}) = \varepsilon. \tag{22}$$

Furthermore,

$$\begin{aligned}&G(x_{\ell(k-1)}, x_{n(k)+1}, x_{n(k)+1}) \\ &\quad \leq G(x_{\ell(k-1)}, x_{\ell(k)}, x_{\ell(k)}) + G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1})\end{aligned} \tag{23}$$

and

$$\begin{aligned}&G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}) \\ &\quad \leq G(x_{\ell(k)}, x_{\ell(k-1)}, x_{\ell(k-1)}) + G(x_{\ell(k-1)}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)+1}, x_{n(k)}, x_{n(k)}).\end{aligned} \tag{24}$$

Passing to the limit when $k \to \infty$ and using (10) and (16), we obtain that

$$\lim_{k\to\infty} G(x_{\ell(k)-1}, x_{n(k)+1}, x_{n(k)+1}) = \varepsilon. \tag{25}$$

By regarding the assumptions (G3) and (G5) together with the expression (13), we derive the following:

$$\begin{aligned}\varepsilon &\le G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}) \le G(x_{n(k)}, x_{\ell(k)}, x_{\ell(k)+1}) \\ &\le G(x_{n(k)}, x_{\ell(k)}, x_{\ell(k)}) + G(x_{\ell(k)}, x_{\ell(k)}, x_{\ell(k)+1}).\end{aligned} \tag{26}$$

Letting $k \to \infty$ in the inequality above and using (12) and (16), we conclude that

$$\lim_{k\to\infty} G(x_{n(k)}, x_{\ell(k)}, x_{\ell(k)+1}) = \varepsilon. \tag{27}$$

□

Theorem 2.1 *Let (X, G) be a G-complete G-metric space and $\{A_j\}_{j=1}^m$ be a family of nonempty G-closed subsets of X with $Y = \bigcup_{j=1}^m A_j$. Let $T : Y \to Y$ be a map satisfying*

$$T(A_j) \subseteq A_{j+1}, \quad j = 1, 2, \ldots, m, \text{ where } A_{m+1} = A_1. \tag{28}$$

Suppose that there exist functions $\phi \in \Phi$ and $\psi \in \Psi$ such that the map T satisfies the inequality

$$\psi\big(G(Tx, Ty, Ty)\big) \le \psi\big(M(x, y, y)\big) - \phi\big(M(x, y, y)\big) \tag{29}$$

for all $x \in A_j$ and $y \in A_{j+1}$, $j = 1, 2, \ldots, m$, where

$$\begin{aligned}M(x, y, y) = \max\Big\{&G(x, y, y), G(x, Tx, Tx), G(y, Ty, Ty), G(x, y, Tx), \\ &\frac{1}{3}\big[2G(x, Ty, Ty) + G(y, Tx, Tx)\big], \frac{1}{3}\big[G(x, Ty, Ty) + 2G(y, Tx, Tx)\big]\Big\}.\end{aligned} \tag{30}$$

Then T has a unique fixed point in $\bigcap_{j=1}^m A_j$.

Proof First we show the existence of a fixed point of the map T. For this purpose, we take an arbitrary $x_0 \in A_1$ and define a sequence $\{x_n\}$ in the following way:

$$x_n = Tx_{n-1}, \quad n = 1, 2, 3, \ldots. \tag{31}$$

We have $x_0 \in A_1, x_1 = Tx_0 \in A_2, x_2 = Tx_1 \in A_3, \ldots$ since T is a cyclic mapping. If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \mathbb{N}$, then, clearly, the fixed point of the map T is x_{n_0}. From now on, assume that $x_{n+1} \ne x_n$ for all $n \in \mathbb{N}$. Consider the inequality (29) by letting $x = x_n$ and $y = x_{n+1}$,

$$\begin{aligned}\psi\big(G(Tx_n, Tx_{n+1}, Tx_{n+1})\big) &= \psi\big(G(x_{n+1}, x_{n+2}, x_{n+2})\big) \\ &\le \psi\big(M(x_n, x_{n+1}, x_{n+1})\big) - \phi\big(M(x_n, x_{n+1}, x_{n+1})\big),\end{aligned} \tag{32}$$

where

$$\begin{aligned}M(x_n, x_{n+1}, x_{n+1}) = \max\Big\{&G(x_n, x_{n+1}, x_{n+1}), G(x_n, Tx_n, Tx_n), G(x_{n+1}, Tx_{n+1}, Tx_{n+1}), \\ &G(x_n, x_{n+1}, Tx_n), \frac{1}{3}\big[2G(x_n, Tx_{n+1}, Tx_{n+1}) + G(x_{n+1}, Tx_n, Tx_n)\big],\end{aligned}$$

$$
\begin{aligned}
&\frac{1}{3}\big[G(x_n, Tx_{n+1}, Tx_{n+1}) + 2G(x_{n+1}, Tx_n, Tx_n)\big]\Big\} \\
&= \max\Big\{G(x_n,x_{n+1},x_{n+1}), G(x_n,x_{n+1},x_{n+1}), G(x_{n+1},x_{n+2},x_{n+2}), \\
&\quad G(x_n,x_{n+1},x_{n+1}), \frac{1}{3}\big[2G(x_n,x_{n+2},x_{n+2}) + G(x_{n+1},x_{n+1},x_{n+1})\big], \\
&\quad \frac{1}{3}\big[G(x_n,x_{n+2},x_{n+2}) + 2G(x_{n+1},x_{n+1},x_{n+1})\big]\Big\} \\
&= \max\Big\{G(x_n,x_{n+1},x_{n+1}), G(x_{n+1},x_{n+2},x_{n+2}), \frac{2}{3}G(x_n,x_{n+2},x_{n+2})\Big\} \\
&\le \max\big\{G(x_n,x_{n+1},x_{n+1}), G(x_{n+1},x_{n+2},x_{n+2})\big\}.
\end{aligned}
\tag{33}
$$

If $M(x_n,x_{n+1},x_{n+1}) = G(x_{n+1},x_{n+2},x_{n+2})$, then the expression (32) implies that

$$
\psi\big(G(x_{n+1},x_{n+2},x_{n+2})\big) \le \psi\big(G(x_{n+1},x_{n+2},x_{n+2})\big) - \phi\big(G(x_{n+1},x_{n+2},x_{n+2})\big).
\tag{34}
$$

So, the inequality (34) yields $\phi(G(x_{n+1},x_{n+2},x_{n+2})) = 0$. Thus, we conclude that

$$
G(x_{n+1},x_{n+2},x_{n+2}) = 0.
$$

This contradicts the assumption $x_n \ne x_{n+1}$ for all $n \in \mathbb{N}$. So, we derive that

$$
M(x_n,x_{n+1},x_{n+1}) = G(x_n,x_{n+1},x_{n+1}).
\tag{35}
$$

Hence the inequality (32) turns into

$$
\begin{aligned}
\psi\big(G(x_{n+1},x_{n+2},x_{n+2})\big) &\le \psi\big(G(x_n,x_{n+1},x_{n+1})\big) - \phi\big(G(x_n,x_{n+1},x_{n+1})\big) \\
&\le \psi\big(G(x_n,x_{n+1},x_{n+1})\big).
\end{aligned}
\tag{36}
$$

Thus, $\{G(x_n,x_{n+1},x_{n+1})\}$ is a nonnegative, nonincreasing sequence that converges to $L \ge 0$. We will show that $L = 0$. Suppose, on the contrary, that $L > 0$. Taking $\limsup_{n\to+\infty}$ in (36), we derive that

$$
\begin{aligned}
&\limsup_{n\to+\infty} \psi\big(G(x_{n+1},x_{n+2},x_{n+2})\big) \\
&\quad \le \limsup_{n\to+\infty} \psi\big(G(x_n,x_{n+1},x_{n+1})\big) - \liminf_{n\to+\infty} \phi\big(G(x_n,x_{n+1},x_{n+1})\big) \\
&\quad \le \limsup_{n\to+\infty} \psi\big(G(x_n,x_{n+1},x_{n+1})\big).
\end{aligned}
\tag{37}
$$

By the continuity of ψ and the lower semi-continuity of ϕ, we get

$$
\psi(L) \le \psi(L) - \phi(L).
\tag{38}
$$

Then it follows that $\phi(L) = 0$. Therefore, we get $L = 0$, that is,

$$
\lim_{n\to\infty} G(x_n,x_{n+1},x_{n+1}) = 0.
\tag{39}
$$

Lemma 1.1 with $x = x_n$ and $y = x_{n-1}$ implies that

$$G(x_n, x_{n-1}, x_{n-1}) \leq 2G(x_{n-1}, x_n, x_n). \tag{40}$$

So, we get that

$$\lim_{n\to\infty} G(x_n, x_{n-1}, x_{n-1}) = 0. \tag{41}$$

Next, we will show that $\{x_n\}$ is a G-Cauchy sequence in (X, G). Suppose, on the contrary, that $\{x_n\}$ is not G-Cauchy. Then, due to Proposition 1.2, there exist $\varepsilon > 0$ and corresponding subsequences $\{n(k)\}$ and $\{\ell(k)\}$ of $\mathbb{N}$ satisfying $n(k) > \ell(k) > k$ for which

$$G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}) \geq \varepsilon, \tag{42}$$

where $n(k)$ is chosen as the smallest integer satisfying (42), that is,

$$G(x_{\ell(k)}, x_{n(k)-1}, x_{n(k)-1}) < \varepsilon. \tag{43}$$

By (42), (43) and the rectangle inequality (G5), we easily derive that

$$\begin{aligned}
\varepsilon &\leq G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}) \leq G(x_{\ell(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) \\
&< \varepsilon + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}).
\end{aligned} \tag{44}$$

Letting $k \to \infty$ in (44) and using (39), we get

$$\lim_{k\to\infty} G(x_{\ell(k)}, x_{n(k)}, x_{n(k)}) = \varepsilon. \tag{45}$$

Notice that for every $k \in \mathbb{N}$ there exists $s(k)$ satisfying $0 \leq s(k) \leq m$ such that

$$n(k) - \ell(k) + s(k) \equiv 1(m). \tag{46}$$

Thus, for large enough values of k, we have $r(k) = \ell(k) - s(k) > 0$, and $x_{r(k)}$ and $x_{n(k)}$ lie in the adjacent sets A_j and A_{j+1} respectively for some $0 \leq j \leq m$. When we substitute $x = x_{r(k)}$ and $y = x_{n(k)}$ in the expression (29), we get that

$$\psi\bigl(G(Tx_{r(k)}, Tx_{n(k)}, Tx_{n(k)})\bigr) \leq \psi\bigl(M(x_{r(k)}, x_{n(k)}, x_{n(k)})\bigr) - \phi\bigl(M(x_{r(k)}, x_{n(k)}, x_{n(k)})\bigr), \tag{47}$$

where

$$\begin{aligned}
M(x_{r(k)}, x_{n(k)}, x_{n(k)}) = \max\Bigl\{ & G(x_{r(k)}, x_{n(k)}, x_{n(k)}), G(x_{r(k)}, x_{r(k)+1}, x_{r(k)+1}), \\
& G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}), G(x_{r(k)}, x_{n(k)}, x_{r(k)+1}), \\
& \frac{1}{3}\bigl[2G(x_{r(k)}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)}, x_{r(k)+1}, x_{r(k)+1})\bigr], \\
& \frac{1}{3}\bigl[G(x_{r(k)}, x_{n(k)+1}, x_{n(k)+1}) + 2G(x_{n(k)}, x_{r(k)+1}, x_{r(k)+1})\bigr]\Bigr\}.
\end{aligned} \tag{48}$$

By using Lemma 2.1, we obtain that

$$\lim_{k\to\infty}\frac{1}{3}\big[2G(x_{r(k)},x_{n(k)+1},x_{n(k)+1})+G(x_{n(k)},x_{r(k)+1},x_{r(k)+1})\big]=\varepsilon, \tag{49}$$

and

$$\lim_{k\to\infty}\frac{1}{3}\big[G(x_{r(k)},x_{n(k)+1},x_{n(k)+1})+2G(x_{n(k)},x_{r(k)+1},x_{r(k)+1})\big]=\varepsilon. \tag{50}$$

So, we obtain that

$$\psi(\varepsilon)\leq\psi\big(\max\{\varepsilon,0,0,\varepsilon,\varepsilon,\varepsilon\}\big)-\phi\big(\max\{\varepsilon,0,0,\varepsilon,\varepsilon,\varepsilon\}\big)=\psi(\varepsilon)-\phi(\varepsilon). \tag{51}$$

So, we have $\phi(\varepsilon)=0$. We deduce that $\varepsilon=0$. This contradicts the assumption that $\{x_n\}$ is not G-Cauchy. As a result, the sequence $\{x_n\}$ is G-Cauchy. Since (X,G) is G-complete, it is G-convergent to a limit, say $w\in X$. It easy to see that $w\in\bigcap_{j=1}^m A_j$. Since $x_0\in A_1$, then the subsequence $\{x_{m(n-1)}\}_{n=1}^\infty\in A_1$, the subsequence $\{x_{m(n-1)+1}\}_{n=1}^\infty\in A_2$ and, continuing in this way, the subsequence $\{x_{m(n-1)}\}_{n=1}^\infty\in A_m$. All the m subsequences are G-convergent in the G-closed sets A_j and hence they all converge to the same limit $w\in\bigcap_{j=1}^m A_j$. To show that the limit w is the fixed point of T, that is, $w=Tw$, we employ (29) with $x=x_n$, $y=w$. This leads to

$$\psi\big(G(Tx_n,Tw,Tw)\big)\leq\psi\big(M(x_n,w,w)\big)-\phi\big(M(x_n,w,w)\big), \tag{52}$$

where

$$\begin{aligned}M(x_n,w,w)=\max\Big\{&G(x_n,w,w),G(x_n,x_{n+1},x_{n+1}),G(w,Tw,Tw),\\&G(x_n,w,x_{n+1}),\frac{1}{3}\big[2G(x_n,Tw,Tw)+G(w,x_{n+1},x_{n+1})\big],\\&\frac{1}{3}\big[G(x_n,Tw,Tw)+2G(w,x_{n+1},x_{n+1})\big]\Big\}.\end{aligned} \tag{53}$$

Passing to limsup as $n\to\infty$, we get

$$\psi\big(G(w,Tw,Tw)\big)\leq\psi\big(G(w,Tw,Tw)\big)-\phi\big(G(w,Tw,Tw)\big). \tag{54}$$

Thus, $\phi(G(w,Tw,Tw))=0$ and hence $G(w,Tw,Tw)=0$, that is, $w=Tw$.

Finally, we prove that the fixed point is unique. Assume that $v\in X$ is another fixed point of T such that $v\neq w$. Then, since both v and w belong to $\bigcap_{j=1}^m A_j$, we set $x=v$ and $y=w$ in (29), which yields

$$\psi\big(G(Tv,Tw,Tw)\big)\leq\psi\big(M(v,w,w)\big)-\phi\big(\big(M(v,w,w)\big)\big), \tag{55}$$

where

$$\begin{aligned}M(v,w,w)=\max\Big\{&G(v,w,w),G(v,Tv,Tv),G(w,Tw,Tw),\\&\frac{1}{3}\big[2G(v,Tw,Tw)+G(w,Tv,Tv)\big],\frac{1}{3}\big[G(v,Tw,Tw)+2G(w,Tv,Tv)\big]\Big\}.\end{aligned} \tag{56}$$

On the other hand, by setting $x = w$ and $y = v$ in (29), we obtain that

$$\psi\big(G(Tw,Tv,Tv)\big) \leq \psi\big(M(w,v,v)\big) - \phi\big(\big(M(w,v,v)\big)\big), \tag{57}$$

where

$$M(w,v,v) = \max\Big\{G(w,v,v), G(w,Tw,Tw), G(v,Tv,Tv), G(w,v,Tw), \\ \frac{1}{3}\big[2G(w,Tv,Tv) + G(v,Tw,Tw)\big], \frac{1}{3}\big[G(w,Tv,Tv) + 2G(v,Tw,Tw)\big]\Big\}. \tag{58}$$

If $G(v,w,w) = G(w,v,v)$, then $v = w$. Indeed, by definition, we get that $d_G(v,w) = 0$. Hence $v = w$. If $G(v,w,w) > G(w,v,v)$, then by (56) $M(v,w,w) = G(v,w,w)$ and by (55),

$$\psi\big(G(v,w,w)\big) \leq \psi\big(G(v,w,w)\big) - \phi\big(\big(G(v,w,w)\big)\big), \tag{59}$$

and, clearly, $G(v,w,w) = 0$. So, we conclude that $v = w$. Otherwise, $G(w,v,v) > G(v,w,w)$. Then by (58), $M(w,v,v) = G(w,v,v)$ and by (57),

$$\psi\big(G(w,v,v)\big) \leq \psi\big(G(w,v,v)\big) - \phi\big(\big(G(w,v,v)\big)\big), \tag{60}$$

and, clearly, $G(w,v,v) = 0$. So, we conclude that $v = w$. Hence the fixed point of T is unique. □

Remark 2.1 We notice that some fixed point result in the context of G-metric can be obtained by usual (well-known) fixed point theorems (see, *e.g.*, [50, 51]). In fact, this is not a surprising result due to strong relationship between the usual metric and G-metric space (see, *e.g.*, [2, 3, 5]). Note that a G-metric space tells about the distance of three points instead of two points, which makes it original. We also emphasize that the techniques used in [50, 51] are not applicable to our main theorem.

To illustrate Theorem 2.1, we give the following example.

Example 2.1 Let $X = [-1,1]$ and let $T : X \to X$ be given as $Tx = \frac{-x}{8}$. Let $A = [-1,0]$ and $B = [0,1]$. Define the function $G : X \times X \times X \to [0,\infty)$ as

$$G(x,y,z) = |x-y| + |y-z| + |z-x|. \tag{61}$$

Clearly, the function G is a G-metric on X. Define also $\phi : [0,\infty) \to [0,\infty)$ as $\phi(t) = \frac{t}{8}$ and $\psi : [0,\infty) \to [0,\infty)$ as $\psi = \frac{t}{2}$. Obviously, the map T has a unique fixed point $x = 0 \in A \cap B$.

It can be easily shown that the map T satisfies the condition (29). Indeed,

$$G(Tx,Ty,Ty) = |Tx - Ty| + |Ty - Ty| + |Ty - Tx| = 2|Tx - Ty| = \frac{|y-x|}{4},$$

which yields

$$\psi\big(G(Tx,Ty,Ty)\big) = \frac{|y-x|}{8}. \tag{62}$$

Moreover, we have

$$\begin{aligned}
M(x,y,y) &= \max\Big\{|x-y|+|y-y|+|y-x|, |x-Tx|+|Tx-Tx|+|Tx-x|,\\
&\qquad |y-Ty|+|Ty-Ty|+|Ty-y|, |x-y|+|Tx-y|+|Tx-x|,\\
&\qquad \frac{1}{3}\big[2\big(|x-Ty|+|Ty-Ty|+|Ty-x|\big)+|y-Tx|+|Tx-Tx|+|Tx-y|\big],\\
&\qquad \frac{1}{3}\big[|x-Ty|+|Ty-Ty|+|Ty-x|+2\big(|y-Tx|+|Tx-Tx|+|Tx-y|\big)\big]\Big\}\\
&= \max\Big\{2|x-y|, 2|Tx-x|, 2|Ty-y|,\\
&\qquad \frac{1}{3}\big[4|Ty-x|+2|Tx-y|\big], \frac{1}{3}\big[2|Ty-x|+4|Tx-y|\big]\Big\}.
\end{aligned} \tag{63}$$

We derive from (63) that

$$2|x-y| \le M(x,y,y). \tag{64}$$

On the other hand, we have the following inequality:

$$\psi\big(M(x,y,y)\big)-\phi\big(M(x,y,y)\big)=\frac{M(x,y,y)}{2}-\frac{M(x,y,y)}{8}=\frac{3M(x,y,y)}{8}. \tag{65}$$

By elementary calculation, we conclude from (65) and (64) that

$$\frac{3|x-y|}{4} \le \frac{3M(x,y,y)}{8} = \psi\big(M(x,y,y)\big)-\phi\big(M(x,y,y)\big). \tag{66}$$

Combining the expressions (62) and (65), we obtain that

$$\begin{aligned}
\psi\big(G(Tx,Ty,Ty)\big) &= \frac{|y-x|}{8}\\
&\le \frac{3|x-y|}{4} \le \frac{3M(x,y,y)}{8} = \psi\big(M(x,y,y)\big)-\phi\big(M(x,y,y)\big).
\end{aligned} \tag{67}$$

Hence, all conditions of Theorem 2.1 are satisfied. Notice that 0 is the unique fixed point of T.

For particular choices of the functions ϕ, ψ, we obtain the following corollaries.

Corollary 2.1 *Let (X,G) be a G-complete G-metric space and $\{A_j\}_{j=1}^m$ be a family of nonempty G-closed subsets of X with $Y=\bigcup_{j=1}^m A_j$. Let $T: Y\to Y$ be a map satisfying*

$$T(A_j)\subseteq A_{j+1}, \quad j=1,2,\ldots,m, \text{ where } A_{m+1}=A_1. \tag{68}$$

Suppose that there exists a constant $k\in(0,1)$ such that the map T satisfies

$$G(Tx,Ty,Ty) \le kM(x,y,y) \tag{69}$$

for all $x \in A_j$ *and* $y \in A_{j+1}$, $j = 1, 2, \ldots, m$, *where*

$$M(x,y,y) = \max\biggl\{G(x,y,y), G(x,Tx,Tx), G(y,Ty,Ty), \frac{1}{3}\bigl[2G(x,Ty,Ty) + G(y,Tx,Tx)\bigr], \\ \frac{1}{3}\bigl[G(x,Ty,Ty) + 2G(y,Tx,Tx)\bigr]\biggr\}. \tag{70}$$

Then T *has a unique fixed point in* $\bigcap_{j=1}^m A_j$.

Proof The proof is obvious by choosing the functions ϕ, ψ in Theorem 2.1 as $\phi(t) = (1-k)t$ and $\psi(t) = t$. □

Corollary 2.2 *Let* (X, G) *be a G-complete G-metric space and* $\{A_j\}_{j=1}^m$ *be a family of nonempty G-closed subsets of X with* $Y = \bigcup_{j=1}^m A_j$. *Let* $T : Y \to Y$ *be a map satisfying*

$$T(A_j) \subseteq A_{j+1}, \quad j = 1, 2, \ldots, m, \text{ where } A_{m+1} = A_1. \tag{71}$$

Suppose that there exist constants a, b, c, d and e with $0 < a + b + c + d + e < 1$ *and there exists a function* $\psi \in \Psi$ *such that the map T satisfies the inequality*

$$\begin{aligned}\psi\bigl(G(Tx,Ty,Ty)\bigr) \le{}& aG(x,y,y) + bG(x,Tx,Tx) + cG(y,Ty,Ty) \\ &+ d\biggl(\frac{1}{3}\bigl[2G(x,Ty,Ty) + G(y,Tx,Tx)\bigr]\biggr) \\ &+ e\biggl(\frac{1}{3}\bigl[G(x,Ty,Ty) + 2G(y,Tx,Tx)\bigr]\biggr)\end{aligned} \tag{72}$$

for all $x \in A_j$ *and* $y \in A_{j+1}$, $j = 1, 2, \ldots, m$. *Then* T *has a unique fixed point in* $\bigcap_{j=1}^m A_j$.

Proof Clearly, we have

$$\begin{aligned}&aG(x,y,y) + bG(x,Tx,Tx) + cG(y,Ty,Ty) + d\biggl(\frac{1}{3}\bigl[2G(x,Ty,Ty) + G(y,Tx,Tx)\bigr]\biggr) \\ &\quad + e\biggl(\frac{1}{3}\bigl[G(x,Ty,Ty) + 2G(y,Tx,Tx)\bigr]\biggr) \\ &\le (a + b + c + d + e)M(x,y,y),\end{aligned} \tag{73}$$

where

$$M(x,y,y) = \max\biggl\{G(x,y,y), G(x,Tx,Tx), G(y,Ty,Ty), \frac{1}{3}\bigl[2G(x,Ty,Ty) + G(y,Tx,Tx)\bigr], \\ \frac{1}{3}\bigl[G(x,Ty,Ty) + 2G(y,Tx,Tx)\bigr]\biggr\}. \tag{74}$$

By Corollary 2.1, the map T has a unique fixed point. □

Corollary 2.3 *Let* (X, G) *be a G-complete G-metric space and* $\{A_j\}_{j=1}^m$ *be a family of nonempty G-closed subsets of X with* $Y = \bigcup_{j=1}^m A_j$. *Let* $T : Y \to Y$ *be a map satisfying*

$$T(A_j) \subseteq A_{j+1}, \quad j = 1, 2, \ldots, m, \text{ where } A_{m+1} = A_1.$$

Suppose that there exist functions $\phi \in \Phi$ and $\psi \in \Psi$ such that the map T satisfies the inequality

$$\psi\big(G(Tx,Ty,Tz)\big) \le \psi\big(M(x,y,z)\big) - \phi\big(M(x,y,z)\big)$$

for all $x \in A_j$ and $y \in A_{j+1}$, $j = 1,2,\ldots,m$, where

$$\begin{aligned} M(x,y,z) = \max\Big\{ & G(x,y,z), G(x,Tx,Tx), G(y,Ty,Ty), G(z,Tz,Tz), \\ & \frac{1}{3}\big[G(x,Ty,Ty) + G(y,Tx,Tx) + G(z,Tx,Tx)\big], \\ & \frac{1}{3}\big[G(x,Tz,Tz) + G(z,Tx,Tx) + G(y,Tx,Tx)\big], \\ & \frac{1}{3}\big[G(y,Tx,Tx) + G(x,Ty,Ty) + G(z,Ty,Ty)\big], \\ & \frac{1}{3}\big[G(y,Tz,Tz) + G(z,Ty,Ty) + G(x,Ty,Ty)\big], \\ & \frac{1}{3}\big[G(z,Tx,Tx) + G(x,Tz,Tz) + G(y,Tz,Tz)\big], \\ & \frac{1}{3}\big[G(z,Ty,Ty) + G(y,Tz,Tz) + G(x,Tz,Tz)\big]\Big\}. \end{aligned} \tag{75}$$

Then T has a unique fixed point in $\bigcap_{j=1}^m A_j$.

Proof The expression (75) coincides with the expression (30). Following the lines in the proof of Theorem 2.1, by letting $x = x_n$ and $y = z = x_{n+1}$, we get the desired result. □

Cyclic maps satisfying integral type contractive conditions are amongst common applications of fixed point theorems. In this context, we consider the following applications.

Corollary 2.4 *Let (X,G) be a G-complete G-metric space and $\{A_j\}_{j=1}^m$ be a family of nonempty G-closed subsets of X with $Y = \bigcup_{j=1}^m A_j$. Let $T : Y \to Y$ be a map satisfying*

$$T(A_j) \subseteq A_{j+1}, \quad j = 1,2,\ldots,m, \text{ where } A_{m+1} = A_1.$$

Suppose also that there exist functions $\phi \in \Phi$ and $\psi \in \Psi$ such that the map T satisfies

$$\psi\left(\int_0^{G(Tx,Ty,Ty)} ds\right) \le \psi\left(\int_0^{M(x,y,y)} ds\right) - \phi\left(\int_0^{M(x,y,y)} ds\right),$$

where

$$\begin{aligned} M(x,y,y) = \max\Big\{ & G(x,y,y), G(x,Tx,Tx), G(y,Ty,Ty), \frac{1}{3}\big[2G(x,Ty,Ty) + G(y,Tx,Tx)\big], \\ & \frac{1}{3}\big[G(x,Ty,Ty) + 2G(y,Tx,Tx)\big]\Big\} \end{aligned}$$

for all $x \in A_j$ and $y \in A_{j+1}$, $j = 1,2,\ldots,m$. Then T has a unique fixed point in $\bigcap_{j=1}^m A_j$.

Corollary 2.5 *Let (X, G) be a G-complete G-metric space and $\{A_j\}_{j=1}^m$ be a family of nonempty G-closed subsets of X with $Y = \bigcup_{j=1}^m A_j$. Let $T : Y \to Y$ be a map satisfying*

$$T(A_j) \subseteq A_{j+1}, \quad j = 1, 2, \dots, m, \text{ where } A_{m+1} = A_1.$$

Suppose also that

$$\int_0^{G(Tx,Ty,Ty)} ds \le k \int_0^{M(x,y,y)} ds,$$

where $k \in (0, 1)$ and

$$M(x,y,y) = \max\biggl\{G(x,y,y), G(x,Tx,Tx), G(y,Ty,Ty), \frac{1}{3}\bigl[2G(x,Ty,Ty) + G(y,Tx,Tx)\bigr], \frac{1}{3}\bigl[G(x,Ty,Ty) + 2G(y,Tx,Tx)\bigr]\biggr\}$$

for all $x \in A_j$ and $y \in A_{j+1}$, $j = 1, 2, \dots, m$. Then T has a unique fixed point in $\bigcap_{j=1}^m A_j$.

Proof The proof is obvious by choosing the function ϕ, ψ in Corollary 2.4 as $\phi(t) = (1 - k)t$ and $\psi(t) = t$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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