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Anti-periodic extremal problems for a class of nonlinear evolution inclusions in R^N

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Abstract

The aim of this paper is to establish the existence of anti-periodic solutions to the following nonlinear anti-periodic problem: $\dot{x} + A(t,x) \in \operatorname{Ext} F(t,x)$ a.e. $t \in I$, $x(T) = -x(0)$, in R^N where $\operatorname{Ext} F(t,x)$ denotes the extremal point set of the multifunction $F(t,x)$, and $A(t,x)$ is a nonlinear map from R^N to R^N. Sufficient conditions for the existence of extremal solutions are presented. Also, we prove that the extremal point set of this problem is compact in $C(I, R^N)$ and dense in the solution set of nonlinear evolution problems with a convex valued perturbation which is multivalued. We apply our results on the control system with *a priori* feedback.

Keywords: anti-periodic solution; evolution inclusion; the Schauder fixed point theorem; continuous selection

1 Introduction

In this paper, we consider the following anti-periodic problems:

$$\begin{cases} \dot{x} + A(t,x) \in \operatorname{Ext} F(t,x) & \text{a.e. } t \in I, \\ x(T) = -x(0), \end{cases} \tag{1}$$

in R^N where $I = [0, T]$, $F(t,x) : I \times R^N \to 2^{R^N}$ satisfies some conditions mentioned later, $A(t,x)$ is a nonlinear map from $I \times R^N$ to R^N. Anti-periodic problems of evolution inclusions have important applications in many fields, such as auto-control, partial differential equations and engineering, *etc.* The study of anti-periodic solutions for nonlinear evolution equations was initiated by Okochi [1]. Since then, many authors devoted themselves to the investigation of the existence of anti-periodic solutions to nonlinear evolution equations in Hilbert spaces. For the details, see [2–7] and the references therein. In [5], Chen studied the anti-periodic solution for the following first order semilinear evolution equation:

$$\begin{cases} \dot{u} + Au(t) \in F(t, u), & t \in R, \\ u(t + T) = -u(t), \end{cases}$$

where $A : R^N \to R^N$ is a matrix, $f : R \times R^N \to R^N$ is a continuous function satisfying $f(t + T, u) = -f(t, -u)$ for all $(t, u) \in R \times R^N$. Chen *et al.* [8] also studied this problem in a real separable Hilbert space when A is a dense self-adjoint operator which only has a point spectrum. Recently, Q Liu [9], ZH Liu [10] and Wang [11] considered anti-periodic problem of nonlinear evolution equation in a real reflexive Banach space and obtained some

existence results by applying the theory of pseudo-monotone perturbations of maximal monotone mappings and the theory of evolution operators. However, there are few results about the extremal anti-periodic problem, which is connected with the 'Bang-Bang' control problem in the above works. For the case of periodic problems, we refer to the work of Li-Xue [12], Xue-Yu [13] in Hilbert and Xue-Cheng [14] in Banach space. Inspired by [15], which considers the problem in Banach space, we continue to consider the existence of solutions for a nonlinear evolution inclusion in R^N by relaxing the constraint conditions. Our approach will be based on techniques and results of the theory of the extremal continuous selection theorem and the Schauder fixed point theorem.

The paper is divided into four parts. In Section 2, we introduce some notation, and definitions we need for the results. In Section 3, we present some basic assumptions and main results, the proofs of the main results are given based on the Schauder fixed point theorem and the extremal continuous selection theorem. Finally, an example is presented for our results in Section 4.

2 Preliminaries

For convenience, we introduce some notation as follows. In Euclidean space, $(\cdot, \cdot)$ expresses inner product, while $\| \cdot \|$ expresses the Euclidean norm. Let $L^2([0, T]; R^N)$ denote the set of the map $x : [0, T] \to R^N$ which satisfies $\int_0^T |x|^2 \, dt < \infty$, and the norm in $L^2([0, T]; R^N)$ is denoted by $\|x\|_2 = (\int_0^T |x|^2 \, dt)^{\frac{1}{2}}$. If $F = [0, b]$, the $\operatorname{Ext} F = \{0, b\}$. We recall some basic definitions and facts from multivalued analysis which we shall need in the sequel. For details we refer to the book of Zeidler [16]. Let $I = [0, T]$, (I, Σ) be the Lebesgue measurable space and X be a separable Banach space. Denote

$$P_f(X) = \{A \subset X : \text{nonempty and closed}\},$$

$$P_k(X) = \{A \subset X : \text{nonempty and compact}\},$$

$$P_{wkc}(X) = \{A \subset X : \text{nonempty, weakly compact and convex}\}.$$

Let $A \subset P_f(X)$, $x \in X$, then the distance from x to A is given by $d(x, A) = \inf\{|x - a| : a \in A\}$. A multifunction $F : I \to P_f(X)$ is said to be measurable if and only if, for every $z \in X$, the function $t \to d(z, F(t)) = \inf\{\|z - x\| : x \in F(t)\}$ is measurable. A multifunction $G : I \to 2^X \setminus \{\emptyset\}$ is said to be graph measurable, if $\operatorname{Gr} G = \{(t, x) : x \in G(t)\} \in \Sigma \times \mathcal{B}(X)$ with $\mathcal{B}(X)$ being the Borel σ-field of X. On $P_f(X)$ we can define a generalized metric, known in the literature as the 'Hausdorff metric', by setting

$$h(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\}$$

for all $A, B \in P_f(X)$. It is well known that $(P_f(X), h)$ is a complete metric space and $P_{fc}(X)$ is a closed subset of it. When Z is a Hausdorff topological space, a multifunction $G : Z \to P_f(X)$ is said to be h-continuous if it is continuous as a function from Z into $(P_f(X), h)$.

Let Y, Z be Hausdorff topological spaces and $G : Y \to 2^Z \setminus \{\phi\}$. We say that $G(\cdot)$ is 'upper semicontinuous (USC)' (resp., 'lower semicontinuous (LSC)'), if for all $C \subseteq Z$ nonempty closed, $G^-(C) = \{y \in Y : G(y) \cap C \neq \phi\}$ (resp., $G^+(C) = \{y \in Y : G(y) \subseteq C\}$) is closed in Y. An USC multifunction has a closed graph in $Y \times Z$, while the converse is true if G is locally compact (*i.e.* for every $y \in Y$ there exists a neighborhood U of y such that $\overline{F(U)}$ is compact

in Z). A multifunction which is both USC and LSC is said to be 'continuous'. If Y, Z are both metric spaces, then the above definition of LSC is equivalent to saying that for all $z \in Z$, $y \to d_Z(z, G(y)) = \inf\{d_Z(z, v) : v \in G(y)\}$ has upper semicontinuity as an R_+-valued function. Also, lower semicontinuity is equivalent to saying that if $y_n \to y$ in Y as $n \to \infty$, then

$$G(y) \subseteq \underline{\lim} G(y_n) = \{z \in Z : \lim d_Z(z, G(y_n)) = 0\}$$
$$= \{z \in Z : z = \lim z_n, z_n \in G(y_n), n \geq 1\}.$$

A set $D \subseteq L^2(I, X)$ is said to be 'decomposable', if for every $g_1, g_2 \in D$ and for every $J \subseteq I$ measurable we have $\chi_J g_1 + \chi_{J^c} g_2 \in D$. Let X be a Banach space and let $L^2(I, X)$ be the Banach space of all functions $u : I \to X$, which are Bochner integrable. $D(L^2(I, X))$ denotes the collection of nonempty decomposable subsets of $L^2(I, X)$. The following lemmas are still needed in the proof of our main theorems.

Lemma 2.1 (see [17]) *Let X be a separable metric space and let $F : X \to D(L^2(I, X))$ be a lower semicontinuous multifunction with closed decomposable values. Then F has a continuous selection.*

Let X be a separable Banach Space and $C(I, X)$ be the Banach space of all continuous functions. A multifunction $F : I \times X \to P_{wkc}(X)$ is said to be Carathéodory type, if for every $x \in X$, $F(\cdot, x)$ is measurable, and for almost all $t \in I$, $F(t, \cdot)$ is h-continuous. (*i.e.* it is continuous form X to the metric space $(P_f(X), h)$ where h is Hausdorff metric). Let $M \subset C(I, X)$. A multifunction $F : I \times X \to P_{wkc}(X)$ is called integrably bounded on M if there exists a function $\lambda : I \to R_+$ such that for almost all $t \in I$, $\sup\{\|y\| : y \in F(t, x(t)), x(\cdot) \in M\} \leq \lambda(t)$. A nonempty subset $M_0 \subset C(I, X)$ is called σ-compact if there is a sequence $\{M_k\}_{k \geq 1}$ of compact subsets M_k such that $M_0 = \bigcup_{k \geq 1} M_k$. Let $M_0 \subset M$, such that M_0 is dense in M and σ-compact. The following continuous selection theorem in the extreme point case is due to Tolstonogov [18].

Lemma 2.2 (see [18]) *Let the multifunction $F : I \times X \to P_{wkc}(X)$ be of Carathéodory type and integrably bounded. Then there exists a continuous function $g : M \to L_p(I, X)$ such that for almost all $t \in I$, if $x(\cdot) \in M_0$, then $g(x)(t) \in \text{Ext} F(t, x(t))$, and if $x(\cdot) \in M \setminus M_0$, then $g(x)(t) \in \overline{\text{Ext}} F(t, x(t))$.*

3 Main results

Let $I = [0, T]$ and $C(I; R^N)$ be all the continuous functions from I to R^N with the max norm. We let $C_\beta = \{v(\cdot) \in C(I; R^N) : v(0) = -v(T)\}$, and $W^{1,2}(I; R^N) = \{u(\cdot) \in C_\beta : \dot{u}(\cdot) \in L^2(I; R^N)\}$. $W^{1,2}(I; R^N)$ is a separable Banach space under the norm $\|\cdot\|_{1,2}$.

Consider the following anti-periodic problem:

$$\begin{cases} \dot{x}(t) + A(t, x(t)) \in \text{Ext} F(t, x(t)) & \text{a.e. } t \in I, \\ x(0) = -x(T), \end{cases}$$

where $A : I \times R^N \to R^N$ is a hemicontinuous function, $F : I \times R^N \to 2^{R^N}$ is a multifunction. By a solution x of problem (1), we mean a function $x \in W^{1,2}(I, R^N)$ and there exists a

function $f(t) \in \mathrm{Ext}\, F(t, x(t))$ such that

$$\langle \dot{x}(t), v \rangle + \langle A(t, x(t)), v \rangle = \langle f(t), v \rangle$$

for all $v \in R^N$ and almost all $t \in I$. The precise hypotheses on the data of problem (1) are the following:

$H(A)$: $A : I \times R^N \to R^N$ is a nonlinear function such that

  (i) $t \to A(t, x)$ is measurable;

  (ii) for each $t \in I$, the operator $A(t, \cdot) : R^N \to R^N$ is uniformly monotone and hemicontinuous, that is, there exists a constant $p > 0$ such that
  $(A(t, x_1) - A(t, x_2), x_1 - x_2) \geq p \|x_1 - x_2\|^2$ for all $x_1, x_2 \in R^N$.

$H(F)$: $F : R \times R^N \to P_{kc}(R^N)$ is a multifunction such that

  (i) $(t, x) \to F(t, x)$ is graph measurable;

  (ii) for almost all $t \in I$, $x \to F(t, x)$ is h-continuous;

  (iii) there exist a nonnegative function $b(\cdot) \in L_+^2(I)$ and a constant $c_1 > 0$ such that

$$\left| F(t, x) \right| = \sup \left\{ \|f\| : f \in F(t, x) \right\} \leq b(t) + c\|x\|^\alpha$$

  for all $x \in R^N$, $t \in T$, where $\alpha < 1$.

We still need the following lemma.

Lemma 3.1 (see [19]) *If the hypothesis $H(A)$ holds, consider the equation*

$$\dot{x} + A(t, x) = f(t) \quad a.e.\ t \in I, \tag{2}$$

where $f \in L^2([0, T]; R^N)$. Then problem (2) has a unique T-anti-periodic solution.

Theorem 3.1 *If hypotheses $H(A)$ and $H(F)$ hold, then problem (1) has at least one solution.*

Proof We define $L : W^{1,2}(I; R^N) \to L^2([0, T]; R^N)$ as $Lx = \dot{x} + A(t, x)$ and $x(0) = -x(T)$. By Lemma 3.1, we have $L : W^{1,2}(I; R^N) \to L^2([0, T]; R^N)$ is one to one and surjective, and so $L^{-1} : L^2([0, T]; R^N) \to W^{1,2}(I; R^N)$ is well defined. As in the proof of Theorem 3.1, we obtain *a priori* bound for S_e which denotes the solution set of problem (1). We know that there exist $M_i > 0$, $i = 1, 2$, such that $\|x\|_{1,2} < M_1$ and $\|x\|_{C(I,H)} < M_2$ for all $x \in S_e$. Let $\psi(t) = b(t) + CM_2$, $\psi(t) \in L_2^+(I)$. We may assume that $|F(t, x)| \leq \psi(t)$, a.e. on I for all $x \in R^N$. So let

$$W = \left\{ v \in L_q(I, H) : \left\| v(t) \right\|_H \leq \psi(t) \text{ a.e. on } I \right\},$$

then $\hat{K} = L^{-1}(W) \subseteq W^{1,2}(I; R^N)$ is compact convex subset in $C(I, H)$. Obviously $\hat{K}$ is convex. We only need to show the compactness. Let $\{x_n\}_{n \geq 1} \subset \hat{K}$, then there exists $h_n \in W$ such that $L(x_n) = h_n$, i.e. $\dot{x}_n = h_n - A(t, x_n)$. By the definition of W, so W is uniformly bounded in $L^2([0, T]; R^N)$. By the Dunford-Pettis theorem, passing to a subsequence if necessary, we may assume that $h_n \rightharpoonup h$ in $L^2([0, T]; R^N)$ for some $h \in W$. From the definition of W, we have

$$\|x_n\|_{W_{1,2}} = \left\| L^{-1}(Lx_n) \right\|_{W_{1,2}} = \left\| L^{-1} h_n \right\|_{W_{1,2}} \leq M_1.$$

Therefore, the sequence $\{x_n\}_{n \geq 1} \subset W^{1,2}(I;R^N)$ is bounded. Because of the compactness of the embedding $W^{1,2}(I;R^N) \subset L^2([0,T];R^N)$, we find that the sequence $\{x_n\}_{n \geq 1} \subset L^2([0,T];R^N)$ is relatively compact. So by passing to a subsequence if necessary, we may assume that $x_n \to x$ in $L^2([0,T];R^N)$. Moreover, by the boundedness of the sequence $\{x_n\}_{n \geq 1} \subset W^{1,2}(I;R^N)$, it follows that the sequence $\{\dot{x}_n\}_{n \geq 1} \subset L^2([0,T];R^N)$ is uniformly bounded and passing to a subsequence if necessary, we may assume that $\dot{x}_n \rightharpoonup \dot{x}$ in $L^2([0,T];R^N)$. Since the embedding $W^{1,2}(I;R^N) \subset C(I;R^N)$ is continuous and $W^{1,2}(I;R^N) \subset L^2([0,T];R^N)$ is compact, it follows that $x_n \to x$ in $C(I;R^N)$ and $x_n \to x$ in $L^2([0,T];R^N)$. Hence, $x_n \to x$ in R^N for all $t \in I \setminus \Lambda$, $m(\Lambda) = 0$ (m being the Lebesgue measure on R). Since A is hemicontinuous and monotone. Thus, $A(t,x_n) \rightharpoonup A(t,x)$ in $L^2([0,T];R^N)$ and as $n \to \infty$, we obtain $\dot{x} + A(t,x) = h$ a.e. on I and $x(0) = -x(T)$. Note that

$$\dot{x}_n - \dot{x} + \big(A(t,x_n) - A(t,x)\big) = h_n - h.$$

Taking the inner product above with $x_n - x$ and integrating from 0 to T, one can see that

$$
\begin{aligned}
\int_0^T & \big(A(t,x_n) - A(t,x), x_n - x\big)\, dt \\
&= \int_0^T (h_n - h, x_n - x)\, dt - \int_0^T (\dot{x}_n - \dot{x}, x_n - x)\, dt \\
&\leq 2\int_0^T \|\psi\| \|x_n - x\|\, dt \\
&\leq 2\|\psi\|_{L^2} \|x_n - x\|_{L^2} \to 0 \quad \text{as } n \to \infty.
\end{aligned}
\tag{3}
$$

By hypothesis $H(A)$, it follows that

$$
\begin{aligned}
\int_0^T \big(A(t,x_n) - A(t,x), x_n - x\big)\, dt &\geq p\int_0^T \|x_n - x\|^2\, dt \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}
\tag{4}
$$

So, we can find $\tau \in I \setminus \Lambda$ such that

$$\big\|x_n(\tau) - x(\tau)\big\| \to 0 \quad \text{as } n \to \infty.$$

Using the integration by parts formula for functions in $W^{1,2}(I;R^N)$, for any $t \in I$ we have

$$
\begin{aligned}
\big\|x_n(t) - x(t)\big\|^2 &= \big\|x_n(\tau) - x(\tau)\big\|^2 + 2\int_\tau^t \big(\dot{x}_n(s) - \dot{x}(s), x_n(s) - x(s)\big)\, ds \\
&\leq \big\|x_n(\tau) - x(\tau)\big\|^2 + 4\big\|\varphi(t)\big\|_{L^2} \big\|x_n(t) - x(t)\big\|_{L^2}.
\end{aligned}
\tag{5}
$$

By (5), we see that

$$\max_{t \in I} \big\|x_n(t) - x(t)\big\| \to 0 \quad \text{as } n \to \infty.$$

So, $x_n(t) \to x(t)$ in $C(I; R^N)$. Since $x = L^{-1}(h)$ with $h \in W$, we conclude that $L^{-1}(W) \subseteq C(I; R^N)$ is compact. From Lemma 2.2, we can find a continuous map $f : \hat{K} \to L^2(I; R^N)$ such that $f(x)(t) \in \text{Ext}\, F(t, x(t))$ a.e. on I for all $x \in \hat{K}$. Then $L^{-1} \circ f$ is a compact operator. On applying the Schauder fixed point theorem, there exists a $x \in \hat{K}$ such that $x = L^{-1} \circ f(x)$. This is a solution of (1), and so $S_e \neq \emptyset$ in $W^{1,2}(I; R^N)$. $\qquad\square$

For the relation theorem of problem (1), we need the following definition and hypotheses.

Definition 3.1 (see [20]) The multifunction mapping $F : I \times R^N \to P_k(R^N)$ is called 'one-sided Lipschitz (OSL)' continuous if there is an integrable function $\omega : I \to R_+$ such that for every $x, y \in R^N$, $t \in I$, and $v \in F(t, x)$ there exists $u \in F(t, y)$ such that

$$(v - u, x - y) \le \omega(t)\|x - y\|^2.$$

Under the above hypotheses, let S denote the solution set of the equation $\dot{x}(t) + A(t, x(t)) \in F(t, x)$, $x(0) = -x(T)$.

Theorem 3.2 *If hypotheses $H(A)$ and $H(F)$ hold and, moreover, $F(t, x)$ satisfies the OSL condition, we have $\overline{S_e} = S$, where the closure is taken in $C(I; R^N)$.*

Proof Let $x \in S$, then there exist $f \in L^2(I; R^N)$ and $f(x)(t) \in F(t, x(t))$ a.e. on I, such that

$$\begin{aligned} \dot{x}(t) + A\big(t, x(t)\big) &= f(t, x), \\ x(0) &= -x(T). \end{aligned} \tag{6}$$

As before let $W = \{v \in L^2(I; R^N) : \|v\| \le \psi(t) \text{ a.e. on } I\}$, then $\hat{K} = L^{-1}(W) \subseteq W^{1,2}(I; R^N)$ is compact convex subset in $C(I; R^N)$. For every $y \in \hat{K}$, we define the multifunction

$$Q_\epsilon(t) = \big\{v \in F(t, y) : (f - v, x - y) \le \omega(t)\|x - y\|^2 + \epsilon\big\}.$$

Clearly, for every $t \in I$, $Q_\epsilon(t) \neq \emptyset$, and it is graph measurable. On applying the Aumann selection theorem, we get a measurable function $v : I \to R^N$ such that $v(t) \in Q_\epsilon(t)$ almost everywhere on I. So we define the multifunction

$$R_\epsilon(y) = \big\{v \in S_{F(\cdot, y)} : (f - v, x - y) \le \omega(t)\|x - y\|^2 + \epsilon\big\}.$$

We see that $R_\epsilon : \hat{K} \to 2^{L^2(I; R^N)}$ has nonempty and decomposable values. It follows from Theorem 3 of [21] that $\overline{R_\epsilon(\cdot)}$ is LSC. Therefore $y \to \overline{R_\epsilon(y)}$ is LSC and has closed and decomposable values. So we apply Lemma 2.1 to get a continuous map $f_\epsilon : \hat{K} \to L^2(I; R^N)$ such that $f_\epsilon(y) \in \overline{R_\epsilon(y)}$ for all $y \in \hat{K}$. Invoking II-Theorem 8.31 of [22] (in [22, p.260]), we can find a continuous map $g_\epsilon : \hat{K} \to L^2(I; R^N)$ such that $g_\epsilon(y)(t) \in \text{Ext}\, F(t, y)$ almost everywhere on I, and $\|f_\epsilon(y) - g_\epsilon(y)\| \le \epsilon$ for all $y \in \hat{K}$. Now let $\epsilon \to 0$ and set $f_{\epsilon_n} = f_\epsilon$, $g_{\epsilon_n} = g_\epsilon$. Note that $\|g_{\epsilon_n}(y)\| \le \psi(t)$ a.e. on I with $\psi \in L^2(I; R^N)$, so we have $g_{\epsilon_n} \rightharpoonup f_{\epsilon_n}$ in $L^2(I; R^N)$. We consider the following problem:

$$\begin{aligned} \dot{x}(t) + A(x)(t) &= g_{\epsilon_n}(x)(t), \\ x(0) &= -x(T), \end{aligned} \tag{7}$$

where $g_{\epsilon_n}(x) \in \operatorname{Ext} R_\epsilon(x)$. We see that $L^{-1}g_{\epsilon_n} : \hat{K} \to \hat{K}$ is a compact operator and by the Schauder fixed point theorem, we obtain a solution $x_{\epsilon_n} \in S_e \subset W^{1,2}(I;R^N)$ of (7). We see that the sequence $\{x_{\epsilon_n}\}_{n\geq 1} \subset \hat{K}$ is uniformly bounded. So by passing to a subsequence if necessary, we may assume that $x_{\epsilon_n} \rightharpoonup \hat{x}$ in $W^{1,2}(I;R^N)$. From the proof of Theorem 3.1, we know that $x_{\epsilon_n} \to \hat{x}$ in $C(I,H)$ and $\hat{x}(0) = -\hat{x}(T)$. Note that $Lx_{\epsilon_n} - Lx = g_{\epsilon_n}(x_{\epsilon_n}) - f(x)$. So, we have

$$\big(\dot{x}_{\epsilon_n}(t) - \dot{x}(t), x_{\epsilon_n}(t) - x(t)\big) + \big(A(x_{\epsilon_n})(t) - A(x)(t), x_{\epsilon_n}(t) - x(t)\big)$$
$$= \big(g_{\epsilon_n}(x_{\epsilon_n})(t) - f(x)(t), x_{\epsilon_n}(t) - x(t)\big).$$

However,

$$\big(A(x_{\epsilon_n})(t) - A(x)(t), x_{\epsilon_n}(t) - x(t)\big) \geq 0 \quad \text{a.e. } I.$$

Then,

$$\big(\dot{x}_{\epsilon_n}(t) - \dot{x}(t), x_{\epsilon_n}(t) - x(t)\big)$$
$$\leq \big(g_{\epsilon_n}(x_{\epsilon_n})(t) - f(x)(t), x_{\epsilon_n}(t) - x(t)\big)$$
$$= \big(g_{\epsilon_n}(x_{\epsilon_n})(t) - f_{\epsilon_n}(x_{\epsilon_n})(t), x_{\epsilon_n}(t) - x(t)\big) + \big(f_{\epsilon_n}(x_{\epsilon_n})(t) - f(x)(t), x_{\epsilon_n}(t) - x(t)\big).$$

By $g_{\epsilon_n} \rightharpoonup f_{\epsilon_n}$ in $L^2(I;R^N)$ and $x_{\epsilon_n} \to \hat{x}$ in $L^2(I;R^N)$, we have

$$\big(g_{\epsilon_n}(x_{\epsilon_n})(t) - f_{\epsilon_n}(x_{\epsilon_n})(t), x_{\epsilon_n}(t) - x(t)\big)$$
$$= \big(g_{\epsilon_n}(x_{\epsilon_n})(t) - f_{\epsilon_n}(x_{\epsilon_n})(t), x_{\epsilon_n}(t) - \hat{x}(t)\big)$$
$$\quad + \big(g_{\epsilon_n}(x_{\epsilon_n})(t) - f_{\epsilon_n}(x_{\epsilon_n})(t), \hat{x}(t) - x(t)\big) \to 0 \quad \text{a.e. } I. \tag{8}$$

Hence, there exists a constant $N_0 > 0$, and one has

$$\big| \big(g_{\epsilon_n}(x_{\epsilon_n})(t) - f_{\epsilon_n}(x_{\epsilon_n})(t), x_{\epsilon_n}(t) - x(t)\big) \big| < \epsilon$$

as $n > N_0$. It follows that

$$\frac{1}{2}\frac{d}{dt}\|x_{\epsilon_n} - x\|^2 = \big(\dot{x}_{\epsilon_n}(t) - \dot{x}(t), x_{\epsilon_n}(t) - x(t)\big)$$
$$\leq \big(g_{\epsilon_n}(x_{\epsilon_n})(t) - f_{\epsilon_n}(x_{\epsilon_n})(t), x_{\epsilon_n}(t) - x(t)\big)$$
$$\quad + \big(f_{\epsilon_n}(x_{\epsilon_n})(t) - f(x)(t), x_{\epsilon_n}(t) - x(t)\big)$$
$$\leq \big(f_{\epsilon_n}(x_{\epsilon_n})(t) - f(x)(t), x_{\epsilon_n}(t) - x(t)\big) + \epsilon$$
$$\leq \omega(t)\|x_{\epsilon_n} - x\|^2 + \epsilon. \tag{9}$$

Let $S_n(t) = \|x_{\epsilon_n} - x\|^2$, then $\dot{S}_n(t) \leq 2\omega(t)S_n(t) + 2\epsilon$. Integrating over (9) from 0 to t, one has

$$S_n(t) \leq \int_0^t 2\omega(s)S_n(s)\,ds + S(0) + 2\epsilon t.$$

By using the Gronwall inequality, we have

$$S_n(t) \le \int_0^t 2\omega(s)\big(S_n(0) + 2\epsilon s\big) \exp^{2\int_0^t \omega(\mu)\,d\mu} ds + S_n(0) + 2\epsilon t.$$

From $g_{\epsilon_n}(x)(t) \in F(t,x)$, we know that $S_n(0) = 0$. Let $\epsilon \to 0$, we have $S_n(t) \to 0$, *i.e.* $\|x_\epsilon(t) - x(t)\| \to 0$ for any $t \in I$. Therefore, $x = \hat{x}$, *i.e.* $x_\epsilon \to x$ and $x_\epsilon \in S_e$, and so $S \subseteq \overline{S_e}$. Also S is closed in $C(I,H)$ (see the proof of Theorem 3.1), thus $S = \overline{S_e}$. □

4 An application

We present an example of a nonlinear anti-periodic distributed parameter control system, with *a priori* feedback (*i.e.* state dependent control constraint set). Let $T = [0,b]$, $\dot{x} = (\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_N)$. Consider the following control system:

$$\begin{cases} \dot{x} + a(t,x)x = g(t,x)u(t) & \text{a.e. on } T, \\ x(0) = -x(b), \\ u(t) \in \operatorname{Ext} U(t,x(t)) & \text{a.e. on } T. \end{cases} \tag{10}$$

The hypotheses on the data (10) are the following.

$H(a)$: $a : T \times R^N \to R^+, g : T \times R^N \to R$ are Carathéodory functions such that for almost all $t \in T$

$$0 < \theta_1 \le a(t,x) \le \theta_2,$$

$$|g(t,x)| \le \eta_1(t) + \eta_2(t)|x|^\alpha,$$

with $\theta_1, \theta_2 > 0$, $0 < \alpha < 1$, $\eta_1(t) \in L_+^2(T)$, $\eta_2(t) \in L^\infty(T)$.

$H(U)$: $U : T \times R^N \to P_{kc}(R^N)$ is a multifunction such that

(i) for all $x \in R^N$, $t \to U(t,x)$ is measurable;
(ii) for all $t \in T$, $x \to U(t,x)$ is h-continuous;
(iii) for almost all $t \in T$ and all $x \in R^N$, $|U(t,x)| \le \gamma$, with $\gamma > 0$.

Let $A : T \times R^N \to R^N$ be the operator defined by $A(t,x) = a(t,x)x$. Evidently, using hypothesis $H(a)$, it is straightforward to check that A satisfies hypothesis $H(A)$. Also, let $F : T \times R^N \to P_{kc}(R^N)$ be defined by

$$F(t,x) = \big\{ y \in R^N : y(t) = g\big(t,x(t)\big)u(t), u(t) \in U\big(t,x(t)\big) \text{ a.e. on } T \big\}.$$

Using hypotheses $H(a)$ and $H(U)$, it is straightforward to check that F satisfies hypothesis $H(F)_1$.

Rewrite problem (10) in the following equivalent evolution inclusion form:

$$\begin{cases} \dot{x} + A(t,x(t)) \in \operatorname{Ext} F(t,x) & \text{a.e. on } T, \\ x(0) = -x(b). \end{cases} \tag{11}$$

We can apply Theorem 3.1 on problem (11) and obtain the following.

Theorem 4.1 *If the hypotheses $H(a)$ and $H(U)$ hold, then problem* (10) *has a solution* $x \in W^{1,2}(I;R^N)$.

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