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Unicity of meromorphic functions and their linear differential polynomials

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Abstract

This paper studies the unicity of meromorphic functions that share an arbitrary nonzero small function with their general linear differential polynomials. The result derived here extends one by Brück in 1996 and others.

MSC: 30D35; 30D30

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1 Introduction

In 1929, R Nevanlinna [1–3] proved his celebrated *five-value* and *four-value* theorems, which gave birth to the study of the unicity of meromorphic functions in the open complex plane $\mathbb{C}$; in 1977, by considering the unicity of an entire function f and its first order derivative f', Rubel and Yang [4] showed that $f \equiv f'$ when they share two distinct, finite values *CM*, which was generalized to meromorphic functions independently by Gundersen [5] as well as by Mues and Steinmetz [6] in 1983. Since these results, there have been many papers on several related problems about f and its derivatives, and we refer the reader to [3, 7–13] and the references therein for more details.

Before we proceed, we spare the reader for a moment and assume the familiarity with the basics of Nevanlinna's value distribution theory of meromorphic functions in $\mathbb{C}$ such as the *first* and *second* main theorems, and the common notations such as the *characteristic function* $T(r,f)$, the *proximity function* $m(r,f)$ and the *counting functions* $N(r,f)$ (with multiplicities) and $\bar{N}(r,f)$ (without multiplicities); $S(r,f)$ denotes any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ except possibly on a set of finite Lebesgue measure, not necessarily the same at each occurrence.

Let a, f, g be some meromorphic functions on $\mathbb{C}$. a is said to be a small function to f, provided $T(r,a) = S(r,f)$. Given a, a small function to both f and g or some value in $\mathbb{C} \cup \{\infty\}$, one says that f and g share a *CM* provided the two meromorphic functions $f - a$ and $g - a$ have the same zeros with counting multiplicities.

Now, we continue our discussion and observe a question that arises naturally: When only f and f' share one value *CM*, then what can one have accordingly? Towards this direction, in 1996, Brück [14] first proved the following result.

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Theorem A *Let f be a nonconstant entire function with $N(r, \frac{1}{f'}) = S(r, f)$. When f and f′ share a finite, nonzero value a CM, then there is a constant $c \neq 0$ such that*

$$\frac{f - a}{f' - a} = c. \tag{1}$$

This interesting result was extended by Zhang [15] in 1998, when he proved, among some related results, the following one.

Theorem B *Let f be a nonconstant meromorphic function such that*

$$\bar{N}(r, f) + N\left(r, \frac{1}{f'}\right) < (\lambda + o(1)) T(r, f') \quad \text{for } \lambda \in \left(0, \frac{1}{2}\right). \tag{2}$$

When f and f′ share a finite, nonzero value a CM, then we have (1).

Quite recently, via replacing the finite, nonzero value a by a small function $a(z)$ related to f, Al-Khaladi [16, 17] obtained the following striking result.

Theorem C *Let f be a nonconstant meromorphic function such that*

$$\bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f'}\right) < (\lambda + o(1)) T(r, f') \quad \text{for } \lambda \in \left(0, \frac{1}{2}\right). \tag{3}$$

When f and f′ share a small function $a(z)$ ($\not\equiv 0, \infty$) CM, then we have $\frac{f-a}{f'-a} = 1 - \frac{c}{a}$, where c is a constant such that $\frac{c}{a} \neq 1$ and $N(r, 1 - \frac{c}{a}) = S(r, f)$.

Remark D There are reasons why the condition on the zeros of f' is posed by Brück [14]: To control the multiple values of f for each finite value, inspired by the exponential function; yet, it is not natural to generalize this condition by the zeros of $f^{(k)}$ when $k \geq 2$ as there is no need to control the multiple values of f'.

Next, we define a linear differential polynomial of order $k \geq 1$, such as

$$\mathcal{L}(f) := a_1 f' + a_2 f'' + \cdots + a_k f^{(k)} \quad (a_k \not\equiv 0), \tag{4}$$

associated with f, where a_j are small functions of f for $j = 1, 2, \ldots, k$. Then we can prove the following theorems, which are the main results of this paper.

Theorem 1 *Let f be a nonconstant meromorphic function satisfying* (3) *with $\lambda \in (0, \frac{1}{k+1})$. When f and $\mathcal{L}(f)$ share a small function $a(z)$ ($\not\equiv 0, \infty$) CM, then we have $\frac{f-a}{\mathcal{L}(f)-a} = 1 - \frac{c}{a}$ for a constant c with $\frac{c}{a} \neq 1$ and $N(r, 1 - \frac{c}{a}) = S(r, f)$.*

Theorem 2 *Under the assumptions of Theorem* 1, *we further suppose that f, $a(z)$ ($\not\equiv 0, \infty$) and a_j, $j = 1, 2, \ldots, k$, are all entire, then we have either $f = \mathcal{L}(f)$, or $\frac{f-a}{\mathcal{L}(f)-a} = c$ for a constant $c \neq 0$ and $a(z)$ must also be a constant as well.*

As suggested by one of the referees, the reader may also feel interested in a very recent paper by Al-Khaladi [18], where in fact some results of Lahiri and Sarkar [19] were generalized that in turn was deeply related to a paper of Yu [20]. One notices the paper [19]

was not suitably cited in the key reference of [18] (not the paper [18] itself), despite the conclusions as well as the methods being very similar.

It is worth to mention that the machinery used in this paper is standard Nevanlinna's value distribution theory while the methods involved are combinations of those already applied in Al-Khaladi [16, 17] as well as in Han and Yi [8].

2 Proofs of the main results

This section is devoted to the detailed proofs of Theorems 1 and 2. We first define the following auxiliary function α as

$$\alpha := \frac{f-a}{\mathcal{L}-a}. \tag{5}$$

Here and hereafter, we use $\mathcal{L}$ for $\mathcal{L}(f)$ for brevity. Our hypotheses imply α is such a meromorphic function that $T(r,\alpha) = O(T(r,f))$ yet $N(r,\alpha) = S(r,f)$.

Case I. a is a finite, nonzero constant.

Rewrite (5) as $f - a = \alpha\mathcal{L} - a\alpha$ and differentiate it to yield

$$f' = (\alpha\mathcal{L})' - a\alpha' = \alpha(\beta f' - \gamma), \tag{6}$$

where we introduced two meromorphic functions $\beta := \frac{(\alpha\mathcal{L})'}{\alpha f'}$ and $\gamma := a\frac{\alpha'}{\alpha}$.

Notice that $\alpha \not\equiv 0$. We get easily

$$\begin{cases} \frac{1}{\alpha} = \beta - \gamma\frac{1}{f'}, \\ -\frac{\alpha'}{\alpha^2} = \beta' + \gamma\frac{f''}{(f')^2} - \gamma'\frac{1}{f'}. \end{cases} \tag{7}$$

Substitute $\frac{1}{\alpha}$ from the top line to the bottom line in (7) to derive

$$\left(\gamma\frac{1}{f'} - \beta\right)\frac{\alpha'}{\alpha} = \beta' + \gamma\frac{f''}{(f')^2} - \gamma'\frac{1}{f'}. \tag{8}$$

That is, for the meromorphic function $\delta := \gamma(\frac{\alpha'}{\alpha} + \frac{\gamma'}{\gamma} - \frac{f''}{f'})$, one has

$$\delta\frac{1}{f'} = \beta' + \beta\frac{\alpha'}{\alpha}. \tag{9}$$

Now, when $\gamma \equiv 0$, then we see α is a constant, so that $\frac{f-a}{\mathcal{L}(f)-a} = c$ for a constant $c = \alpha \neq 0$. As a consequence, we assume in the following that $\gamma \not\equiv 0$.

Next, if $\delta \equiv 0$, then $\frac{\alpha'}{\alpha} + \frac{\gamma'}{\gamma} - \frac{f''}{f'} \equiv 0$; using $\alpha\gamma = a\alpha'$, one derives

$$\alpha\gamma = a\alpha' = c_1 f'. \tag{10}$$

Here, $c_1 \neq 0$ is a constant. Hence, it follows that $N(r,f) = O(N(r,\alpha)) = S(r,f)$, which further yields $N(r,\frac{1}{\alpha}) \leq k\bar{N}(r,f) + S(r,f) = S(r,f)$. On the other hand, combining (6) and (10) provides us with the following identity:

$$(1 + c_1)f' = (\alpha\mathcal{L})'. \tag{11}$$

If $c_1 = -1$, then $\alpha\mathcal{L} = c_2$ for a constant $c_2 \neq 0$; yet, since now $f' = -a\alpha'$, we have $\mathcal{L} = -a\mathcal{L}(\alpha)$ via an easy computation, so that $-a\alpha\mathcal{L}(\alpha) = c_2$. Rewrite it as

$$\frac{c_2}{\alpha^2} = -a\frac{\mathcal{L}(\alpha)}{\alpha}. \tag{12}$$

This then implies that $m(r, \frac{1}{\alpha}) \leq \frac{1}{2}m(r, \frac{\mathcal{L}(\alpha)}{\alpha}) + S(r,f) = S(r,f)$. Consequently, using the first main theorem, one has $T(r,\alpha) = S(r,f)$ and thus $T(r,f) = S(r,f)$ from (10). This is a contradiction. If $c_1 \neq -1$, then from (11) one observes

$$\frac{1 + c_1}{\alpha} = \frac{(\alpha\mathcal{L})'}{\alpha f'} = \beta. \tag{13}$$

An easy calculation says $m(r, \frac{1}{\alpha}) \leq m(r, \beta) + S(r,f) = S(r,f)$, so that $T(r,\alpha) = S(r,f)$ and thus $T(r,f) = S(r,f)$ by (10). This is a contradiction again.

All these foregoing discussions imply that $\delta \not\equiv 0$. So, from (9), we have

$$\begin{aligned}
m\left(r, \frac{1}{f'}\right) &\leq m\left(r, \frac{\alpha'}{\alpha}\right) + m(r, \beta) + m\left(r, \frac{\beta'}{\beta}\right) + m\left(r, \frac{1}{\delta}\right) \\
&= m\left(r, \frac{1}{\delta}\right) + S(r,f) = T(r,\delta) - N\left(r, \frac{1}{\delta}\right) + S(r,f) \\
&= m(r, \delta) + N(r, \delta) - N\left(r, \frac{1}{\delta}\right) + S(r,f).
\end{aligned}$$

Keeping in mind $m(r, \gamma) = S(r,f)$, we have $m(r, \delta) = S(r,f)$ and thus

$$m\left(r, \frac{1}{f'}\right) + N\left(r, \frac{1}{\delta}\right) = N(r, \delta) + S(r,f). \tag{14}$$

Reset $\delta = a(\frac{\alpha'}{\alpha})^2 + a(\frac{\alpha'}{\alpha})' - a\frac{\alpha'}{\alpha}\frac{f''}{f'}$ and note that the zeros of α come from the poles of f. When $f'(z_0) = 0$ with multiplicity $p \geq k + 1$, then $\alpha(z_0) \neq 0$ and $\mathcal{L}(f)(z_0) = (\mathcal{L}(f))'(z_0) = 0$ with multiplicities at least $p - k + 1$ and $p - k$, respectively; using (6), one observes $\alpha'(z_0) = 0$ with multiplicity at least $p - k$ since $f' - (\alpha\mathcal{L})' = a\alpha'$; thus, recalling $\gamma = a\frac{\alpha'}{\alpha}$, we know $\gamma(z_0) = 0$ with multiplicity at least $p - k$. Thereby, $\delta(z_0) = 0$ with multiplicity at least $p - k - 1$. As a result, it follows that

$$N_{(k+1}\left(r, \frac{1}{f'}\right) \leq N\left(r, \frac{1}{\delta}\right) + (k + 1)\bar{N}_{(k+1}\left(r, \frac{1}{f'}\right) + S(r,f). \tag{15}$$

When $\delta(z_1) = \infty$, then either $\alpha(z_1) = 0$ (from $f(z_1) = \infty$ in fact) or $f(z_1) = \infty$ with multiplicity 2 of $\delta(z_1) = \infty$, or $f'(z_1) = 0$ with multiplicity 1 of $\delta(z_1) = \infty$. In view of (15) for the case where $f'(z_1) = 0$, we can thus conclude that

$$N(r, \delta) \leq 2\bar{N}(r,f) + \bar{N}_{k)}\left(r, \frac{1}{f'}\right) + S(r,f). \tag{16}$$

Here, $N_{k)}(r, \frac{1}{f'})$, $\bar{N}_{k)}(r, \frac{1}{f'})$, and $N_{(k+1}(r, \frac{1}{f'})$, $\bar{N}_{(k+1}(r, \frac{1}{f'})$ are the counting functions of the zeros of f' with multiplicities $\leq k$ and $\geq k + 1$, respectively.

Altogether with (14), (15), and (16), we obtain

$$
\begin{aligned}
T(r,f') &= m\left(r,\frac{1}{f'}\right) + N_{k)}\left(r,\frac{1}{f'}\right) + N_{(k+1}\left(r,\frac{1}{f'}\right) \\
&= N(r,\delta) + N_{k)}\left(r,\frac{1}{f'}\right) + (k+1)\bar{N}_{(k+1}\left(r,\frac{1}{f'}\right) + S(r,f) \\
&\leq 2\bar{N}(r,f) + \bar{N}_{k)}\left(r,\frac{1}{f'}\right) + N_{k)}\left(r,\frac{1}{f'}\right) + (k+1)\bar{N}_{(k+1}\left(r,\frac{1}{f'}\right) + S(r,f),
\end{aligned}
$$

which further implies that, as $k \geq 1$ and $N_{k)}(r,\frac{1}{f'}) \leq k\bar{N}_{k)}(r,\frac{1}{f'}) + S(r,f)$,

$$
T(r,f') \leq (k+1)\left\{\bar{N}(r,f) + \bar{N}\left(r,\frac{1}{f'}\right)\right\} + S(r,f). \tag{17}
$$

This obviously contradicts our assumption. As a result, when a is a finite, nonzero constant, it follows that $\frac{f-a}{\mathcal{L}(f)-a} = c$ with a constant $c \neq 0$.

Case II. a is a nonconstant, small function.

Like before, rewrite (5) as $f - a = \alpha\mathcal{L} - a\alpha$ and differentiate it to yield

$$
f' - a' = (\alpha\mathcal{L})' - (\alpha a)'. \tag{18}
$$

When $(f' - a')(z_2) = 0$, then we have $\{(\alpha\mathcal{L})' - (\alpha a)'\}(z_2) = 0$. Recall that the zeros of α come from the poles of f. As a consequence, one deduces that

$$
H(z_2) := \left\{\frac{(\alpha\mathcal{L})'}{\alpha f'} - \frac{(a\alpha)'}{\alpha a'}\right\}(z_2) = 0. \tag{19}
$$

If $H \not\equiv 0$, then we have, noticing that $m(r,H) = S(r,f)$,

$$
\begin{aligned}
\bar{N}\left(r,\frac{1}{f'-a'}\right) &\leq N\left(r,\frac{1}{H}\right) + S(r,f) \\
&= T(r,H) + S(r,f) = N(r,H) + S(r,f).
\end{aligned}
$$

Note when $H(z_3) = \infty$, then we have either $\alpha(z_3) = 0$ (from $f(z_3) = \infty$ actually) or $f(z_3) = \infty$ - with multiplicity k of $H(z_3) = \infty$, or $f'(z_3) = 0$ - with multiplicity $k-1$ of $H(z_3) = \infty$. Thus, we can derive that, similar to (16),

$$
N(r,H) \leq k\left\{\bar{N}(r,f) + \bar{N}\left(r,\frac{1}{f'}\right)\right\} + S(r,f). \tag{20}
$$

Apply the second main theorem for three small functions to f' to yield

$$
T(r,f') \leq \bar{N}\left(r,\frac{1}{f'}\right) + \bar{N}(r,f') + \bar{N}\left(r,\frac{1}{f'-a'}\right) + S(r,f), \tag{21}
$$

which together with (20) yields (17) again. This gives a contradiction.

Thus, $H \equiv 0$. That is, $(\alpha\mathcal{L})' \equiv \frac{f'}{a}(a\alpha)'$. Using (18), one derives

$$
a'(f' - a') \equiv (a\alpha)'(f' - a'), \tag{22}
$$

so that $a' \equiv (a\alpha)'$; in other words, $\alpha = 1 - \frac{c}{a}$ for a constant c with $\frac{c}{a} \neq 1$. We thus conclude that $\frac{f-a}{\mathcal{L}(f)-a} = 1 - \frac{c}{a}$ and $N(r, 1 - \frac{c}{a}) = N(r, \alpha) = S(r, f)$.

Finally, let us prove the associated Theorem 2. When f, a, and a_j ($j = 1, 2, \ldots, k$) are all entire functions, we have $\alpha(z) = e^{\omega(z)}$ for some entire function $\omega(z)$. If a is constant, then we have the same result as in Case I: $\frac{f-a}{\mathcal{L}(f)-a} = c$ with a constant $c \neq 0$; if a is nonconstant, then $\frac{c}{a} = 1 - e^{\omega(z)}$ admits many zeros, so that a cannot be entire unless $c = 0$, which provides us with the case $f = \mathcal{L}(f)$.

Competing interests

Authors' contributions

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Acknowledgements

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