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# Well-posedness for parametric generalized vector quasivariational inequality problems of the Minty type

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## Abstract

In this paper, we introduce the concepts of well-posedness, and of well-posedness in the generalized sense for parametric generalized vector quasivariational inequality problems of the Minty type. The necessary and sufficient conditions for the various kinds of well-posedness of these problems are obtained. Our results are different from some main results in the literature and extend them.

**MSC:** 90C31; 49J53; 49J40; 49J45

**Keywords:** parametric generalized vector quasivariational inequality problems of the Minty type; well-posed; well-posedness in the generalized sense; upper semicontinuity

## 1 Introduction and preliminaries

A vector variational inequality in a finite-dimensional Euclidean space was introduced first by Giannessi [1]. Later, this problem has been extended and studied by many authors in abstract spaces; see [2–6]. Moreover, vector variational inequality problems have many important applications in vector optimization problems [7–9], vector equilibria problems [10, 11], and variational relation problems [12, 13].

The concept of well-posedness for unconstrained scalar optimization problems was first introduced and studied by Tykhonov [8], which has become known as Tykhonov well-posedness. In 1966, Levitin and Polyak [14] introduced the concept of well-posedness for constrained scalar optimization problems. With the development of the theory about optimization problems, the concept of well-posedness has been generalized to several related problems, as vector optimization problems, see [15–20], variational inequality problems, see [15, 21–23], equilibria problems, see [24–33] and the references therein. Recently, Fang and Huang [22] studied the well-posedness for a vector variational inequality of the Minty type and the Stampacchia type. Very recently, Lalitha and Bhatia [23] also studied a quasivariational inequality problem of the Minty type, and the well-posedness for this problem was obtained.

Motivated and inspired by the work mentioned, in this paper, we also study the parametric generalized vector quasivariational inequality problems. However, we only study the well-posedness for generalized vector quasivariational inequality problems of the Minty type. The well-posedness for generalized vector quasivariational inequality problems of the Stampacchia type is the same as the Minty type. Let $X$, $Y$, $\Gamma$, $\Lambda$ be metric spaces and

$C \subset Y$ be a closed, convex, and pointed cone with $\operatorname{int} C \neq \emptyset$. The cone $C$ induces a partial ordering in $Y$ defined by

$$y < x \quad \Leftrightarrow \quad y - x \in -\operatorname{int} C, \quad \forall x, y \in Y,$$

$$y \not< x \quad \Leftrightarrow \quad y - x \notin -\operatorname{int} C, \quad \forall x, y \in Y,$$

where $\operatorname{int} C$ denotes the interior of $C$.

Let $L(X, Y)$ be the space of all linear continuous operators from $X$ into $Y$, and $A \subset X$ be a nonempty subset. Let $K_1 : A \times \Gamma \to 2^A$, $K_2 : A \times \Gamma \to 2^A$, and $T : A \times \Gamma \to 2^{L(X,Y)}$ be set-valued mappings. Let $Q : L(X, Y) \to L(X, Y)$, $\eta : A \times A \times \Lambda \to A$ be continuous single-valued mappings. We denote by $\langle z, x \rangle$ the value of a linear operator $z \in L(X; Y)$ at $x \in X$, and we always assume that $\langle \cdot, \cdot \rangle$ is continuous.

Now we adopt the following notations (see [10, 12, 13]). For subsets $M$ and $N$ under consideration we adopt the notations

$$(u, v) \text{ w } M \times N \quad \text{means } \forall u \in M, \exists v \in N,$$

$$(u, v) \text{ m } M \times N \quad \text{means } \exists v \in N, \forall u \in M,$$

$$(u, v) \text{ s } M \times N \quad \text{means } \forall u \in M, \forall v \in N,$$

$$(u, v) \text{ } \bar{w} \text{ } M \times N \quad \text{means } \exists u \in M, \forall v \in N \text{ and similarly for } \bar{m}, \bar{s}.$$

where w, m, and s are used for weak, middle, and strong, respectively, kinds of considered problems. Let $\alpha \in \{w, m, s\}$, $\bar{\alpha} \in \{\bar{w}, \bar{m}, \bar{s}\}$, and, for $\gamma \in \Gamma$, $\lambda \in \Lambda$. We consider the following parametric generalized vector quasivariational inequality problems of the Minty type (in short: $(\text{MQVIP}^{\gamma\lambda})$).

$(\text{MQVIP}^{\gamma\lambda})$ Find $\bar{x} \in K_1(\bar{x}, \gamma)$ such that $(y, z)\alpha K_2(\bar{x}, \gamma) \times T(y, \gamma)$ satisfies

$$\langle Q(z), \eta(y, \bar{x}, \lambda) \rangle \not< 0.$$

Denote by (MQVIP) the family $\{(\text{MQVIP}^{\gamma\lambda}) : (\gamma, \lambda) \in \Gamma \times \Lambda\}$. For each $\gamma \in \Gamma$, $\lambda \in \Lambda$, and let $E(\gamma) := \{x \in A : x \in K_1(x, \gamma)\}$. We denote by $\Psi_\alpha(\gamma, \lambda)$ the solution sets of $(\text{MQVIP}^{\gamma\lambda})$.

Throughout the article, we assume that $\Psi_\alpha(\gamma, \lambda) \neq \emptyset$ for each $(\gamma, \lambda)$ in the neighborhoods $(\gamma_0, \lambda_0) \in \Gamma \times \Lambda$.

Next, we recall some basic definitions and some of their properties.

**Definition 1.1** ([34, 35]) Let $X$ and $Z$ be two topological vector spaces and let $G : X \to 2^Z$ be a multifunction.

  (i) $G$ is said to be *lower semicontinuous* (*lsc*) *at* $x_0$ if $G(x_0) \cap U \neq \emptyset$ for each open set $U \subseteq Z$ implies the existence of a neighborhood $V$ of $x_0$ such that $G(x) \cap U \neq \emptyset$, $\forall x \in V$.

 (ii) $G$ is said to be *upper semicontinuous* (*usc*) *at* $x_0$ if for each open set $U \supseteq G(x_0)$, there is a neighborhood $V$ of $x_0$ such that $U \supseteq G(x)$, $\forall x \in V$.

(iii) $G$ is said to be *closed at* $x_0$ if for each net $\{(x_n, y_n)\} \in \operatorname{graph} G := \{(x, y) | y \in G(x)\}$, $(x_n, y_n) \to (x_0, y_0)$, it follows that $(x_0, y_0) \in \operatorname{graph} G$.

**Lemma 1.2** ([34, 35]) *Let $X$ and $Z$ be two topological vector spaces and $G : X \to 2^Z$ be a multifunction.*

(i) *If $Z$ is compact and $G$ is closed at $x_0$, then $G$ is usc at $x_0$.*

(ii) *If $G$ is usc at $x_0$ and $G(x_0)$ is closed, then $G$ is closed at $x_0$.*

The structure of this article is as follows. In the remaining part of this section, we recall definitions for later use. In Section 2, we introduce concepts of well-posedness, and well-posedness in the generalized sense for parametric generalized vector quasivariational inequality problems of the Minty type. Moreover, the necessary and sufficient conditions for the various kinds of well-posedness of these problems are obtained.

## 2 Main results

**Definition 2.1** Let $\{(\gamma_n, \lambda_n)\} \subseteq \Gamma \times \Lambda$ converges to $(\gamma_0, \lambda_0)$. A sequence $\{x_n\} \subseteq A$ is said to be an approximating sequence for (MQVIP) corresponding to $\{(\gamma_n, \lambda_n)\}$, if

(i) $x_n \in K_1(x_n, \lambda_n), \forall n$;

(ii) there exists a sequence $\{\varepsilon_n\} \in \operatorname{int} C$ that converges to 0 such that

$$(y, z)\alpha K_2(x_n, \gamma_n) \times T(y, \gamma_n) \quad \text{satisfies} \quad \langle Q(z), \eta(y, x_n, \lambda_n) \rangle + \varepsilon_n \not< 0.$$

**Definition 2.2** The problem (MQVIP) is said to be well-posed at $(\gamma_0, \lambda_0)$ if

(i) the problem (MQVIP) has a unique solution $x_0$, *i.e.*, $\Psi_\alpha(\gamma_0, \lambda_0) = \{x_0\}$;

(ii) for any sequence $\{(\gamma_n, \lambda_n)\} \subseteq \Gamma \times \Lambda$ converges to $(\gamma_0, \lambda_0)$, every approximating sequence $\{x_n\}$ for (MQVIP) corresponding to $\{(\gamma_n, \lambda_n)\}$ converges to $x_0$.

**Definition 2.3** The problem (MQVIP) is said to be well-posed in the generalized sense at $(\gamma_0, \lambda_0)$ if

(i) the solution set $\Psi_\alpha(\gamma_0, \lambda_0)$ of (MQVIP) is nonempty;

(ii) for any sequence $\{(\gamma_n, \lambda_n)\} \subseteq \Gamma \times \Lambda$ that converges to $(\gamma_0, \lambda_0)$, every approximating sequence $\{x_n\}$ for (MQVIP) corresponding to $\{(\gamma_n, \lambda_n)\}$ has a subsequence which converges to some point of $\Psi_\alpha(\gamma_0, \lambda_0)$.

For $\gamma \in \Gamma$, $\lambda \in \Lambda$, and $\varepsilon \in \operatorname{int} C$, we denote the approximate solution set of (MQVIP) by $\Omega(\gamma, \lambda, \varepsilon)$:

$$\Omega(\gamma, \lambda, \varepsilon) := \left\{ x \in K_1(x, \gamma) | (y, z)\alpha K_2(x, \gamma) \times T(y, \gamma) : \langle Q(z), \eta(y, x, \lambda) \rangle + \varepsilon \not< 0 \right\}.$$

**Remark 2.4**

(i) In the special case, where $A = B$, $X = Y$, $\Gamma = \Lambda$, $K_1(x, \gamma) = K_2(x, \gamma) = A$, $\eta(y, x, \lambda) = y - x$, and $Q$ is an identity map, let $T : A \times \Gamma \to L(X, Y)$ be a single-valued mapping, then the problem (MQVIP$^{\gamma\lambda}$) reduces to the problem (MVVI$^\lambda$) studied in [22].

(ii) In the special case as in Remark 2.4(i), then Definitions 2.1, 2.2, and 2.3 reduce to Definitions 2.2, 2.5, and 2.6, respectively, of Fang and Huang in [22].

(iii) Well-posedness for vector problems has been defined in different ways. In this paper, we denote $\varepsilon \in \operatorname{int} C$ instead of $\epsilon e$, with $\epsilon$ being positive numbers and $e \in \operatorname{int} C$, *i.e.*, only a fixed direction $e$ is allowed (see [24, 28]).

**Remark 2.5** ([36]) Let $X$ and $Z$ be two metric spaces and $G : X \to 2^Z$ be a multifunction. If $G(x_0)$ is compact, then $G$ is usc at $x_0$ if and only if for any sequence $\{x_n\}$ that converges to $x_0$ and for any sequence $\{y_n\} \subseteq G(x_n)$, there is a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ converging to some $y_0 \in G(x_0)$. If, in addition, $G(x_0) = \{y_0\}$ is a singleton, then the above limit point $y$ must be $y_0$ and the whole $\{y_n\}$ converges to $y_0$.

The following theorem gives sufficient conditions for the well-posedness and the well-posedness in the generalized sense for (MQVIP).

**Theorem 2.6** *Assume for problem* (MQVIP) *that*

(i) *$E$ is usc at $\gamma_0$ and $E(\gamma_0)$ is a compact set;*

(ii) *in $K_1(A, \Gamma) \times \{\gamma_0\}$, $K_2$ is lsc;*

(iii) *in $K_2(K_1(A, \Gamma), \Gamma) \times \{\gamma_0\}$, $T$ is usc and compact-valued if $\alpha = $ w (or $\alpha = $ m), and lsc if $\alpha = $ s.*

*Then* (MQVIP) *is well-posed in the generalized sense at $(\gamma_0, \lambda_0)$. Moreover, if $\Psi_\alpha(\gamma_0, \lambda_0)$ is a singleton, then this problem is well-posed at $(\gamma_0, \lambda_0)$.*

*Proof* Since $\alpha = \{$w, m, s$\}$, we have in fact three cases. However, the proof techniques are similar. We consider only the case $\alpha = $ s. We first prove that $\Omega_s$ is upper semicontinuous at $(\gamma_0, \lambda_0, 0)$. Indeed, we suppose to the contrary the existence of an open subset $V$ of $\Omega_s(\gamma_0, \lambda_0, 0)$ such that for all $\{(\gamma_n, \lambda_n, \varepsilon_n)\} \subseteq \Gamma \times \Lambda \times C$ it converges to $\{(\gamma_0, \lambda_0, 0)\}$, that is, $x_n \in \Omega_s(\gamma_n, \lambda_n, \varepsilon_n)$, $x_n \notin V$, for all $n$. Since $E$ is usc and is compact-valued at $\gamma_0$, we can assume that $x_n$ tends to $x_0$ for some $x_0 \in E(\gamma_0)$. If $x_0 \notin \Omega_s(\gamma_0, \lambda_0, 0) = \Psi(\gamma_0, \lambda_0)$, $\exists y_0 \in K_2(x_0, \gamma_0)$, $\exists z_0 \in T(y_0, \gamma_0)$ such that

$$\langle Q(z_0), \eta(y_0, x_0, \lambda_0) \rangle < 0.$$

By the lower semicontinuity of $K_2$, $T$ at $(x_0, \gamma_0)$ and $(y_0, \gamma_0)$, $\forall y_0 \in K_2(x_0, \gamma_0)$, $\forall z_0 \in T(y_0, \gamma_0)$ there exists $y_n \in K_2(x_n, \gamma_n)$, $z_n \in T(y_n, \gamma_n)$ such that $y_n \to y_0$, $z_n \to z_0$. Since $x_n \in \Omega(\gamma_n, \lambda_n, \varepsilon_n)$, we have

$$\langle Q(z_n), \eta(y_n, x_n, \lambda_n) \rangle + \varepsilon_n \not< 0. \tag{2.1}$$

Let id $: C \to C$ be an identity map, by the continuity of $\eta$, $Q$, and $\langle \cdot, \cdot \rangle$, it follows that $\langle \cdot, \cdot \rangle + $ id is continuous (where id is continuous). So (2.1) implies

$$\langle Q(z_0), \eta(y_0, x_0, \lambda_0) \rangle \not< 0,$$

which is impossible. Hence, $x_0$ belongs to $\Omega_s(\gamma_0, \lambda_0, 0) \subseteq V$, which is again a contradiction, since $x_n \notin V$, for all $n$. Therefore, $\Omega_s$ is usc at $(\gamma_0, \lambda_0, 0)$.

Now we prove that $\Omega_s(\gamma_0, \lambda_0, 0)$ is compact, by checking its closedness. Indeed, let $x_n \in \Omega_s(\gamma_0, \lambda_0, 0)$, $x_n \to x_0$. This proof is similar to above and so we have $x_0 \in \Omega_s(\gamma_0, \lambda_0, 0)$ and hence $\Omega_s(\gamma_0, \lambda_0, 0)$ is compact. By Remark 2.5, we complete the proof. □

The following example shows that the upper semicontinuity and compactness of $E$ are essential.

**Example 2.7** Let $A = B = X = Y = \mathbb{R}$, $\Gamma = \Lambda = [0,1]$, $C = \mathbb{R}_+$, $\gamma_0 = 0$, $Q$ be an identity map, $K_1, K_2 : A \times \Gamma \to 2^A$, $T : A \times \Gamma \to 2^{L(X,Y)}$, and $\eta : A \times A \times \Gamma \to A$ be defined by

$$K_1(x, \gamma) = \left(-\gamma - \frac{1}{2}, \gamma\right],$$

$$\eta(y, x, \gamma) = \left\{\gamma^2 + \gamma + 2 + \varepsilon\right\},$$

$$T(y, \gamma) = \left\{\frac{1}{2^{\gamma+2}}\right\},$$

$$K_2(x, \gamma) = \left[0, 2^{\gamma^2+1}\right].$$

Then we have $E(0) = (-\frac{1}{2}, 0]$ and $E(\gamma) = (-\gamma - \frac{1}{2}, \gamma]$, $\forall \gamma \in (0, 1]$. We show that assumptions (ii) and (iii) of Theorem 2.6 are fulfilled. But the family $\{(\text{MQVIP}^{\gamma\lambda}) : (\gamma, \lambda) \in \Gamma \times \Lambda\}$ is not well-posed in the generalized sense at $(0, 0)$. The reason is that $E$ is not usc at 0 and $E(0)$ is not compact. In fact

$$\Omega_\alpha(\gamma, \lambda, \varepsilon) = \begin{cases} (-\frac{1}{2}, 0], & \text{if } \gamma = 0, \\ (-\gamma - \frac{1}{2}, \gamma], & \text{if } \gamma \in (0, 1]. \end{cases}$$

The following example shows that the lower semicontinuity of $K_2$ is essential.

**Example 2.8** Let $A = B = [-1, 1]$, $X = Y = \mathbb{R}$, $\Gamma = \Lambda = [0, 1]$, $C = \mathbb{R}_+$, $\gamma_0 = 0$, $Q$ be an identity map, $K_1, K_2 : A \times \Gamma \to 2^A$, $T : A \times \Gamma \to 2^{L(X,Y)}$, and $\eta : A \times A \times \Gamma \to A$ be defined by

$$K_1(x, \gamma) = [0, 1],$$

$$\eta(y, x, \gamma) = \{x + y - \varepsilon\},$$

$$T(y, \gamma) = \{1\},$$

$$K_2(x, \gamma) = \begin{cases} \{-1, 0, 1\}, & \text{if } \gamma = 0, \\ \{0, 1\}, & \text{otherwise.} \end{cases}$$

We have $E(\gamma) = [0, 1]$, $\forall \gamma \in [0, 1]$. Hence $E$ is usc at 0 and $E(0)$ is compact and the conditions (ii) and (iii) of Theorem 2.6 are easily seen to be fulfilled. But the family $\{(\text{MQVIP}^{\gamma\lambda}) : (\gamma, \lambda) \in \Gamma \times \Lambda\}$ is not well-posed in the generalized sense at $(0, 0)$. The reason is that $K_2$ is not lower semicontinuous at $(x, 0)$. In fact

$$\Omega_\alpha(\gamma, \lambda, \varepsilon) = \begin{cases} \{1\}, & \text{if } \gamma = 0, \\ [0, 1], & \text{if } \gamma \in (0, 1]. \end{cases}$$

**Theorem 2.9** *Assume for problem* (MQVIP) *the assumptions* (ii) *and* (iii) *as in Theorem* 2.6 *and replace* (i) *by* (i′):

(i′) *A is compact, $K_1$ is closed in $A \times \{\gamma_0\}$.*

*Then* (MQVIP) *is well-posed in the generalized sense at* $(\gamma_0, \lambda_0)$. *Moreover, if* $\Psi_\alpha(\gamma_0, \lambda_0)$ *is a singleton, then this problem is well-posed at* $(\gamma_0, \lambda_0)$.

*Proof* We omit the proof since the technique is similar to that for Theorem 2.6 with suitable modifications. □

The following example shows that the compactness of $A$ cannot be dropped.

**Example 2.10** Let $A = B = X = Y = (-\infty, +\infty)$, $\Gamma = \Lambda = [0, +\infty)$, $C = [0, +\infty)$, $\gamma_0 = 0$, $H$ be the identity map, $K_1, K_2 : A \times \Gamma \to 2^A$, $T : A \times \Gamma \to 2^B$, and $\eta : A \times A \times \Gamma \to A$ be defined by

$$K_1(x, \gamma) = \begin{cases} \{0\}, & \text{if } \gamma = 0, \\ \{2^{1+\gamma}\}, & \text{if } \gamma \neq 0, \end{cases}$$

$$K_2(x, \gamma) = [0, 1],$$

$$\eta(y, x, \lambda) = \left\{ \frac{1}{2^{\gamma + \gamma^2}} - \varepsilon 3^{\cos^2(\gamma) + \gamma + 1} \right\},$$

$$T(y, \gamma) = \left\{ \frac{1}{3^{\cos^2(\gamma) + \gamma + 1}} \right\}.$$

We see that $K_1$ is closed at $(x, 0)$, the assumptions (ii) and (iii) of Theorem 2.9 are satisfied. But the family $\{(\text{MQVIP}^{\gamma\lambda}) : (\gamma, \lambda) \in \Gamma \times \Lambda\}$ is not well-posed in the generalized sense at $(0, 0)$. The reason is that $A$ is not compact. In fact, $\Omega_\alpha(0, 0, 0) = \{0\}$ and $\Omega_\alpha(\gamma, \lambda, \varepsilon) = \{2^{1+\gamma}\}$, $\forall \gamma \in (0, +\infty)$.

The following example shows that the closedness of $K_1$ is essential.

**Example 2.11** Let $A = B = X = Y = [-3, 3]$, $\Gamma = \Lambda = [0, 1]$, $C = \mathbb{R}_+$, $\gamma_0 = 0$, $H$ be an identity map, $K_1, K_2 : A \times \Gamma \to 2^A$, $T : A \times \Gamma \to 2^B$, and $\eta : A \times A \times \Gamma \to A$ be defined by

$$K_1(x, \gamma) = (-3\gamma, 3],$$

$$K_2(x, \gamma) = [0, 3],$$

$$\eta(y, x, \gamma) = \left\{ x^2 - yx - \varepsilon \right\},$$

$$T(y, \gamma) = \{1\}.$$

We show that $A$ is compact and the conditions (ii), (iii) of Theorem 2.9 are easily seen to be fulfilled. But the family $\{(\text{MQVIP}^{\gamma\lambda}) : (\gamma, \lambda) \in \Gamma \times \Lambda\}$ is not well-posed in the generalized sense at $(0, 0)$. The reason is that $K_1$ is not closed at $(x, 0)$. In fact,

$$\Omega_\alpha(\gamma, \lambda, \varepsilon) = \begin{cases} \{3\}, & \text{if } \gamma = 0, \\ \{0, 3\}, & \text{if } \gamma \neq 0. \end{cases}$$

The following example shows that all assumptions of Theorem 2.6 are satisfied.

**Example 2.12** Let $X = Y = \mathbb{R}$, $A = B = [0, 3]$, $\Gamma = \Lambda = [0, 1]$, $C = \mathbb{R}_+$, $\gamma_0 = 0$, $H$ be an identity map, and let $K_1, K_2 : A \times \Gamma \to 2^A$, $T : A \times \Gamma \to 2^{L(X,Y)}$, and $\eta : A \times A \times \Gamma \to A$ be defined

by

$$K_1(x, \gamma) = K_2(x, \gamma) = [0, 1],$$

$$\eta(y, x, \gamma) = \gamma^2 + 2\gamma + 1 - \varepsilon,$$

$$T(y, \gamma) = \{1\}.$$

Then $E(\gamma) = [0, 1]$, $\forall \gamma \in [0, 1]$. We see that all assumptions of Theorem 2.9 are satisfied. So, the family $\{(\text{MQVIP}^{\gamma\lambda}) : (\gamma, \lambda) \in \Gamma \times \Lambda\}$ is well-posed in the generalized sense at $(0, 0)$. In fact, $\Omega(\gamma, \lambda, \varepsilon) = [0, 1]$, $\forall \gamma \in [0, 1]$.

For $(\gamma, \lambda) \in \Gamma \times \Lambda$, $\varepsilon \in \text{int } C$, and positive $\xi$, we define the following sets of approximate solutions of the family $\{(\text{MQVIP}^{\gamma\lambda}) : (\gamma, \lambda) \in \Gamma \times \Lambda\}$:

$$\Sigma_\alpha^{\gamma_0\lambda_0}(\xi, \varepsilon)$$

$$= \bigcup_{\gamma \in B(\gamma_0, \xi), \lambda \in B(\lambda_0, \xi)} \Omega_\alpha(\gamma, \lambda, \varepsilon)$$

$$= \bigcup_{\gamma \in B(\gamma_0, \xi), \lambda \in B(\lambda_0, \xi)} \{x \in K_1(x, \gamma) | (y, z)\alpha K_2(x, \gamma) \times T(y, \gamma) : \langle Q(z), \eta(y, x, \lambda) \rangle + \varepsilon \not< 0\},$$

where $B(\gamma_0, \xi)$ and $B(\lambda_0, \xi)$ are the closed balls centered at $\gamma_0$ and $\lambda_0$ with radius $\xi$.

Observe that, for every $(\gamma, \lambda) \in \Gamma \times \Lambda$,

(i) $\Sigma_\alpha^{\gamma_0\lambda_0}(0, 0) = \Omega_\alpha(\gamma_0, \lambda_0, 0) = \Psi_\alpha(\gamma_0, \lambda_0)$;

(ii) $\Psi_\alpha(\gamma_0, \lambda_0) \subseteq \Omega_\alpha(\gamma_0, \lambda_0, \varepsilon) \subseteq \Sigma_\alpha^{\gamma_0\lambda_0}(\xi, \varepsilon)$.

**Theorem 2.13** *Assume X is complete and the following conditions hold*:

(i) $K_1$ *is closed in* $A \times \{\gamma_0\}$, *and in* $K_1(A, \Gamma) \times \{\gamma_0\}$, $K_2$ *is lsc*;

(ii) *in* $K_2(K_1(A, \Gamma), \Gamma) \times \{\gamma_0\}$, $T$ *is usc and compact-valued if* $\alpha = \text{w}$ (*or* $\alpha = \text{m}$), *and lsc if* $\alpha = \text{s}$.

*Then* (MQVIP) *is well-posed at* $(\gamma_0, \lambda_0)$ *if and only if*

$$\Sigma_\alpha^{\gamma_0\lambda_0}(\xi, \varepsilon) \neq \emptyset, \quad \forall \xi > 0, \varepsilon \in \text{int } C \quad \text{and} \quad \text{diam } \Sigma_\alpha^{\gamma_0\lambda_0}(\xi, \varepsilon) \to 0 \quad \text{as } (\xi, \varepsilon) \to (0, 0).$$

*Proof* Similar arguments can be applied to the three cases. We present only the proof for the case where $\alpha = \text{s}$. If (MQVIP) is well-posed at $(\gamma_0, \lambda_0)$, then (MQVIP) has a unique solution $x_0 \in \Psi_s(\gamma_0, \lambda_0)$ and hence $\Sigma_s^{\gamma_0\lambda_0}(\xi, \varepsilon) \neq \emptyset$, $\forall \xi > 0$, $\varepsilon \in \text{int } C$ as $\Psi_s(\gamma_0, \lambda_0) \subseteq \Sigma_s^{\gamma_0\lambda_0}(\xi, \varepsilon)$. If diam $\Sigma_s^{\gamma_0\lambda_0}(\xi, \varepsilon) \not\to 0$ as $(\xi, \varepsilon) \to (0, 0)$, then there exist $q > 0$ and $\xi_n > 0$, $\varepsilon_n \in \text{int } C$, such that $\varepsilon_n \to 0$, $\xi_n \to 0$, and

$$\text{diam } \Sigma_s^{\gamma_0\lambda_0}(\xi_n, \varepsilon_n) > q > 0, \quad \forall n \in \mathbb{N}.$$

Then there exist $x_n^1, x_n^2 \in \Sigma_s^{\gamma_0\lambda_0}(\xi_n, \varepsilon_n)$ such that $d(x_n^1, x_n^2) > \frac{q}{2} > 0$. Hence there exist $\gamma_n^1, \gamma_n^2 \in B(\gamma_0, \xi_n)$, and $\lambda_n^1, \lambda_n^2 \in B(\lambda_0, \xi_n)$ such that $\forall y \in K_2(x_n^1, \gamma_n^1)$, $\forall z \in T(y, \gamma_n^1)$ satisfy

$$\langle Q(z), \eta(y, x_n^1, \lambda_n^1) \rangle + \varepsilon_n \not< 0,$$

and $\forall y \in K_2(x_n^2, \gamma_n^2)$, $\forall z \in T(y, \gamma_n^2)$ satisfy

$$\langle Q(z), \eta(y, x_n^2, \lambda_n^2) \rangle + \varepsilon_n \not< 0,$$

*i.e.*, $\{x_n^1\}$ and $\{x_n^2\}$ are approximating sequences for (MQVIP) corresponding to $\{(\gamma_n^1, \lambda_n^1)\}$ and $\{(\gamma_n^2, \lambda_n^2)\}$, respectively. Hence, the sequences $\{x_n^1\}$ and $\{x_n^2\}$ converges to the unique solution $x_0$ of (MQVIP$^{\gamma_0 \lambda_0}$), contradicting the fact that $d(x_n^1, x_n^2) > \frac{q}{2} > 0$, $\forall n \in \mathbb{N}$.

Conversely, let $\{\gamma_n\} \to \gamma_0$ and $\{\lambda_n\} \to \lambda_0$, and $\{x_n\}$ be approximating sequences for (MQVIP) corresponding to $\{\gamma_n\}$ and $\{\lambda_n\}$. Then there is $\{\varepsilon_n\} \to 0$ such that $\forall y \in K_2(x_n, \gamma_n)$, $\forall z \in T(y, \gamma_n)$ satisfying

$$\langle Q(z), \eta(y, x_n, \lambda_n) \rangle + \varepsilon_n \not< 0, \quad \forall n \in \mathbb{N}.$$

This yields $x_n \in \Sigma_s^{\gamma_0 \lambda_0}(\xi_n, \varepsilon_n)$ with $\{\xi_n\} = \max\{d(\gamma_n, \gamma_0), d(\lambda_n, \lambda_0)\} \to 0$, as $n \to +\infty$. Since diam $\Sigma_s^{\gamma_0 \lambda_0}(\xi_n, \varepsilon_n) \to 0$ as $(\xi_n, \varepsilon_n) \to (0, 0)$, it follows that $\{x_n\}$ is Cauchy and converges to a point $x_0$. By the closedness of $K_1$ at $(x_0, \gamma_0)$, $x_0 \in K_1(x_0, \gamma_0)$.

Next, we verify that $x_0 \in \Psi_s(\gamma_0, \lambda_0)$. Using the same argument as for Theorem 2.6, we deduce that $x_0 \in \Psi_s(\gamma_0, \lambda_0)$.

Now we prove that (MQVIP$^{\gamma_0 \lambda_0}$) has a unique solution. If $\Psi_s(\gamma_0, \lambda_0)$ has two distinct solutions $x_1$ and $x_2$, it is not hard to see that $x_1, x_2 \in \Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon)$, $\forall \xi > 0$, $\varepsilon \in \text{int} \, C$. It follows that

$$0 < d(x_1, x_2) \le \Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon) \to 0,$$

which is impossible. Hence, (MQVIP) is well-posed at $(\gamma_0, \lambda_0)$. $\qquad\square$

The following example shows that the uniqueness of well-posed is essential.

**Example 2.14** Let $X = Y = \mathbb{R}$, $A = B = [-1, 1]$, $\Gamma = \Lambda = [0, 1]$, $C = \mathbb{R}_+$, $\gamma_0 = 0$, $H$ be an identity map, and let $K_1, K_2 : A \times \Gamma \to 2^A$, $T : A \times \Gamma \to 2^{L(X,Y)}$, and $\eta : A \times A \times \Gamma \to A$ be defined by

$$K_1(x, \gamma) = K_2(x, \gamma) = [0, 1],$$

$$\eta(y, x, \gamma) = \gamma + 1 - \varepsilon,$$

$$T(y, \gamma) = \{1\}.$$

We show that the conditions (i) and (ii) of Theorem 2.13 are easily seen to be fulfilled and the family $\{(\text{MQVIP}^{\gamma \lambda}) : (\gamma, \lambda) \in \Gamma \times \Lambda\}$ is well-posed at $(0, 0)$. But diam $\Sigma_\alpha^{\gamma_0 \lambda_0}(\xi, \varepsilon) = [0, 1] \not\to 0$ as $(\xi, \varepsilon) \to (0, 0)$.

The following example shows that all assumptions of Theorem 2.13 are satisfied.

**Example 2.15** Let $A = B = X = Y = \mathbb{R}$, $\Gamma = \Lambda = [0, 1]$, $C = \mathbb{R}_+$, $\gamma_0 = 0$, $H$ be an identity map, and let $K_1, K_2 : A \times \Gamma \to 2^A$, $T : A \times \Gamma \to 2^{L(X,Y)}$, and $\eta : A \times A \times \Gamma \to A$ be defined by

$$K_1(x, \gamma) = [0, +\infty),$$

$$\eta(y, x, \gamma) = y - x + \gamma,$$

$$T(y, \gamma) = \{1\},$$

$$K_2(x, \gamma) = [0, 1].$$

We show that the conditions (i) and (ii) of Theorem 2.13 are easily seen to be fulfilled and $\Psi_\alpha(0,0) = \{0\}$ and

$$
\Omega_\alpha(\gamma, \lambda, \varepsilon) = \begin{cases} [0, \varepsilon], & \text{if } \gamma = 0, \\ [0, \gamma + \varepsilon], & \text{if } \gamma \in (0,1] \end{cases}
$$

and $\Sigma_\alpha^{\gamma_0 \lambda_0}(\xi, \varepsilon) = [0, \varepsilon]$ and the family $\{(\text{MQVIP}^{\gamma\lambda}) : (\gamma, \lambda) \in \Gamma \times \Lambda\}$ is well-posed at $(0,0)$, and diam $\Sigma_\alpha^{\gamma_0 \lambda_0}(\xi, \varepsilon) \to 0$ as $(\xi, \varepsilon) \to (0,0)$.

Next, we consider the following notions of measures of noncompactness.

**Definition 2.16** ([37, 38]) Let $X$ is complete. The Kuratowski measure of the set $A \subseteq X$ is defined by

$$
\zeta(A) = \inf\left\{ \vartheta > 0 \,\middle|\, A \subseteq \bigcup_{i=1}^n L_i, \text{diam } L_i < \vartheta, i = 1, 2, \ldots, n, \text{for some } n \in \mathbb{N} \right\}.
$$

**Definition 2.17** ([37, 38]) $A$, $B$ be nonempty subsets of $X$. The Hausdorff metric $H(\cdot, \cdot)$ between $A$ and $B$ is defined by

$$
H(A, B) = \max\left\{ H^*(A, B), H^*(B, A) \right\},
$$

where $H^*(A, B) = \sup_{a \in A} d(a, B)$ with $d(a, B) = \inf_{b \in B} \|a - b\|$.

By the definitions of $\zeta$ and $H$, we have

$$
\zeta(A) \leq \zeta(B) + 2H(A, B),
$$

for every all bounded sets $A$ and $B$.

**Remark 2.18** ([39, 40]) The function $\zeta$ is a regular measure of noncompactness defined by $\zeta : 2^X \to [0, +\infty]$ that satisfies the following conditions:
   (i) $\zeta(D) = +\infty$ if and only if the set $D$ is unbounded;
   (ii) $\zeta(D) = \zeta(\text{cl}(D))$;
   (iii) from $\zeta(D) = 0$ it follows that $D$ is a totally bounded set;
   (iv) from $P \subseteq Q$ it follows that $\zeta(P) \leq \zeta(Q)$;
   (v) if $X$ is a complete space, and if $\{B_n\}$ is a sequence of closed subsets of $X$ such that $B_{n+1} \subseteq B_n$ for each $n \in \mathbb{N}$ and $\lim_{n \to +\infty} \zeta(B_n) = 0$, then $M = \bigcap_{n \in N} B_n$ is a nonempty compact set and $\lim_{n \to +\infty} H(B_n, M) = 0$, where $H$ is a Hausdorff metric.

**Lemma 2.19** *Assume we have problem* (MQVIP). *Let* $\Gamma$, $\Lambda$ *be finite dimensional and the following conditions hold*:
   (i) $K_1$ *is closed in* $A \times \{\gamma_0\}$, *and in* $K_1(A, \Gamma) \times \{\gamma_0\}$, $K_2$ *is lsc*;
   (ii) *in* $K_2(K_1(A, \Gamma), \Gamma) \times \{\gamma_0\}$, $T$ *is usc and compact-valued if* $\alpha = \text{w}$ (*or* $\alpha = \text{m}$), *and lsc if* $\alpha = \text{s}$.
*Then* $\Sigma_\alpha^{\gamma_0 \lambda_0}(\xi, \varepsilon)$ *is closed, for all* $\xi > 0$, $\varepsilon \in \text{int } C$.

*Proof* Similar arguments can be applied in the three cases. We present only the proof for the case where $\alpha = \text{s}$. We let $x_n \in \Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon)$ such that $x_n \to x$. Hence, for all $n \in \mathbb{N}$, there

exist $\gamma_n \in B(\gamma_0, \xi)$ and $\lambda_n \in B(\lambda_0, \xi)$ and $\forall y \in K_2(x_n, \gamma_n)$, $\forall z \in T(y, \gamma_n)$ such that

$$\langle Q(z), \eta(y, x_n, \lambda_n) \rangle + \varepsilon \not< 0, \quad \forall n \in \mathbb{N}.$$

Since $B(\gamma_0, \xi)$ and $B(\lambda_0, \xi)$ are compact, we can assume that $\{\gamma_n\} \to \gamma \in B(\gamma_0, \xi)$ and $\{\lambda_n\} \to \lambda \in B(\lambda_0, \xi)$. By the closedness of $K_1$ at $(x, \gamma)$, we find that $x \in K_1(x, \gamma)$. We show that $\forall y \in K_2(x, \gamma)$, $\forall z \in T(y, \gamma)$ such that

$$\langle Q(z), \eta(y, x, \lambda) \rangle + \varepsilon \not< 0,$$

*i.e.*, $x \in \Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon)$. Indeed, if $x \notin \Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon)$, then $\exists y \in K_2(x, \gamma)$, $\exists z \in T(y, \gamma)$ such that

$$\langle Q(z), \eta(y, x, \lambda) \rangle + \varepsilon < 0.$$

By the lower semicontinuity of $K_2$ and $T$, there exist $y_n \in K_2(x_n, \gamma_n)$, $z_n \in T(y_n, \gamma_n)$ such that $\{y_n\} \to y$, $\{z_n\} \to z$, for all $n$. As $x_n \in \Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon)$ we have

$$\langle Q(z_n), \eta(y_n, x_n, \lambda_n) \rangle + \varepsilon \not< 0.$$

By the continuity of $Q$, $\eta$, $\langle \cdot, \cdot \rangle$ and id, it follows that $\langle \cdot, \cdot \rangle + $ id is continuous. So we have

$$\langle Q(z), \eta(y, x, \lambda) \rangle + \varepsilon \not< 0,$$

and we see a contradiction. Hence $x \in \Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon)$. Thus, $\Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon)$ is closed. $\qquad \square$

Next, we provide sufficient conditions for the two sets to coincide.

**Lemma 2.20** *Assume for problem* (MQVIP) *the following conditions to hold*:
  (i) $K_1(x, \cdot)$ *is closed at* $\gamma_0$, *and* $K_2(x, \cdot)$ *is lsc at* $\gamma_0$;
  (ii) *in* $K_2(K_1(A, \Gamma), \Gamma) \times \{\gamma_0\}$, $T$ *is usc and compact-valued if* $\alpha = $ w (*or* $\alpha = $ m), *and lsc if* $\alpha = $ s.
*Then* $\Psi_\alpha(\gamma_0, \lambda_0) = \bigcap_{\varepsilon \in \operatorname{int} C, \xi > 0} \Sigma_\alpha^{\gamma_0 \lambda_0}(\xi, \varepsilon)$, *for every* $(\gamma_0, \lambda_0) \in \Gamma \times \Lambda$.

*Proof* We present only the proof for the case where $\alpha = $ s. We first prove that $\bigcap_{\varepsilon \in \operatorname{int} C} \Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon) = \Omega_s(\gamma_0, \lambda_0, \varepsilon)$. It is easy to see that $\bigcap_{\varepsilon \in \operatorname{int} C} \Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon) \supseteq \Omega_s(\gamma_0, \lambda_0, \varepsilon)$. Thus, we only need to show that $\bigcap_{\varepsilon \in \operatorname{int} C} \Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon) \subseteq \Omega_s(\gamma_0, \lambda_0, \varepsilon)$. Indeed, let $x \in \bigcap_{\varepsilon \in \operatorname{int} C} \Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon)$, there are $\gamma_n \in B(\gamma_0, \xi)$ and $\lambda_n \in B(\lambda_0, \xi)$ such that $\forall y \in K_2(x, \gamma_n)$, $\forall z \in T(y, \gamma_n)$ satisfying

$$\langle Q(z), \eta(y, x, \lambda_n) \rangle + \varepsilon \not< 0.$$

Since $x \in K_1(x, \gamma_n)$, $\gamma_n \to \gamma_0$ and $K_1$ is closed, we have $x \in K_1(x, \gamma_0)$. Now we verify that $x \in \Omega(\gamma_0, \lambda_0, \varepsilon)$. Indeed, for each $y \in K_2(x, \gamma_0)$, by the semicontinuity of $K_2(x, \cdot)$ at $\gamma_0$ and the semicontinuity of $T$ at $(y, \gamma_0)$, there exist $y_n \in K_2(x, \gamma_n)$ and $z_n \in T(y_n, \gamma_n)$ such that $\{y_n\} \to y$, $\{z_n\} \to z$. As $x \in \Omega_s(\gamma_n, \lambda_n, \varepsilon)$, we have

$$\langle Q(z_n), \eta(y_n, x, \lambda_n) \rangle + \varepsilon \not< 0.$$

By the continuity of $Q$, $\eta$, $\langle \cdot, \cdot \rangle$, $\langle \cdot, \cdot \rangle + \mathrm{id}$, and $(y_n, z_n, \gamma_n, \lambda_n) \to (y, z, \gamma_0, \lambda_0)$. We have

$$\langle Q(z), \eta(y, x, \lambda_0) \rangle + \varepsilon \not< 0,$$

*i.e.*,

$$\bigcap_{\varepsilon \in \mathrm{int}\, C} \Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon) \subseteq \Omega_s(\gamma_0, \lambda_0, \varepsilon).$$

Hence

$$\bigcap_{\varepsilon \in \mathrm{int}\, C} \Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon) = \Omega_s(\gamma_0, \lambda_0, \varepsilon).$$

It is clear that

$$\Psi_s(\gamma_0, \lambda_0) = \bigcap_{\xi > 0} \Omega_s(\gamma_0, \lambda_0, \varepsilon) = \bigcap_{\xi > 0, \varepsilon \in \mathrm{int}\, C} \Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon). \qquad \square$$

The following theorem shows the well-posedness in the generalized sense at $(\gamma_0, \lambda_0)$ for (MQVIP) by using the Kuratowski measure $\zeta$.

**Theorem 2.21** *Let $X$ be complete, $\Gamma$, $\Lambda$ be finite dimensional and the following conditions hold*:

(i) *$K_1$ is closed in $A \times \{\gamma_0\}$, and in $K_1(A, \Gamma) \times \{\gamma_0\}$, $K_2$ is lsc*;

(ii) *in $K_2(K_1(A, \Gamma), \Gamma) \times \{\gamma_0\}$, $T$ is usc and compact-valued if $\alpha = \mathrm{w}$ (or $\alpha = \mathrm{m}$), and lsc if $\alpha = \mathrm{s}$.*

*Then (MQVIP) is well-posed in the generalized sense at $(\gamma_0, \lambda_0)$ if and only if*

$$\Sigma_\alpha^{\gamma_0 \lambda_0}(\xi, \varepsilon) \neq \emptyset, \quad \forall \xi > 0, \varepsilon \in \mathrm{int}\, C \quad \text{and} \quad \zeta\left(\Sigma_\alpha^{\gamma_0 \lambda_0}(\xi, \varepsilon)\right) \to 0 \quad \text{as } (\xi, \varepsilon) \to (0, 0).$$

*Proof* Similar arguments can be applied in the three cases. We present only the proof for the case where $\alpha = \mathrm{s}$. Now we suppose that (MQVIP) is well-posed in the generalized sense at $(\gamma_0, \lambda_0)$. Let $\Psi_s$ be a solution set of (MQVIP$^{\gamma \lambda}$) for all $(\gamma, \lambda) \in \Gamma \times \Lambda$. Then, from Theorem 2.6, we see that $\Psi_s(\gamma_0, \lambda_0)$ is a nonempty compact. Clearly $\Psi_s(\gamma_0, \lambda_0) \subseteq \Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon)$, $\forall \xi > 0$, $\varepsilon \in \mathrm{int}\, C$. Now we show that

$$\zeta\left(\Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon)\right) \to 0 \quad \text{as } (\xi, \varepsilon) \to (0, 0).$$

Indeed, since $\Psi_s(\gamma_0, \lambda_0) \subseteq \Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon)$, $\forall \xi > 0$, $\varepsilon \in \mathrm{int}\, C$. Using the concept of Hausdorff metric, we have

$$H\left(\Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon), \Psi_s(\gamma_0, \lambda_0)\right)$$
$$= \max\left\{H^*\left(\Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon), \Psi_s(\gamma_0, \lambda_0)\right), H^*\left(\Psi_s(\gamma_0, \lambda_0), \Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon)\right)\right\}$$
$$= H^*\left(\Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon), \Psi_s(\gamma_0, \lambda_0)\right).$$

Suppose that $\Psi_s(\gamma_0, \lambda_0) \subseteq \bigcup_{i=1}^n L_i$, $\mathrm{diam}\, L_i < \vartheta$, $i = 1, 2, \ldots, n$, for some $n \in \mathbb{N}$.

We set $\Delta_i = \{t \in A \mid d(t, L_i) \le H(\Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon), \Psi_s(\gamma_0, \lambda_0))\}$.

We claim that $\Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon) \subseteq \bigcup_{i=1}^{n} \Delta_i$. Indeed, let $x \in \Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon)$. Then $d(x, \Psi_s(\gamma_0, \lambda_0)) \le H(\Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon), \Psi_s(\gamma_0, \lambda_0))$. Since $\Psi_s(\gamma_0, \lambda_0) \subseteq \bigcup_{i=1}^{n} L_i$, we see that

$$d\left(x, \bigcup_{i=1}^{n} L_i\right) \le H\left(\Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon), \Psi_s(\gamma_0, \lambda_0)\right).$$

Hence, there is $k$ such that $d(x, L_k) \le H(\Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon), \Psi_s(\gamma_0, \lambda_0))$, i.e., $x \in \Delta_k$. So

$$\Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon) \subseteq \bigcup_{i=1}^{n} \Delta_i.$$

Note further that

$$\text{diam}\, \Delta_i = \text{diam}\, L_i + 2H\left(\Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon), \Psi_s(\gamma_0, \lambda_0)\right)$$
$$\le \vartheta + 2H\left(\Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon), \Psi_s(\gamma_0, \lambda_0)\right).$$

Hence,

$$\zeta\left(\Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon)\right) \le 2H\left(\Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon), \Psi_s(\gamma_0, \lambda_0)\right) + \zeta\left(\Psi_s(\gamma_0, \lambda_0)\right).$$

Since $\Psi_s(\gamma_0, \lambda_0)$ is compact, $\zeta(\Psi_s(\gamma_0, \lambda_0)) = 0$, so we have

$$\zeta\left(\Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon)\right) \le 2H^*\left(\Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon), \Psi_s(\gamma_0, \lambda_0)\right).$$

Now we prove that

$$H^*\left(\Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon)\right) \to 0 \quad \text{as } (\xi, \varepsilon) \to (0, 0).$$

Suppose to the contrary that

$$H^*\left(\Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon)\right) \nrightarrow 0 \quad \text{as } (\xi, \varepsilon) \to (0, 0).$$

There are $\theta > 0$, $(\xi_n, \varepsilon_n) \to (0, 0)$, and $x_n \in \Sigma_s^{\gamma_0 \lambda_0}(\xi_n, \varepsilon_n)$ such that

$$d\left(x_n, \Psi_s(\gamma_0, \lambda_0)\right) \ge \theta > 0, \quad \forall n \in \mathbb{N}.$$

$\{x_n\}$ is an approximating sequence of (MQVIP). By the well-posedness in the generalized sense of (MQVIP) at $(\gamma_0, \lambda_0)$, there is a subsequence $\{x_k\}$ of $\{x_n\}$ converging to some point of $\Psi_s(\gamma_0, \lambda_0)$, which is impossible as $d(x_n, \Psi_s(\gamma_0, \lambda_0)) \ge \theta > 0$, $\forall n \in \mathbb{N}$. Hence

$$\zeta\left(\Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon)\right) \to 0 \quad \text{as } (\xi, \varepsilon) \to (0, 0).$$

Conversely, $\zeta(\Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon)) \to 0$ as $(\xi, \varepsilon) \to (0, 0)$. By Lemma 2.19, we see that $\Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon)$ is closed, for all $\xi > 0$, $\varepsilon \in \text{int}\, C$. By Lemma 2.20, we have

$$\Psi_s(\gamma_0, \lambda_0) = \bigcap_{\varepsilon \in \text{int}\, C, \xi > 0} \Sigma_s^{\gamma_0 \lambda_0}(\xi, \varepsilon).$$

Since $\zeta(\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon)) \to 0$ as $(\xi,\varepsilon) \to (0,0)$, the regular measure properties of $\zeta$ imply that $\Psi_s(\gamma_0,\lambda_0)$ is compact and

$$H\left(\Sigma_s^{\gamma_0\lambda_0}(\xi,\varepsilon),\Psi_s(\gamma_0,\lambda_0)\right) \to 0 \quad \text{as } (\xi,\varepsilon) \to (0,0).$$

Let $\{x_n\}$ be an approximating sequence for (MQVIP) corresponding to $\{(\gamma_n,\lambda_n)\}$, where $\{\gamma_n\} \to \gamma_0$ and $\{\lambda_n\} \to \lambda_0$. There is $\{\varepsilon_n\} \to 0$ such that $\forall y \in K_2(x_n,\gamma_n)$, $\forall z \in T(y,\gamma_n)$ satisfying

$$\langle Q(z),\eta(y,x_n,\lambda_n)\rangle + \varepsilon_n \nless 0, \quad \forall n \in \mathbb{N}.$$

This means that $x_n \in \Sigma_s^{\gamma_0\lambda_0}(\xi_n,\varepsilon_n)$ with $\xi_n := \max\{d(\gamma_0,\gamma_n),d(\lambda_0,\lambda_n)\}$. We see that

$$d\left(x_n,\Psi_s(\gamma_0,\lambda_0)\right) \leq H\left(\Sigma_s^{\gamma_0\lambda_0}(\xi_n,\varepsilon_n),\Psi_s(\gamma_0,\lambda_0)\right) \to 0 \quad \text{as } n \to +\infty.$$

Hence, there is $\bar{x}_n \in \Psi_s(\gamma_0,\lambda_0)$ such that

$$d(x_n,\bar{x}_n) \to 0 \quad \text{as } n \to +\infty.$$

By the compactness of $\Psi_s(\gamma_0,\lambda_0)$, there is a subsequence $\{\bar{x}_{n_k}\}$ of $\{\bar{x}_n\}$ convergent to some point $x_0$ of $\Psi_s(\gamma_0,\lambda_0)$. Therefore, the corresponding subsequence $\{x_{n_k}\}$ of $\{x_n\}$ tends to $x_0$. Hence, (MQVIP) is well-posed in the generalized sense at $(\gamma_0,\lambda_0)$. □

**Remark 2.22** In cases as in Remark 2.4(i), Theorems 3.3, 3.4, and 3.5-3.6 in [22] are particular cases of Theorems 2.13, 2.21, and 2.9, respectively. However, the assumptions and our proof methods are very different from Theorems 3.3, 3.4, and 3.5-3.6 in [22].

The following example shows that the closedness of $K_1$ in Theorem 2.21 cannot be dropped.

**Example 2.23** Let $X = Y = \mathbb{R}$, $A = B = [-5,5]$, $\Gamma = \Lambda = [0,1]$, $C = \mathbb{R}_+$, $\gamma_0 = 0$, $H$ be an identity map, and let $K_1, K_2 : A \times \Gamma \to 2^A$, $T : A \times \Gamma \to 2^{L(X,Y)}$, and $\eta : A \times A \times \Gamma \to A$ be defined by

$$K_1(x,\gamma) = [-5\gamma,5],$$

$$\eta(y,x,\gamma) = x(x-y),$$

$$T(y,\gamma) = \{1\},$$

$$K_2(x,\gamma) = [0,5].$$

We show that $K_2$ is lsc in $K_1(A,\Gamma) \times \Gamma$ and the condition (ii) of Theorem 2.21 is easily seen to be fulfilled and $\Sigma_\alpha^{\gamma_0\lambda_0}(\xi,\varepsilon) \subseteq [-5,5]$. Hence, $\zeta(\Sigma_\alpha^{\gamma_0\lambda_0}(\xi,\varepsilon)) \to 0$ as $(\xi,\varepsilon) \to (0,0)$. But the family $\{(\text{MQVIP}^{\gamma\lambda}) : (\gamma,\lambda) \in \Gamma \times \Lambda\}$ is not well-posed in the generalized sense at $(0,0)$. The reason is that $K_1$ is not closed at $(A,0)$. Indeed, we let $\gamma_n = x_n = \frac{1}{n} \to 0$, as $n \to \infty$ and $t_n = \frac{1}{n} \in K_1(x_n,\gamma_n) = (-\frac{5}{n},5]$, $\forall n \in \mathbb{N}$. It is clear that $\{t_n\}$ is convergent to $0 \notin K_1(0,0) = (0,5]$. In fact, $\Omega_\alpha(\gamma_0,\lambda_0,\varepsilon) = \Sigma_\alpha^{\gamma_0\lambda_0}(\xi,\varepsilon) = \{5\}$.

The following example shows that the lower semicontinuity of $K_2$ in Theorem 2.21 is essential.

**Example 2.24** Let $X = Y = \mathbb{R}$, $A = B = [-2, 2]$, $\Gamma = \Lambda = [0, 1]$, $C = \mathbb{R}_+$, $\varepsilon \in \operatorname{int} C$, $\xi > 0$, $\gamma_0 = 0$, $H$ be an identity map, and let $K_1, K_2 : A \times \Gamma \to 2^A$, $T : A \times \Gamma \to 2^{L(X,Y)}$, and $\eta : A \times A \times \Gamma \to A$ be defined by

$$K_1(x, \gamma) = [0, 2],$$

$$\eta(y, x, \gamma) = x + y,$$

$$T(y, \gamma) = \{1\},$$

$$K_2(x, \gamma) = \begin{cases} \{-2, 0, 2\}, & \text{if } \gamma = 0, \\ \{0, 2\}, & \text{otherwise.} \end{cases}$$

We show that $K_2$ is lsc in $K_1(A, \Gamma) \times \Gamma$ and the condition (ii) of Theorem 2.21 is easily seen to be fulfilled and $\Sigma_\alpha^{\gamma_0 \lambda_0}(\xi, \varepsilon) \subseteq [-2, 2]$. Hence, $\zeta(\Sigma_\alpha^{\gamma_0 \lambda_0}(\xi, \varepsilon)) \to 0$ as $(\xi, \varepsilon) \to (0, 0)$. But the family $\{(\mathrm{MQVIP}^{\gamma\lambda}) : (\gamma, \lambda) \in \Gamma \times \Lambda\}$ is not well-posed in the generalized sense at $(0, 0)$. The reason is that $K_2$ is not lower semicontinuous. In fact

$$\Omega_\alpha(\gamma, \lambda, \varepsilon) = \Sigma_\alpha^{\gamma\lambda}(\xi, \varepsilon) = \begin{cases} [2 - \varepsilon, 2] \cap [0, 2], & \text{if } \gamma = 0, \\ [0, 2], & \text{if } \gamma \in (0, 1]. \end{cases}$$

The following example shows that all assumptions of Theorem 2.21 are fulfilled.

**Example 2.25** Let $X = Y = \mathbb{R}$, $A = B = \Gamma = \Lambda = [0, 2]$, $C = \mathbb{R}_+$, $\varepsilon \in \operatorname{int} C$, $\xi > 0$, $\gamma_0 = 0$, $H$ be an identity map, and let $K_1, K_2 : A \times \Gamma \to 2^A$, $T : A \times \Gamma \to 2^{L(X,Y)}$, and $\eta : A \times A \times \Gamma \to A$ be defined by

$$K_1(x, \gamma) = K_2(x, \gamma) = [\gamma, \gamma + 2],$$

$$\eta(y, x, \gamma) = \left\{ 2^{\gamma^2 + 1} - \varepsilon \right\},$$

$$T(y, \gamma) = \left\{ \frac{1}{2^{\gamma^2 + 1}} \right\}.$$

We show that the assumptions (i) and (ii) of Theorem 2.21 are easily seen to be fulfilled and

$$\Omega_\alpha(\gamma, \lambda, \varepsilon) = \begin{cases} [\gamma, \gamma + 2], & \text{if } \gamma \in (0, 1], \\ [0, 2], & \text{if } \gamma = 0, \end{cases}$$

and $\Sigma_\alpha^{\gamma_0 \lambda_0}(\xi, \varepsilon) \subseteq [0, 2]$. Hence, $\zeta(\Sigma_\alpha^{\gamma_0 \lambda_0}(\xi, \varepsilon)) \to 0$ as $(\xi, \varepsilon) \to (0, 0)$, and the family $\{(\mathrm{MQVIP}^{\gamma\lambda}) : (\gamma, \lambda) \in \Gamma \times \Lambda\}$ is well-posed in the generalized sense at $(0, 0)$.

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