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Hybrid methods for a mixed equilibrium problem and fixed points of a countable family of multivalued nonexpansive mappings

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Abstract

In this paper, we prove a strong convergence theorem for a new hybrid method, using shrinking projection method introduced by Takahashi and a fixed point method for finding a common element of the set of solutions of mixed equilibrium problem and the set of common fixed points of a countable family of multivalued nonexpansive mappings in Hilbert spaces. We also apply our main result to the convex minimization problem and the fixed point problem of a countable family of multivalued nonexpansive mappings.

MSC: 47H09; 47H10

Keywords: multivalued nonexpansive mappings; mixed equilibrium problem; shrinking projection method

1 Introduction

The mixed equilibrium problem (MEP) includes several important problems arising in optimization, economics, physics, engineering, transportation, network, Nash equilibrium problems in noncooperative games, and others. Variational inequalities and mathematical programming problems are also viewed as the abstract equilibrium problems (EP) (*e.g.*, [1, 2]). Many authors have proposed several methods to solve the EP and MEP, see, for instance, [1–9] and the references therein.

Fixed point problems for multivalued mappings are more difficult than those of single-valued mappings and play very important role in applied science and economics. Recently, many authors have proposed their fixed point methods for finding a fixed point of both multivalued mapping and a family of multivalued mappings. All of those methods have only weak convergence.

It is known that Mann's iterations have only weak convergence even in the Hilbert spaces. To overcome this problem, Takahashi [10] introduced a new method, known as shrinking projection method, which is a hybrid method of Mann's iteration, and the projection method, and obtained strong convergence results of such method. In this paper, we use the shrinking projection method defined by Takahashi [10] and our new method to define a new hybrid method for MEP and a fixed point problem for a family of nonexpansive multivalued mappings.

An element $p \in K$ is called a *fixed point* of a single-valued mapping T if $p = Tp$ and of a multivalued mapping T if $p \in Tp$. The set of fixed points of T is denoted by $F(T)$.

Let X be a real Banach space. A subset K of X is called *proximinal* if for each $x \in X$, there exists an element $k \in K$ such that

$$d(x,k) = d(x,K),$$

where $d(x,K) = \inf\{\|x-y\| : y \in K\}$ is the distance from the point x to the set K.

Let X be a uniformly convex real Banach space, and let K be a nonempty closed convex subset of X, and let $CB(K)$ be a family of nonempty closed bounded subsets of K, and let $P(K)$ be a nonempty proximinal bounded subsets of K.

For multivalued mappings $T : K \to P(K)$, define $P_T(x) := \{y \in T(x) : \|x-y\| = d(x, T(x))\}$ for all $x \in K$.

The *Hausdorff metric* on $CB(X)$ is defined by

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}$$

for all $A, B \in CB(X)$.

A multivalued mapping $T : K \to CB(K)$ is said to be *nonexpansive* if $H(Tx, Ty) \leq \|x-y\|$ for all $x, y \in K$.

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let D be a nonempty closed convex subset of H. Let $F : D \times D \to \mathbb{R}$ be a bifunction, and let $\varphi : D \to \mathbb{R} \cup \{+\infty\}$ be a function such that $D \cap \operatorname{dom}\varphi \neq \emptyset$, where $\mathbb{R}$ is the set of real numbers and $\operatorname{dom}\varphi = \{x \in H : \varphi(x) < +\infty\}$.

Flores-Bazán [11] introduced the following mixed equilibrium problem:

$$\text{Find } x \in D \text{ such that } F(x,y) + \varphi(y) \geq \varphi(x), \quad \forall y \in D. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $MEP(F,\varphi)$.

If $\varphi \equiv 0$, then the mixed equilibrium problem (1.1) reduces to the following equilibrium problem:

$$\text{Find } x \in D \text{ such that } F(x,y) \geq 0, \quad \forall y \in D. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $EP(F)$ (see Combettes and Hirstoaga [12]).

If $F \equiv 0$, then the mixed equilibrium problem (1.1) reduces to the following convex minimization problem:

$$\text{Find } x \in D \text{ such that } \varphi(y) \geq \varphi(x), \quad \forall y \in D. \tag{1.3}$$

The set of solutions of (1.3) is denoted by $CMP(\varphi)$.

In an infinite-dimensional Hilbert space, the Mann iteration algorithms have only a weak convergence. In 2003, Nakajo and Takahashi [13] introduced the method, called CQ method, to modify Mann's iteration to obtain the strong convergence theorem for nonexpansive mapping in a Hilbert space. The CQ method has been studied extensively by many authors, for instance, Marino and Xu [14]; Zhou [15]; Zhang and Cheng [16].

In 2008, Takahashi *et al.* [10] introduced the following iteration scheme, which is usually called the shrinking projection method. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ and $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence $\{x_n\}$ of D as follows:

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T_n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \geq 1, \end{cases}$$

where P_{C_n} is the metric projection of H onto C_n and $\{T_n\}$ is a family of nonexpansive mappings. They proved that the sequence $\{x_n\}$ converges strongly to $z = P_{F(T)}x_0$, where $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. The shrinking projection method has been studied widely by many authors, for example, Tada and Takahashi [17]; Aoyama *et al.* [18]; Yao *et al.* [19]; Kang *et al.* [20]; Cholamjiak and Suantai [21]; Ceng *et al.* [22]; Tang *et al.* [23]; Cai and Bu [24]; Kumam *et al.* [25]; Kimura *et al.* [26]; Shehu [27, 28]; Wang *et al.* [29].

In 2009, Wangkeeree and Wangkeeree [30] proved a strong convergence theorem of an iterative algorithm based on extragradient method for finding a common element of the set of solutions of a mixed equilibrium problem, the set of common fixed points of a family of infinitely nonexpansive mappings and the set of the variational inequality for a monotone Lipschitz continuous mapping in a Hilbert space.

In 2011, Rodjanadid [31] introduced another iterative method modified from an iterative scheme of Klin-eam and Suantai [32] for finding a common element of the set of solutions of mixed equilibrium problems and the set of common fixed points of countable family of nonexpansive mappings in real Hilbert spaces. The mixed equilibrium problems have been studied by many authors, for instance, Peng and Yao [33]; Zeng *et al.* [34]; Peng *et al.* [35]; Wangkeeree and Kamraksa [36]; Jaiboon and Kumam [37]; Chamnarnpan and Kumam [38]; Cholamjiak *et al.* [39].

Nadler [40] started to study fixed points of multivalued contractions and nonexpansive mapping by using the Hausdorff metric.

Sastry and Babu [41] defined Mann and Ishikawa iterates for a multivalued map T with a fixed point p, and proved that these iterates converge strongly to a fixed point q of T under the compact domain in a real Hilbert space. Moreover, they illustrated that fixed point q may be different from p.

Panyanak [42] generalized results of Sastry and Babu [41] to uniformly convex Banach spaces and proved a strong convergence theorem of Mann iterates for a mapping defined on a noncompact domain and satisfying some conditions. He also obtained a strong convergence result of Ishikawa iterates for a mapping defined on a compact domain.

Hussain and Khan [43], in 2003, introduced the best approximation operator P_T to find fixed points of *-nonexpansive multivalued mapping and proved strong convergence of its iterates on a closed convex unbounded subset of a Hilbert space, which is not necessarily compact.

Hu *et al.* [44] obtained common fixed point of two nonexpansive multivalued mappings satisfying certain contractive conditions.

Cholamjiak and Suantai [45] proved strong convergence theorems of two new iterative procedures with errors for two quasi-nonexpansive multivalued mappings by using the best approximation operator and the end point condition in uniformly convex Banach spaces. Later, Cholamjiak *et al.* [46] introduced a modified Mann iteration and obtained weak and strong convergence theorems for a countable family of nonexpansive multivalued mappings by using the best approximation operator in a Banach space. They also gave some examples of multivalued mappings T such that P_T are nonexpansive.

Later, Eslamian and Abkar [47] generalized and modified the iteration of Abbas *et al.* [48] from two mappings to the infinite family of multivalued mappings $\{T_i\}$ such that each P_{T_i} satisfies the condition (C).

In this paper, we introduce a new hybrid method for finding a common element of the set of solutions of a mixed equilibrium problem and the set of common fixed points of a countable family of multivalued nonexpansive mappings in Hilbert spaces. We obtain a strong convergence theorem for the sequences generated by the proposed method without the assumption of compactness of the domain and other conditions imposing on the mappings.

In Section 2, we give some preliminaries and lemmas, which will be used in proving the main results. In Section 3, we introduce a new hybrid method and a fixed point method defined by (3.1) and prove strong convergence theorem for finding a common element of the set of solutions between mixed equilibrium problem and common fixed point problems of a countable family of multivalued nonexpansive mappings in Hilbert spaces. We also give examples of the control sequences satisfying the control conditions in main results. In Section 4, we summarize the main results of this paper.

2 Preliminaries

Let D be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in D, denoted by P_Dx, such that

$$\|x - P_Dx\| \leq \|x - y\|, \quad \forall y \in D.$$

P_D is called the *metric projection* of H onto D. It is known that P_D is a nonexpansive mapping of H onto D. It is also know that P_D satisfies $\langle x-y, P_Dx-P_Dy\rangle \geq \|P_Dx-P_Dy\|^2$ for every $x, y \in H$. Moreover, P_Dx is characterized by the properties: $P_Dx \in D$ and $\langle x-P_Dx, P_Dx-y\rangle \geq 0$ for all $y \in D$.

Lemma 2.1 [13] *Let D be a nonempty closed convex subset of a real Hilbert space H and $P_D : H \to D$ be the metric projection from H onto D. Then the following inequality holds*:

$$\|y - P_Dx\|^2 + \|x - P_Dx\|^2 \leq \|x - y\|^2, \quad \forall x \in H, \forall y \in D.$$

Lemma 2.2 [14] *Let H be a real Hilbert space. Then the following equations hold*:
(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$, $\forall x, y \in H$;
(ii) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2$, $\forall t \in [0, 1]$ *and* $x, y \in H$.

Lemma 2.3 [21] *Let H be a real Hilbert space. Then for each $m \in \mathbb{N}$*

$$\left\|\sum_{i=1}^{m} t_i x_i\right\|^2 = \sum_{i=1}^{m} t_i\|x_i\|^2 - \sum_{i=1, i\neq j}^{m} t_i t_j\|x_i - x_j\|^2,$$

$x_i \in H$ and $t_i, t_j \in [0, 1]$ for all $i, j = 1, 2, \ldots, m$ with $\sum_{i=1}^{m} t_i = 1$.

Lemma 2.4 [49] *Let D be a nonempty closed and convex subset of a real Hilbert space H. Given $x, y, z \in H$ and also given $a \in \mathbb{R}$, the set*

$$\{v \in D : \|y - v\|^2 \le \|x - v\|^2 + \langle z, v\rangle + a\}$$

is convex and closed.

For solving the mixed equilibrium problem, we assume the bifunction F, φ and the set D satisfy the following conditions:

(A1) $F(x, x) = 0$ for all $x \in D$;
(A2) F is monotone, that is, $F(x, y) + F(y, x) \le 0$ for all $x, y \in D$;
(A3) for each $x, y, z \in D$, $\limsup_{t\downarrow 0} F(tz + (1 - t)x, y) \le F(x, y)$;
(A4) $F(x, \cdot)$ is convex and lower semicontinuous for each $x \in D$;
(B1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq D$ and $y_x \in D \cap \operatorname{dom}\varphi$ such that for any $z \in D \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z);$$

(B2) D is a bounded set.

Lemma 2.5 [35] *Let D be a nonempty closed and convex subset of a real Hilbert space H. Let $F : D \times D \to \mathbb{R}$ be a bifunction satisfying conditions* (A1)-(A4) *and $\varphi : D \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function such that $D \cap \operatorname{dom}\varphi \neq \emptyset$. For $r > 0$ and $x \in D$, define a mapping $T_r : H \to D$ as follows*:

$$T_r(x) = \left\{z \in D : F(z, y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \ge \varphi(z), \forall y \in D\right\}$$

for all $x \in H$. Assume that either (B1) *or* (B2) *holds. Then the following conclusions hold*:

(1) *for each $x \in H$, $T_r(x) \neq \emptyset$*;
(2) *T_r is single-valued*;
(3) *T_r is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\|T_r(x) - T_r(y)\|^2 \le \langle T_r(x) - T_r(y), x - y\rangle;$$

(4) *$F(T_r) = MEP(F, \varphi)$*;
(5) *$MEP(F, \varphi)$ is closed and convex.*

As in ([21], Lemma 2.7), the following lemma holds true for multivalued mapping. To avoid repetition, we omit the details of proof.

Lemma 2.6 *Let D be a closed and convex subset of a real Hilbert space H. Let $T : D \to P(D)$ be a multivalued nonexpansive mapping with $F(T) \neq \emptyset$ such that P_T is nonexpansive. Then $F(T)$ is a closed and convex subset of D.*

3 Main results

In the following theorem, we prove strong convergence of the sequence $\{x_n\}$ defined by (3.1) to a common element of the set of solutions of a mixed equilibrium problem and the set of common fixed points of a countable family of multivalued nonexpansive mappings.

Theorem 3.1 *Let D be a nonempty closed and convex subset of a real Hilbert space H. Let F be a bifunction from $D \times D$ to $\mathbb{R}$ satisfying* (A1)-(A4), *and let φ be a proper lower semicontinuous and convex function from D to $\mathbb{R} \cup \{+\infty\}$ such that $D \cap \operatorname{dom}\varphi \neq \emptyset$. Let $T_i : D \to P(D)$ be multivalued nonexpansive mappings for all $i \in \mathbb{N}$ with $\Omega := \bigcap_{i=1}^{\infty} F(T_i) \cap MEP(F,\varphi) \neq \emptyset$ such that all P_{T_i} are nonexpansive. Assume that either* (B1) *or* (B2) *holds and $\{\alpha_{n,i}\} \subset [0,1)$ satisfies the condition $\liminf_{n\to\infty} \alpha_{n,i}\alpha_{n,0} > 0$ for all $i \in \mathbb{N}$. Define the sequence $\{x_n\}$ as follows: $x_1 \in D = C_1$,*

$$\begin{cases} F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in D, \\ y_n = \alpha_{n,0} u_n + \sum_{i=1}^{n} \alpha_{n,i} x_{n,i}, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 0, \end{cases} \tag{3.1}$$

where the sequences $r_n \in (0,\infty)$ with $\liminf_{n\to\infty} r_n > 0$ and $\{\alpha_{n,i}\} \subset [0,1)$ satisfying $\sum_{i=0}^{n} \alpha_{n,i} = 1$ and $x_{n,i} \in P_{T_i} u_n$ for $i \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges strongly to $P_\Omega x_0$.

Proof We split the proof into six steps.

Step 1. Show that $P_{C_{n+1}} x_0$ is well defined for every $x_0 \in D$.

By Lemmas 2.5-2.6, we obtain that $MEP(F,\varphi)$ and $\bigcap_{i=1}^{\infty} F(T_i)$ is a closed and convex subset of D. Hence Ω is a closed and convex subset of D. It follows from Lemma 2.4 that C_{n+1} is a closed and convex for each $n \geq 0$. Let $v \in \Omega$. Then $P_{T_i}(v) = \{v\}$ for all $i \in \mathbb{N}$. Since $u_n = T_{r_n} x_n \in \operatorname{dom}\varphi$, we have

$$\|u_n - v\| = \|T_{r_n} x_n - T_{r_n} v\| \leq \|x_n - v\|,$$

for every $n \geq 0$. Then

$$\begin{aligned} \|y_n - v\| &= \left\| \alpha_{n,0} u_n + \sum_{i=1}^{n} \alpha_{n,i} x_{n,i} - v \right\| \\ &\leq \alpha_{n,0}\|u_n - v\| + \sum_{i=1}^{n} \alpha_{n,i}\|x_{n,i} - v\| \\ &= \alpha_{n,0}\|u_n - v\| + \sum_{i=1}^{n} \alpha_{n,i} d(x_{n,i}, P_{T_i} v) \\ &\leq \alpha_{n,0}\|u_n - v\| + \sum_{i=1}^{n} \alpha_{n,i} H(P_{T_i} u_n, P_{T_i} v) \\ &\leq \alpha_{n,0}\|u_n - v\| + \sum_{i=1}^{n} \alpha_{n,i}\|u_n - v\| \\ &= \|u_n - v\| \leq \|x_n - v\|. \end{aligned} \tag{3.2}$$

Hence $v \in C_{n+1}$, so that $\Omega \subset C_{n+1}$. Therefore, $P_{C_{n+1}} x_0$ is well defined.

Step 2. Show that $\lim_{n\to\infty}\|x_n - x_0\|$ exists.

Since Ω is a nonempty closed convex subset of H, there exists a unique $v \in \Omega$ such that $v = P_\Omega x_0$. Since $x_n = P_{C_n}x_0$ and $x_{n+1} \in C_{n+1} \subset C_n$, $\forall n \geq 0$, we have

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|, \quad \forall n \geq 0.$$

On the other hand, as $v \in \Omega \subset C_n$, we obtain

$$\|x_n - x_0\| \leq \|v - x_0\|, \quad \forall n \geq 0.$$

It follows that the sequence $\{x_n\}$ is bounded and nondecreasing. Therefore, $\lim_{n\to\infty}\|x_n - x_0\|$ exists.

Step 3. Show that $\lim_{n\to\infty} x_n = w \in D$.

For $m > n$, by the definition of C_n, we get $x_m = P_{C_m}x_0 \in C_m \subset C_n$. By applying Lemma 2.1, we have

$$\|x_m - x_n\|^2 \leq \|x_m - x_0\|^2 - \|x_n - x_0\|^2.$$

Since $\lim_{n\to\infty}\|x_n - x_0\|$ exists, it follows that $\{x_n\}$ is Cauchy. Hence there exists $w \in D$ such that $\lim_{n\to\infty} x_n = w$.

Step 4. Show that $\|x_{n,i} - x_n\| \to 0$ as $n \to \infty$ for every $i \in \mathbb{N}$.

From $x_{n+1} \in C_{n+1}$, we have

$$\begin{aligned}\|x_n - y_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \\ &\leq 2\|x_n - x_{n+1}\| \to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{3.3}$$

For $v \in \Omega$, by Lemma 2.3 and (3.2), we get

$$\begin{aligned}\|y_n - v\|^2 &= \left\|\alpha_{n,0}(u_n - v) + \sum_{i=1}^{n}\alpha_{n,i}(x_{n,i} - v)\right\|^2 \\ &\leq \alpha_{n,0}\|u_n - v\|^2 + \sum_{i=1}^{n}\alpha_{n,i}\|x_{n,i} - v\|^2 - \sum_{i=1}^{n}\alpha_{n,i}\alpha_{n,0}\|x_{n,i} - u_n\|^2 \\ &= \alpha_{n,0}\|u_n - v\|^2 + \sum_{i=1}^{n}\alpha_{n,i}d(x_{n,i}, P_{T_i}v)^2 - \sum_{i=1}^{n}\alpha_{n,i}\alpha_{n,0}\|x_{n,i} - u_n\|^2 \\ &\leq \alpha_{n,0}\|u_n - v\|^2 + \sum_{i=1}^{n}\alpha_{n,i}H(P_{T_i}u_n, P_{T_i}v)^2 - \sum_{i=1}^{n}\alpha_{n,i}\alpha_{n,0}\|x_{n,i} - u_n\|^2 \\ &\leq \alpha_{n,0}\|u_n - v\|^2 + \sum_{i=1}^{n}\alpha_{n,i}\|u_n - v\|^2 - \sum_{i=1}^{n}\alpha_{n,i}\alpha_{n,0}\|x_{n,i} - u_n\|^2 \\ &= \|u_n - v\|^2 - \sum_{i=1}^{n}\alpha_{n,i}\alpha_{n,0}\|x_{n,i} - u_n\|^2 \\ &\leq \|x_n - v\|^2 - \sum_{i=1}^{n}\alpha_{n,i}\alpha_{n,0}\|x_{n,i} - u_n\|^2.\end{aligned}$$

This implies that

$$\begin{aligned}\alpha_{n,i}\alpha_{n,0}\|x_{n,i}-u_n\|^2 &\le \sum_{i=1}^{n}\alpha_{n,i}\alpha_{n,0}\|x_{n,i}-u_n\|^2\\ &\le \|x_n-v\|^2-\|y_n-v\|^2\\ &\le M\|x_n-y_n\|,\end{aligned}$$

where $M=\sup_{n\ge 0}\{\|x_n-v\|+\|y_n-v\|\}$. By the given control condition on $\{\alpha_{n,i}\}$ and (3.3), we obtain

$$\lim_{n\to\infty}\|x_{n,i}-u_n\|=0,\quad \forall i\in\mathbb{N}.$$

By Lemma 2.5, we have

$$\begin{aligned}\|u_n-v\|^2 &= \|T_{r_n}x_n-T_{r_n}v\|^2\\ &\le \langle T_{r_n}x_n-T_{r_n}v, x_n-v\rangle\\ &= \langle u_n-v, x_n-v\rangle\\ &= \frac{1}{2}\bigl\{\|u_n-v\|^2+\|x_n-v\|^2-\|x_n-u_n\|^2\bigr\}.\end{aligned}$$

Hence $\|u_n-v\|^2\le\|x_n-v\|^2-\|x_n-u_n\|^2$. By Lemma 2.3, we get

$$\begin{aligned}\|y_n-v\|^2 &= \left\|\alpha_{n,0}u_n+\sum_{i=1}^{n}\alpha_{n,i}x_{n,i}-v\right\|^2\\ &= \alpha_{n,0}\|u_n-v\|^2+\sum_{i=1}^{n}\alpha_{n,i}\|x_{n,i}-v\|^2-\sum_{i=1}^{n}\alpha_{n,i}\alpha_{n,0}\|x_{n,i}-u_n\|^2\\ &\le \alpha_{n,0}\|u_n-v\|^2+\sum_{i=1}^{n}\alpha_{n,i}\|x_{n,i}-v\|^2\\ &= \alpha_{n,0}\|u_n-v\|^2+\sum_{i=1}^{n}\alpha_{n,i}d(x_{n,i},P_{T_i}v)^2\\ &\le \alpha_{n,0}\|u_n-v\|^2+\sum_{i=1}^{n}\alpha_{n,i}H(P_{T_i}u_n,P_{T_i}v)^2\\ &\le \alpha_{n,0}\|u_n-v\|^2+\sum_{i=1}^{n}\alpha_{n,i}\|u_n-v\|^2\\ &= \|u_n-v\|^2\\ &\le \|x_n-v\|^2-\|x_n-u_n\|^2.\end{aligned}$$

This implies that

$$\begin{aligned}\|x_n-u_n\|^2 &\le \|x_n-v\|^2-\|y_n-v\|^2\\ &\le M\|x_n-y_n\|,\end{aligned}$$

where $M = \sup_{n\geq 0}\{\|x_n - v\| + \|y_n - v\|\}$. From (3.3), we get $\lim_{n\to\infty} \|x_n - u_n\| = 0$. It follows that

$$\begin{aligned}\|x_{n,i} - x_n\| &\leq \|x_{n,i} - u_n\| + \|u_n - x_n\| \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Step 5. Show that $w \in \Omega$.

By $\liminf_{n\to\infty} r_n > 0$, we have

$$\left\| \frac{x_n - u_n}{r_n} \right\| = \frac{1}{r_n}\|x_n - u_n\| \to 0, \quad n \to \infty. \tag{3.4}$$

From $\lim_{n\to\infty} x_n = w$, we obtain $\lim_{n\to\infty} u_n = w$.

We will show that $w \in MEP(F,\varphi)$. Since $u_n = T_{r_n}x_n \in \operatorname{dom}\varphi$, we have

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in D.$$

It follows by (A2) that

$$\varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n), \quad \forall y \in D.$$

Hence

$$\varphi(y) - \varphi(u_n) + \left\langle y - u_n, \frac{u_n - x_n}{r_n}\right\rangle \geq F(y, u_n), \quad \forall y \in D.$$

It follows from (3.4), (A4) and the lower semicontinuous of φ that

$$F(y, w) + \varphi(w) - \varphi(y) \leq 0, \quad \forall y \in D.$$

For t with $0 < t \leq 1$ and $y \in D$, let $y_t = ty + (1-t)w$. Since $y, w \in D$ and D is convex, then $y_t \in D$ and hence

$$F(y_t, w) + \varphi(w) - \varphi(y_t) \leq 0.$$

This implies by (A1), (A4) and the convexity of φ, that

$$\begin{aligned}0 &= F(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\ &\leq tF(y_t, y) + (1-t)F(y_t, w) + t\varphi(y) + (1-t)\varphi(w) - \varphi(y_t) \\ &\leq t\big[F(y_t, y) + \varphi(y) - \varphi(y_t)\big].\end{aligned}$$

Dividing by t, we have

$$F(y_t, y) + \varphi(y) - \varphi(y_t) \geq 0, \quad \forall y \in D.$$

Letting $t \to 0$, it follows from the weakly semicontinuity of φ that

$$F(w,y) + \varphi(y) - \varphi(w) \geq 0, \quad \forall y \in D.$$

Hence $w \in MEP(F,\varphi)$. Next, we will show that $w \in \bigcap_{i=1}^{\infty} F(T_i)$. For each $i = 1,2,\dots,n$, we have

$$\begin{aligned} d(w,T_i w) &\leq d(w,x_n) + d(x_n,x_{n,i}) + d(x_{n,i},T_i w) \\ &\leq d(w,x_n) + d(x_n,x_{n,i}) + H(T_i u_n,T_i w) \\ &\leq d(w,x_n) + d(x_n,x_{n,i}) + d(u_n,w). \end{aligned}$$

By Steps 3-4, we have $d(w,T_i w) = 0$. Hence $w \in T_i w$ for all $i = 1,2,\dots,n$.

Step 6. Show that $w = P_\Omega x_0$.

Since $x_n = P_{C_n} x_0$, we get

$$\langle z - x_n, x_0 - x_n \rangle \leq 0, \quad \forall z \in C_n.$$

Since $w \in \Omega \subset C_n$, we have

$$\langle z - w, x_0 - w \rangle \leq 0, \quad \forall z \in \Omega.$$

Now, we obtain that $w = P_\Omega x_0$.

This completes the proof. □

Setting $\varphi \equiv 0$ in Theorem 3.1, we have the following result.

Corollary 3.2 *Let D be a nonempty closed and convex subset of a real Hilbert space H. Let F be a bifunction from $D \times D$ to $\mathbb{R}$ satisfying* (A1)-(A4). *Let $T_i : D \to P(D)$ be multivalued nonexpansive mappings for all $i \in \mathbb{N}$ with $\Omega := \bigcap_{i=1}^{\infty} F(T_i) \cap EP(F) \neq \emptyset$ such that all P_{T_i} are nonexpansive. Assume that $\{\alpha_{n,i}\} \subset [0,1)$ satisfies the condition $\liminf_{n\to\infty} \alpha_{n,i}\alpha_{n,0} > 0$ for all $i \in \mathbb{N}$. Define the sequence $\{x_n\}$ as follows: $x_1 \in D = C_1$,*

$$\begin{cases} F(u_n,y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in D, \\ y_n = \alpha_{n,0} u_n + \sum_{i=1}^{n} \alpha_{n,i} x_{n,i}, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 0, \end{cases} \tag{3.5}$$

where the sequences $r_n \in (0,\infty)$ with $\liminf_{n\to\infty} r_n > 0$ and $\{\alpha_{n,i}\} \subset [0,1)$ satisfying $\sum_{i=0}^{n} \alpha_{n,i} = 1$ and $x_{n,i} \in P_{T_i} u_n$ for $i \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges strongly to $P_\Omega x_0$.

Setting $F \equiv 0$ in Theorem 3.1, we have the following result.

Corollary 3.3 *Let D be a nonempty closed and convex subset of a real Hilbert space H. Let φ be a proper lower semicontinuous and convex function from D to $\mathbb{R} \cup \{+\infty\}$ such that $D \cap \operatorname{dom}\varphi \neq \emptyset$. Let $T_i : D \to P(D)$ be multivalued nonexpansive mappings for all $i \in \mathbb{N}$ with $\Omega := \bigcap_{i=1}^{\infty} F(T_i) \cap CMP(\varphi) \neq \emptyset$ such that all P_{T_i} are nonexpansive. Assume that either* (B1) *or* (B2) *holds, and $\{\alpha_{n,i}\} \subset [0,1)$ satisfies the condition $\liminf_{n\to\infty} \alpha_{n,i}\alpha_{n,0} > 0$ for all $i \in \mathbb{N}$.*

Define the sequence $\{x_n\}$ *as follows*: $x_1 \in D = C_1$,

$$\begin{cases} \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in D, \\ y_n = \alpha_{n,0}u_n + \sum_{i=1}^{n} \alpha_{n,i}x_{n,i}, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \geq 0, \end{cases} \tag{3.6}$$

where the sequences $r_n \in (0,\infty)$ *with* $\liminf_{n\to\infty} r_n > 0$ *and* $\{\alpha_{n,i}\} \subset [0,1)$ *satisfying* $\sum_{i=0}^{n} \alpha_{n,i} = 1$ *and* $x_{n,i} \in P_{T_i}u_n$ *for* $i \in \mathbb{N}$. *Then the sequence* $\{x_n\}$ *converges strongly to* $P_\Omega x_0$.

Remark 3.4

(i) Let $\{\alpha_{n,i}\}$ be double sequence in $(0,1]$. Let (a) and (b) be the following conditions:

(a) $\liminf_{n\to\infty} \alpha_{n,i}\alpha_{n,0} > 0$ for all $i \in \mathbb{N}$,

(b) $\lim_{n\to\infty} \alpha_{n,i}$ exist and lie in $(0,1]$ for all $i = 0,1,2,\ldots$.

It is easy to see that if $\{\alpha_{n,i}\}$ satisfies the condition (a), then it satisfies the condition (b). So Theorem 3.1 and Corollaries 3.2-3.3 hold true when the control double sequence $\{\alpha_{n,i}\}$ satisfies the condition (a).

(ii) The following double sequences are examples of the control sequences in Theorem 3.1 and Corollaries 3.2-3.3:

(1)

$$\alpha_{n,k} = \begin{cases} \frac{1}{2^k}(\frac{n}{n+1}), & n \geq k; \\ 1 - \frac{n}{n+1}(\sum_{k=1}^{n} \frac{1}{2^k}), & n = k-1; \\ 0, & n < k-1, \end{cases}$$

that is,

$$\alpha_{n,k} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \frac{3}{8} & \frac{3}{16} & \frac{3}{32} & \frac{11}{32} & 0 & 0 & \cdots & 0 & \cdots \\ \frac{2}{5} & \frac{1}{5} & \frac{1}{10} & \frac{1}{20} & \frac{1}{4} & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \\ \frac{n}{2(n+1)} & \frac{n}{4(n+1)} & \frac{n}{8(n+1)} & \frac{n}{16(n+1)} & \frac{n}{32(n+1)} & \frac{n}{64(n+1)} & \cdots & \frac{n}{2^k(n+1)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \end{pmatrix}.$$

We see that $\lim_{n\to\infty} \alpha_{n,k} = \frac{1}{2^k}$ and $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,k} = \frac{1}{2^{k+1}}$ for $k = 1,2,3,\ldots$.

(2)

$$\alpha_{n,k} = \begin{cases} \frac{1}{2^k}(\frac{n}{n+1}), & n \geq k \text{ and } n \text{ is odd;} \\ \frac{1}{2^{k+1}}(\frac{n}{n+1}), & n \geq k \text{ and } n \text{ is even;} \\ 1 - \frac{n}{n+1}(\sum_{k=1}^{n} \frac{1}{2^k}), & n = k-1 \text{ and } n \text{ is odd;} \\ 1 - \frac{n}{n+1}(\sum_{k=1}^{n} \frac{1}{2^{k+1}}), & n = k-1 \text{ and } n \text{ is even;} \\ 0, & n < k-1, \end{cases}$$

that is,

$$\alpha_{n,k} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \frac{1}{6} & \frac{1}{12} & \frac{3}{4} & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \frac{3}{8} & \frac{3}{16} & \frac{3}{32} & \frac{11}{32} & 0 & 0 & \cdots & 0 & \cdots \\ \frac{1}{5} & \frac{1}{10} & \frac{1}{20} & \frac{1}{40} & \frac{5}{8} & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \\ \frac{2n-1}{2(2n)} & \frac{2n-1}{4(2n)} & \frac{2n-1}{8(2n)} & \frac{2n-1}{16(2n)} & \frac{2n-1}{32(2n)} & \frac{2n-1}{64(2n)} & \cdots & \frac{2n-1}{2^k(2n)} & \cdots \\ \frac{2n}{4(2n+1)} & \frac{2n}{8(2n+1)} & \frac{2n}{16(2n+1)} & \frac{2n}{32(2n+1)} & \frac{2n}{64(2n+1)} & \frac{2n}{128(2n+1)} & \cdots & \frac{2n}{2^{k+1}(2n+1)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \end{pmatrix}.$$

We see that $\lim_{n\to\infty} \alpha_{n,k}$ does not exist and $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,k} = \frac{1}{2^{k+3}}$ for $k = 1, 2, 3, \ldots$.

4 Conclusions

We use the shrinking projection method defined by Takahashi [10] together with our method for finding a common element of the set of solutions of mixed equilibrium problem and common fixed points of a countable family of multivalued nonexpansive mappings in Hilbert spaces. The main results of paper can be applied for solving convex minimization problems and fixed point problems.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
AB studied and researched a nonlinear analysis and also wrote this article. SS participated in the process of the study and helped to draft the manuscript. All authors read and approved the final manuscript.

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