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A note on '(G, F)-Closed set and tripled point of coincidence theorems for generalized compatibility in partially metric spaces'

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Abstract

Recently, some (common) multidimensional fixed theorems in partially ordered complete metric spaces have appeared as a generalization of existing (usual) fixed point results. Unexpectedly, we realized that most of such (common) coupled fixed theorems are either weaker or equivalent to existing fixed point results in the literature. In particular, we prove that the results included in the very recent paper (Charoensawan and Thangthong in Fixed Point Theory Appl. 2014:245, 2014) can be considered as a consequence of existing fixed point theorems on the topic in the literature.

MSC: 47H10; 54H25

Keywords: fixed point; coincidence point; tripled coincidence point; partial order; compatible mappings

1 Introduction and preliminaries

Multidimensional fixed point theory was initiated in 2006 by Gnana Bhaskar and Lakshmikantham [1]. In fact, the authors [1] investigated the existence and uniqueness of a coupled fixed point of certain operators in the context of a partially ordered set to solve a periodic boundary value problem. Since then, multidimensional fixed point theorems have been investigated heavily by several authors; see, *e.g.*, [1–29] and related references therein.

In this short note, we underline the fact that most of the multidimensional fixed point theorems can be derived from the existing (uni-dimensional) fixed point results in the literature. In particular, we shall show that the result in the recent report [6] can be considered in this frame.

For the sake of completeness, we recollect some basic definitions, notations and results on the topic in the literature. Throughout the paper, let X be a nonempty set. Given a positive integer n, let X^n be the product space $X \times X \times \overset{n}{\cdots} \times X$. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of all nonnegative integers. In the sequel, n, m and k will be used to denote nonnegative integers. Unless otherwise stated, 'for all n' will mean 'for all $n \geq 0$'.

Definition 1.1 (Roldán and Karapınar [22]) A preorder (or a quasiorder) $\preccurlyeq$ on X is a binary relation on X that is *reflexive* (*i.e.*, $x \preccurlyeq x$ for all $x \in X$) and *transitive* (if $x, y, z \in X$ verify $x \preccurlyeq y$ and $y \preccurlyeq z$, then $x \preccurlyeq z$). In such a case, we say that $(X, \preccurlyeq)$ is a preordered space (or a preordered set). If a preorder $\preccurlyeq$ is also *antisymmetric* ($x \preccurlyeq y$ and $y \preccurlyeq x$ imply $x = y$), then $\preccurlyeq$ is called a partial order.

Throughout this manuscript, let (X,d) be a metric space, and let $\preccurlyeq$ be a preorder (or a partial order) on X. In the sequel, $T,g : X \to X$ and $F : X^n \to X$ will denote mappings.

Definition 1.2 A point $(x_1, x_2, \dots, x_n) \in X^n$ is:

- a *coupled coincidence point of F and g* if $n = 2$,

$$F(x_1,x_2) = gx_1 \quad \text{and} \quad F(x_2,x_1) = gx_2;$$

- a *tripled coincidence point of F and g* if $n = 3$,

$$F(x_1,x_2,x_3) = gx_1, \qquad F(x_2,x_1,x_2) = gx_2 \quad \text{and} \quad F(x_3,x_2,x_1) = gx_3;$$

- a *quadrupled coincidence point of F and g* if $n = 4$,

$$F(x_1,x_2,x_3,x_4) = gx_1, \qquad F(x_2,x_3,x_4,x_1) = gx_2,$$
$$F(x_3,x_4,x_1,x_2) = gx_3 \quad \text{and} \quad F(x_4,x_1,x_2,x_3) = gx_4.$$

Notice that when we take g as the identity mapping on X, then a point verifying the related conditions above is a *coupled* (respectively, *tripled*, *quadrupled*) *fixed point of F* due to Gnana Bhaskar and Lakshmikantham [1] (respectively, Berinde and Borcut [9], Karapınar [13]).

Definition 1.3 (Al-Mezel *et al.* [21]) If $(X, \preccurlyeq)$ is a preordered space and $T,g : X \to X$ are two mappings, we will say that T is a *$(g, \preccurlyeq)$-nondecreasing mapping* if $Tx \preccurlyeq Ty$ for all $x,y \in X$ such that $gx \preccurlyeq gy$. If g is the identity mapping on X, T is *$\preccurlyeq$-nondecreasing*.

In [28], $(g, \preccurlyeq)$-nondecreasing mappings were called *g-isotone* mappings (in particular, when X is a product space X^n).

Definition 1.4 A *fixed point* of a self-mapping $T : X \to X$ is a point $x \in X$ such that $Tx = x$. A *coincidence point* between two mappings $T,g : X \to X$ is a point $x \in X$ such that $Tx = gx$. A *common fixed point* of $T,g : X \to X$ is a point $x \in X$ such that $Tx = gx = x$.

Definition 1.5 We will say that T and g are *commuting* if $gTx = Tgx$ for all $x \in X$, and we will say that F and g are *commuting* if $gF(x_1,x_2,\dots,x_n) = F(gx_1,gx_2,\dots,gx_n)$ for all $x_1,\dots,x_n \in X$.

Remark 1.1 If $T,g : X \to X$ are commuting and $x_0 \in X$ is a coincidence point of T and g, then Tx_0 is also a coincidence point of T and g.

In 2003, Ran and Reurings characterized the Banach contraction mapping principle in the context of partially ordered metric space.

Theorem 1.1 (Ran and Reurings [20]) *Let $(X, \preccurlyeq)$ be an ordered set endowed with a metric d and $T : X \to X$ be a given mapping. Suppose that the following conditions hold*:

(a) (X,d) *is complete.*
(b) T *is* $\preccurlyeq$*-nondecreasing.*
(c) T *is continuous.*
(d) *There exists* $x_0 \in X$ *such that* $x_0 \preccurlyeq Tx_0$.
(e) *There exists a constant* $k \in (0,1)$ *such that* $d(Tx,Ty) \leq kd(x,y)$ *for all* $x,y \in X$ *with* $x \succcurlyeq y$.

Then T *has a fixed point. Moreover, if for all* $(x,y) \in X^2$ *there exists* $z \in X$ *such that* $x \preccurlyeq z$ *and* $y \preccurlyeq z$*, we obtain uniqueness of the fixed point.*

After Ran and Reurings' result, fixed point theorems have been investigated heavily. One of the interesting results in this direction was reported by Nieto and Rodríguez-López in [19], who slightly modified the hypothesis of the previous result swapping condition (c) with the fact that $(X,d,\preccurlyeq)$ is nondecreasing-regular as follows.

Definition 1.6 Let $(X,\preccurlyeq)$ be an ordered set endowed with a metric d. We will say that $(X,d,\preccurlyeq)$ is *nondecreasing-regular* (respectively, *nonincreasing-regular*) if any $\preccurlyeq$-nondecreasing (respectively, $\preccurlyeq$-nonincreasing) sequence $\{x_m\}$ is d-convergent to $x \in X$, we have that $x_m \preccurlyeq x$ (respectively, $x_m \succcurlyeq x$) for all m. And $(X,d,\preccurlyeq)$ is *regular* if it is both nondecreasing-regular and nonincreasing-regular.

Inspired by Boyd and Wong's theorem [10], Mukherjea [18] introduced the following kind of control functions:

$$\Phi = \left\{\varphi : [0,\infty) \to [0,\infty) : \varphi(t) < t \text{ and } \lim_{r\to t^+} \varphi(r) < t \text{ for each } t > 0\right\},$$

and proved a version of the following result in which the space is not necessarily endowed with a partial order (but the contractivity condition holds over all pairs of points of the space).

Theorem 1.2 *Let* $(X,\preccurlyeq)$ *be an ordered set endowed with a metric* d *and* $T : X \to X$ *be a given mapping. Suppose that the following conditions hold*:
(a) (X,d) *is complete.*
(b) T *is* $\preccurlyeq$*-nondecreasing.*
(c) *Either* T *is continuous or* $(X,d,\preccurlyeq)$ *is nondecreasing-regular.*
(d) *There exists* $x_0 \in X$ *such that* $x_0 \preccurlyeq Tx_0$.
(e) *There exists* $\varphi \in \Phi$ *such that* $d(Tx,Ty) \leq \varphi(d(x,y))$ *for all* $x,y \in X$ *with* $x \succcurlyeq y$.

Then T *has a fixed point. Moreover, if for all* $(x,y) \in X^2$ *there exists* $z \in X$ *such that* $x \preccurlyeq z$ *and* $y \preccurlyeq z$*, we obtain uniqueness of the fixed point.*

A partial order $\preccurlyeq$ on X can be extended to a partial order $\sqsubseteq$ on X^n defining, for all $Y = (y_1,y_2,\ldots,y_n)$, $V = (v_1,v_2,\ldots,v_n) \in X^n$,

$$Y \sqsubseteq V \quad \text{if} \begin{cases} y_i \preccurlyeq v_i, & i = 1,3,5,\ldots, \\ y_i \succcurlyeq v_i, & i = 2,4,6,\ldots. \end{cases} \tag{1}$$

An interesting generalization of the previous result was given by Wang in [28] using this extended partial order on X^n.

Theorem 1.3 (Wang [28], Theorem 3.4) *Let $(X,\preccurlyeq)$ be a partially ordered set and suppose that there is a metric d on X such that (X,d) is a complete metric space. Let $G:X^n\to X^n$ and $T:X^n\to X^n$ be a G-isotone mapping for which there exists $\varphi\in\Phi$ such that for all $Y\in X^n$, $V\in X^n$ with $G(V)\sqsubseteq G(Y)$,*

$$\rho_n\big(T(Y),T(V)\big)\leq\varphi\big(\rho_n\big(G(Y),G(V)\big)\big),$$

where ρ_n is defined, for all $Y=(y_1,y_2,\ldots,y_n)$, $V=(v_1,v_2,\ldots,v_n)\in X^n$, by

$$\rho_n(Y,V)=\frac{1}{n}\big[d(y_1,v_1)+d(y_2,v_2)+\cdots+d(y_n,v_n)\big].$$

Suppose $T(X^n)\subseteq G(X^n)$ and also suppose either

(a) *T is continuous, G is continuous and commutes with T, or*

(b) *$(X,d,\preccurlyeq)$ is regular and $G(X^n)$ is closed.*

If there exists $Y_0\in X^n$ such that $G(Y_0)$ and $T(Y_0)$ are $\sqsubseteq$-comparable, then T and G have a coincidence point.

For further generalizations of the previous result, we refer readers to papers of Romaguera [25] and in Al-Mezel *et al.* [21].

Gnana Bhaskar and Lakshmikantham introduced the following condition in order to guarantee the existence of coupled fixed points

Definition 1.7 (Gnana Bhaskar and Lakshmikantham [1]) Let $(X,\preccurlyeq)$ be a partially ordered set and $F:X\times X\to X$. We say that F has the *mixed monotone property* if $F(x,y)$ is monotone nondecreasing in x and is monotone nonincreasing in y, that is, for any $x,y\in X$,

$$x_1,x_2\in X,\quad x_1\preccurlyeq x_2\quad\Rightarrow\quad F(x_1,y)\preccurlyeq F(x_2,y),$$
$$y_1,y_2\in X,\quad y_1\preccurlyeq y_2\quad\Rightarrow\quad F(x,y_1)\succcurlyeq F(x,y_2).$$

On the other hand, Samet and Vetro [26] succeeded in proving some results in which the mapping F did not necessarily have the mixed monotone property.

Definition 1.8 (Samet and Vetro [26]) Let (X,d) be a metric space and $F:X\times X\to X$ be a given mapping. Let M be a nonempty subset of X^4. We say that M is an *F-invariant subset of X^4* if, for all $x,y,z,w\in X$,

(i) $(x,y,z,w)\in M\Longleftrightarrow(w,z,y,x)\in M$;

(ii) $(x,y,z,w)\in M\Longrightarrow(F(x,y),F(y,x),F(z,w),F(w,z))\in M$.

The following theorem is the main result in [26].

Theorem 1.4 (Samet and Vetro [26]) *Let (X,d) be a complete metric space, $F:X\times X\to X$ be a continuous mapping and M be a nonempty subset of X^4. We assume that*

(i) *M is F-invariant;*

(ii) *there exists $(x_0,y_0)\in X^2$ such that $(F(x_0,y_0),F(y_0,x_0),x_0,y_0)\in M$;*

(iii) *for all* $(x,y,u,v) \in M$, *we have*

$$\begin{aligned} &d\bigl(F(x,y),F(u,v)\bigr) \\ &\quad\leq \frac{\alpha}{2}\bigl[d\bigl(x,F(x,y)\bigr)+d\bigl(y,F(y,x)\bigr)\bigr] \\ &\qquad+\frac{\beta}{2}\bigl[d\bigl(u,F(u,v)\bigr)+d\bigl(v,F(v,u)\bigr)\bigr]+\frac{\theta}{2}\bigl[d\bigl(x,F(u,v)\bigr)+d\bigl(y,F(v,u)\bigr)\bigr] \\ &\qquad+\frac{\gamma}{2}\bigl[d\bigl(u,F(x,y)\bigr)+d\bigl(v,F(y,x)\bigr)\bigr]+\frac{\delta}{2}\bigl[d(x,u)+d(y,v)\bigr], \end{aligned}$$

where α, β, θ, γ, δ *are nonnegative constants such that* $\alpha+\beta+\theta+\gamma+\delta<1$.

Then F has a coupled fixed point, i.e., there exists $(x,y) \in X \times X$ *such that* $F(x,y)=x$ *and* $F(y,x)=y$.

Furthermore, Sintunavarat *et al.* [27] introduced the notion of *transitive property* to reconsider the Lakshmikantham and Ćirić's theorem (see [17]) without the mixed monotone property.

Definition 1.9 (Sintunavarat *et al.* [27]) Let (X,d) be a metric space and M be a subset of X^4. We say that M satisfies the *transitive property* if, for all $x,y,z,w,a,b \in X$,

$$(x,y,z,w) \in M \quad \text{and} \quad (z,w,a,b) \in M \quad \Longrightarrow \quad (x,y,a,b) \in M.$$

Then they proved the following result.

Theorem 1.5 (Sintunavarat *et al.* [27]) *Let* (X,d) *be a complete metric space and M be a nonempty subset of* X^4. *Assume that there is a function* $\varphi:[0,\infty)\to[0,\infty)$ *with* $0=\varphi(0)<\varphi(t)<t$ *and* $\lim_{r\to t^+}\varphi(r)<t$ *for each* $t>0$, *and also suppose that* $F: X\times X\to X$ *is a mapping such that*

$$d\bigl(F(x,y),F(u,v)\bigr) \leq \varphi\left(\frac{d(x,u)+d(y,v)}{2}\right) \tag{2}$$

for all $(x,y,u,v) \in M$. *Suppose that either*

(a) *F is continuous or*

(b) *if for any two sequences* $\{x_m\}$, $\{y_m\}$ *with* $(x_{m+1},y_{m+1},x_m,y_m) \in M$,

$$\{x_m\} \to x, \qquad \{y_m\} \to y,$$

for all $m \geq 1$, *then* $(x,y,x_m,y_m) \in M$ *for all* $m \geq 1$.

If there exists $(x_0,y_0) \in X \times X$ *such that* $(F(x_0,y_0),F(y_0,x_0),x_0,y_0) \in M$ *and M is an F-invariant set which satisfies the transitive property, then there exist* $x,y \in X$ *such that* $x=F(x,y)$ *and* $y=F(y,x)$, *that is, F has a coupled fixed point.*

Recently, Charoensawan [11], based on Batra and Vashistha's results, introduced the tripled case as follows.

Definition 1.10 (Charoensawan [11]) Let $(X,\preccurlyeq)$ be a metric space and $F:X^3\to X$ be a given mapping. Let M be a nonempty subset of X^6. We say that M is an *F-invariant subset of X^6* if and only if, for all $x,y,z,u,v,w\in X$,

$$\begin{aligned}&(x,y,z,u,v,w)\in M\\&\quad\Longrightarrow\quad\bigl(F(x,y,z),F(y,x,y),F(z,y,x),F(u,v,w),F(v,u,v),F(w,v,u)\bigr)\in M.\end{aligned}$$

The following concept is an extension of Definition 1.9.

Definition 1.11 (Charoensawan [11]) Let $(X,\preccurlyeq)$ be a metric space and M be a subset of X^6. We say that M satisfies the *transitive property* if and only if, for all $x,y,z,u,v,w,a,b,c\in X$,

$$(x,y,z,u,v,w)\in M\quad\text{and}\quad(u,v,w,a,b,c)\in M\quad\Longrightarrow\quad(x,y,z,a,b,c)\in M.$$

Definition 1.12 (Charoensawan [11]) Let $(X,\preccurlyeq)$ be a metric space and $F:X^3\to X$, $g:X\to X$ be given mappings. Let M be a nonempty subset of X^6. We say that M is an *(F,g)-invariant subset of X^6* if and only if, for all $x,y,z,u,v,w\in X$,

$$\begin{aligned}&(gx,gy,gz,gu,gv,gw)\in M\\&\quad\Longrightarrow\quad\bigl(F(x,y,z),F(y,x,y),F(z,y,x),F(u,v,w),F(v,u,v),F(w,v,u)\bigr)\in M.\end{aligned}$$

In the previous definitions, it is not necessary to consider either a metric or a partial order on X.

Theorem 1.6 (Charoensawan [11], Theorem 3.7) *Let $(X,\preccurlyeq)$ be a complete metric space and M be a nonempty subset of X^6. Assume that there is a function $\varphi:[0,+\infty)\to[0,+\infty)$ with $0=\varphi(0)<\varphi(t)<t$ and $\lim_{r\to t^+}\varphi(r)<t$ for each $t>0$, and also suppose that $F:X^3\to X$ and $g:X\to X$ are two continuous functions such that*

$$\begin{aligned}&d\bigl(F(x,y,z),F(u,v,w)\bigr)+d\bigl(F(y,x,y),F(v,u,v)\bigr)+d\bigl(F(z,y,x),F(w,v,u)\bigr)\\&\quad\le3\varphi\biggl(\frac{d(gx,gu)+d(gy,gv)+d(gz,gw)}{3}\biggr)\end{aligned}$$

for all $x,y,z,u,v,w\in X$ with $(gx,gy,gz,gu,gv,gw)\in M$ or $(gu,gv,gw,gx,gy,gz)\in M$. Suppose that $F(X^3)\subseteq gX$, g commutes with F.

If there exists $(x_0,y_0,z_0)\in X^3$ such that

$$\bigl(F(x_0,y_0,z_0),F(y_0,x_0,y_0),F(z_0,y_0,x_0),gx_0,gy_0,gz_0\bigr)\in M$$

and M is an (F,g)-invariant set which satisfies the transitive property, then there exist $x,y,z\in X$ such that

$$gx=F(x,y,z),\qquad gy=F(y,x,y)\quad\textit{and}\quad gz=F(z,y,x).$$

Meanwhile, Kutbi *et al.* [16] used a bidimensional extension of an F-invariant subset as follows.

Definition 1.13 (Kutbi *et al.* [16]) We say that M is an *F-closed subset of X^4* if, for all $x, y, u, v \in X$,

$$(x, y, u, v) \in M \quad \Longrightarrow \quad \bigl(F(x, y), F(y, x), F(u, v), F(v, u)\bigr) \in M.$$

The following one is the main result of Kutbi *et al.* [16].

Theorem 1.7 (Kutbi *et al.* [16]) *Let (X, d) be a complete metric space, let $F : X \times X \to X$ be a continuous mapping, and let M be a subset of X^4. Assume that*:

(i) *M is F-closed*;
(ii) *there exists $(x_0, y_0) \in X^2$ such that $(F(x_0, y_0), F(y_0, x_0), x_0, y_0) \in M$*;
(iii) *there exists $k \in [0, 1)$ such that for all $(x, y, u, v) \in M$, we have*

$$d\bigl(F(x, y), F(u, v)\bigr) + d\bigl(F(y, x), F(v, u)\bigr) \le k\bigl(d(x, u) + d(y, v)\bigr).$$

Then F has a coupled fixed point.

2 Main results

In this section we shall indicate our main result. Before stating the main theorem, we give necessary remarks. First of all, we consider the following family:

$$\Phi' = \Bigl\{\varphi : [0, \infty) \to [0, \infty) : 0 = \varphi(0) < \varphi(t) < t \text{ and } \lim_{r \to t^+} \varphi(r) < t \text{ for each } t > 0\Bigr\}.$$

Notice that this family of control functions was employed by Sintunavarat *et al.* in Theorem 1.5 and by Charoensawan in Theorem 1.6. Here, we should mention that it is not as general as Wang's family Φ since the value $\varphi(0)$ is not necessarily determined if $\varphi \in \Phi$. Thus, we have $\Phi' \subset \Phi$ in this sense.

Secondly, we pay attention to the following fact: Charoensawan's notion of *F-invariant set* is similar to Kutbi *et al.*'s notion of *F-closed set*, but it is different from Samet and Vetro's original concept because property (i) in Definition 1.8 is not imposed. Then, coherently with Definition 1.13, we prefer calling these subsets employing the term *F-closed*.

Definition 2.1 Let $T, g : X \to X$ be two mappings and let $M \subseteq X^2$ be a subset. We will say that M is:

- *(T, g)-closed* if $(Tx, Ty) \in M$ for all $x, y \in X$ such that $(gx, gy) \in M$;
- *(T, g)-compatible* if $Tx = Ty$ for all $x, y \in X$ such that $gx = gy$.

Definition 2.2 ([29]) We will say that a subset $M \subseteq X^2$ is *transitive* if $(x, y), (y, z) \in M$ implies that $(x, z) \in M$.

Definition 2.3 ([29]) Let (X, d) be a metric space and let $M \subseteq X^2$ be a subset. We will say that (X, d, M) is *regular* if for all sequence $\{x_m\} \subseteq X$ such that $\{x_m\} \to x$ and $(x_m, x_{m+1}) \in M$ for all m, we have that $(x_m, x) \in M$ for all m.

Definition 2.4 Let (X, d) be a metric space and let $M \subseteq X^2$ be a subset. Two mappings $T, g : X \to X$ are said to be *(O, M)-compatible* if

$$\lim_{m \to \infty} d(gTx_m, Tgx_m) = 0$$

provided that $\{x_m\}$ is a sequence in X such that $(gx_m, gx_{m+1}) \in M$ for all $m \geq 0$ and

$$\lim_{m\to\infty} Tx_m = \lim_{m\to\infty} gx_m \in X.$$

Remark 2.1 If T and g are commuting, then they are also (O,M)-compatible, whatever M.

The main result in [29], using the previous notions, is the following one.

Theorem 2.1 (Karapınar *et al.* [29]) *Let (X,d) be a complete metric space, let $T,g : X \to X$ be two mappings such that $TX \subseteq gX$, and let $M \subseteq X^2$ be a (T,g)-compatible, (T,g)-closed, transitive subset. Assume that there exists $\varphi \in \Phi$ such that*

$$d(Tx, Ty) \leq \varphi\big(d(gx, gy)\big) \quad \text{for all } x, y \in X \text{ such that } (gx, gy) \in M.$$

Also assume that, at least, one of the following conditions holds:
(a) *T and g are M-continuous and (O,M)-compatible*;
(b) *T and g are continuous and commuting*;
(c) *(X,d,M) is regular and gX is closed.*

If there exists a point $x_0 \in X$ such that $(gx_0, Tx_0) \in M$, then T and g have, at least, a coincidence point.

The following one is the main result of [6].

Theorem 2.2 (Charoensawan and Thangthong [6], Theorem 3.1) *Let $(X, \preccurlyeq)$ be a partially ordered set and M be a nonempty subset of X^6, and let d be a metric on X such that (X,d) is a complete metric space. Assume that $F, G : X \times X \times X \to X$ are two generalized compatible mappings such that G is continuous, and for any $x, y, z \in X$, there exist $u, v, w \in X$ such that $F(x,y,z) = G(u,v,w)$, $F(y,z,x) = G(v,w,u)$, and $F(z,x,y) = G(w,u,v)$. Suppose that there exists $\varphi \in \Phi$ such that the following holds*:

$$\begin{aligned} &d\big(F(x,y,z), F(u,v,w)\big) + d\big(F(y,z,x), F(v,w,u)\big) + d\big(F(z,x,y), F(w,u,v)\big) \\ &\quad \leq \varphi\Big(d\big(G(x,y,z), G(u,v,w)\big) + d\big(G(y,z,x), G(v,w,u)\big) + d\big(G(z,x,y), G(w,u,v)\big)\Big) \end{aligned} \tag{3}$$

for all $x, y, z, u, v, w \in X$ with

$$\big(G(x,y,z), G(y,z,x), G(z,x,y), G(u,v,w), G(v,w,u), G(w,u,v)\big) \in M.$$

Also suppose that either
(a) *F is continuous or*
(b) *for any three sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ with*

$$\begin{aligned} &\big(G(x_n,y_n,z_n), G(y_n,z_n,x_n), G(z_n,x_n,y_n), \\ &\quad G(x_{n+1},y_{n+1},z_{n+1}), G(y_{n+1},z_{n+1},x_{n+1}), G(z_{n+1},x_{n+1},y_{n+1})\big) \in M \end{aligned}$$

and

$$\big\{G(x_n,y_n,z_n)\big\} \to x, \qquad \big\{G(y_n,z_n,x_n)\big\} \to y, \qquad \big\{G(z_n,x_n,y_n)\big\} \to z,$$

for all $n \geq 1$ *implies*

$$\big(G(x_n, y_n, z_n), G(y_n, z_n, x_n), G(z_n, x_n, y_n), x, y, z\big) \in M \quad \textit{for all } n \geq 1.$$

If there exist $x_0, y_0, z_0 \in X \times X$ *such that*

$$\big(G(x_0, y_0, z_0), G(y_0, z_0, x_0), G(z_0, x_0, y_0), F(x_0, y_0, z_0), F(y_0, z_0, x_0), F(z_0, x_0, y_0)\big) \in M$$

and M *is an* (G,F)*-closed, then there exist* $(x,y,z) \in X \times X \times X$ *such that* $G(x,y,z) = F(x,y,z)$, $G(y,z,x) = F(y,z,x)$, *and* $G(z,x,y) = F(z,x,y)$, *that is,* F *and* G *have a tripled point of coincidence.*

The following remarks must be done in order to clarify some facts stated in [6] to the reader.

- In the previous theorem, the authors assumed that $(X, \preccurlyeq)$ is a partially ordered set. Clearly, it is a superfluous hypothesis.
- We understand that '$x_0, y_0, z_0 \in X \times X$' is an erratum and that it must be replaced by '$x_0, y_0, z_0 \in X$'.
- In [6], Example 3.2 is invalid since $G(x,y,z) = x + y + z$ does not necessarily belong to $X = [0,1]$ when $x, y, z \in X$ are arbitrary.

Let $Y = X \times X \times X$. It is easy to show that the mappings $\eta, \delta : Y \times Y \to [0,\infty)$, defined by

$$\eta\big((x,y,z),(u,v,w)\big) = d(x,u) + d(y,v) + d(z,w) \quad \text{and}$$
$$\delta\big((x,y,z),(u,v,w)\big) = \max\big\{d(x,u), d(y,v), d(z,w)\big\}$$

for all $(x,y,z),(u,v,w) \in Y$, are metrics on Y.

Now, given a mapping $F : X \times X \times X \to X$, let us define the mapping $T_F : Y \to Y$ by

$$T_F(x,y,z) = \big(F(x,y,z), F(y,z,x), F(z,x,y)\big) \quad \text{for all } (x,y,z) \in Y.$$

It is simple to show the following properties.

Lemma 2.1 (see, *e.g.*, [7]) *The following properties hold*:

(1) (X,d) *is complete if and only if* (Y,η) (*and* (Y,δ)) *is complete*;

(2) F *has the mixed monotone property if and only if* T_F *is monotone nondecreasing with respect to* $\preceq$;

(3) $(x,y,z) \in X \times X \times X$ *is a tripled fixed point of* F *if and only if* (x,y,z) *is a fixed point of* T_F.

(4) $(x,y,z) \in X \times X \times X$ *is a tripled coincidence point of* F *and* G *if and only if* (x,y,z) *is a coincidence point of* T_F *and* T_G.

(5) $(x,y,z) \in X \times X \times X$ *is a tripled common fixed point of* F *and* G *if and only if* (x,y,z) *is a common fixed point of* T_F *and* T_G.

As a consequence of the previous facts, next we show that Theorem 2.2 is not a true extension: indeed, it can be seen as a simple corollary of Theorem 2.1.

Theorem 2.3 *Theorem* 2.2 *follows from Theorem* 2.1.

Proof Notice that condition (3) is equivalent to

$$\eta\bigl(T_F(x,y,z),T_F(u,v,w)\bigr)\leq\varphi\bigl(\eta\bigl(T_G(x,y,z),T_G(u,v,w)\bigr)\bigr)$$

for all $(x,y,z),(u,v,w)\in Y$ such that $(T_G(x,y,z),T_G(u,v,w))\in M$ (notice that $M\subseteq X^6=Y^2$). By Lemma 2.1, all conditions of Theorem 2.1 are satisfied. □

3 Final remarks

In this section, we underline that the common/coincidence point theorem in [6] can be concluded as a fixed point theorem. For this purpose, we first recall the following crucial lemma.

Lemma 3.1 ([30]) *Let X be a nonempty set and $T:X\to X$ be a function. Then there exists a subset $E\subseteq X$ such that $T(E)=T(X)$ and $T:E\to X$ is one-to-one.*

Theorem 3.1 *Let (X,d) be a complete metric space, let $T:X\to X$ be a mapping, and let $M\subseteq X^2$ be a T-closed, transitive subset. Assume that there exists $\varphi\in\Phi$ such that*

$$d(Tx,Ty)\leq\varphi\bigl(d(x,y)\bigr)\quad\textit{for all } x,y\in X \textit{ such that } (x,y)\in M.$$

Assume that either

(a) *T is continuous, or*

(b) *(X,d,M) is regular.*

If there exists a point $x_0\in X$ such that $(x_0,Tx_0)\in M$, then T and g have, at least, a fixed point.

We skip the proof of this theorem since it can be considered as a special case of Theorem 2.1. Indeed, if we take g as the identity map on X, we conclude the result. On the other hand, by the following lemma, we shall show that Theorem 2.1 can be derived from Theorem 3.1.

Theorem 3.2 *Theorem* 2.1 *is a consequence of Theorem* 3.1.

Proof By Lemma 3.1, there exists $E\subseteq X$ such that $g(E)=g(X)$ and $g:E\to X$ is one-to-one. Define a map $h:g(E)\to g(E)$ by $h(gx)=T(x)$. Since g is one-to-one on $g(E)$, we conclude that h is well defined. Note that

$$d(Tx,Ty)=d\bigl(h(gx),h(gy)\bigr)\leq\varphi\bigl(d(x,y)\bigr)\tag{4}$$

for all $gx,gy\in g(E)$. Since $g(E)=g(X)$ is complete, by using Theorem 3.1, there exists $x_0\in X$ such that $h(gx_0)=gx_0$. Hence, T and g have a point of coincidence. It is clear that T and g have a unique common fixed point whenever T and g are weakly compatible. □

From Theorem 2.3 and Theorem 3.2 we conclude the following result.

Theorem 3.3 *Theorem* 2.2 *is a consequence of Theorem* 3.1.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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