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Monotonicity results and inequalities for the inverse hyperbolic sine function

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Abstract
In the paper, the authors present monotonicity results of a function involving the inverse hyperbolic sine. From these, the authors derive some inequalities for bounding the inverse hyperbolic sine.

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Introduction and main results

In [1, Theorem 1.9], the following inequalities were established: for $0 \le x \le r$ and $r > 0$, the double inequality

$$\frac{(a+1)x}{a+\sqrt{1+x^2}} \le \operatorname{arcsinh} x \le \frac{(b+1)x}{b+\sqrt{1+x^2}} \tag{1}$$

holds true if and only if $a \le 2$ and

$$b \ge \frac{\sqrt{1+r^2}\operatorname{arcsinh} r - r}{r - \operatorname{arcsinh} r}. \tag{2}$$

The aim of this paper is to elementarily generalize inequality (1) to monotonicity results and to deduce more inequalities.

Our results may be stated as the following theorems.

Theorem 1 *For $\theta \in \mathbb{R}$, let*

$$f_\theta(x) = \frac{\theta + \sqrt{1+x^2}}{x} \operatorname{arcsinh} x, \quad x > 0. \tag{3}$$

1. *When $\theta \le 2$, the function $f_\theta(x)$ is strictly increasing*;
2. *When $\theta > 2$, the function $f_\theta(x)$ has a unique minimum.*

As straightforward consequences of Theorem 1, the following inequalities are inferred.

Theorem 2 *Let $0 \le x \le r$ and $r > 0$.*

1. *For* $-1 < \theta \le 2$, *the double inequality*

$$\frac{(1+\theta)x}{\theta+\sqrt{1+x^2}} < \operatorname{arcsinh} x \le \frac{(\theta+\sqrt{1+r^2})\operatorname{arcsinh} r}{r}\,\frac{x}{\theta+\sqrt{1+x^2}} \tag{4}$$

   holds true on $(0, r]$, *where the scalars* $1+\theta$ *and*

$$\frac{(\theta+\sqrt{1+r^2})\operatorname{arcsinh} r}{r}$$

   in (4) *are best possible.*
2. *For* $\theta > 2$, *the double inequality*

$$\begin{aligned}\frac{4(1-1/\theta^2)x}{\theta+\sqrt{1+x^2}} &\le \operatorname{arcsinh} x\\ &\le \max\left\{1+\theta, \frac{(\theta+\sqrt{1+r^2})\operatorname{arcsinh} r}{r}\right\}\frac{x}{\theta+\sqrt{1+x^2}}\end{aligned} \tag{5}$$

   holds true on $(0, r]$.

Remarks

Before proving our theorems, we give several remarks on them.

Remark 1 Replacing $\operatorname{arcsinh} x$ by x in (4) and (5) yields

$$\frac{\sinh x}{x} < \begin{cases} \frac{\theta+\cosh x}{1+\theta}, & -1 < \theta \le 2,\\ \frac{\theta+\cosh x}{4(1-1/\theta^2)}, & \theta > 2 \end{cases} \tag{6}$$

and

$$\frac{\sinh x}{x} > \begin{cases} \frac{r(\theta+\cosh x)}{(\theta+\sqrt{1+r^2})\operatorname{arcsinh} r}, & -1 < \theta \le 2,\\ \frac{r(\theta+\cosh x)}{\max\{r(1+\theta),(\theta+\sqrt{1+r^2})\operatorname{arcsinh} r\}}, & \theta > 2 \end{cases} \tag{7}$$

for $x \in (0, \operatorname{arcsinh} r)$. These can be regarded as Oppenheim-type inequalities for the hyperbolic sine and cosine functions. For information on Oppenheim's double inequality for the sine and cosine functions, please refer to [2], [3, Sections 1.7 and 7.6] and closely related references therein.

Remark 2 It is clear that the left-hand side inequality in (4) recovers the left-hand side inequality in (1), while the right-hand side inequalities in (1) and (4) do not include each other.

Remark 3 Let

$$h_x(\theta) = \frac{1-1/\theta^2}{\theta+\sqrt{1+x^2}}$$

for $\theta > 2$ and $x > 0$. Then

$$h'_x(\theta) = \frac{2\sqrt{x^2+1} + 3\theta - \theta^3}{\theta^3(\theta + \sqrt{x^2+1})^2}.$$

Therefore, the function $h_x(\theta)$ attains its maximum

$$\frac{\sqrt[3]{\sqrt{x^2+1}-x}\{[\sqrt[3]{(\sqrt{1+x^2}-x)^2}+1]^2 - \sqrt[3]{(\sqrt{1+x^2}-x)^2}\}}{[\sqrt[3]{(\sqrt{1+x^2}-x)^2}+1]^2[\sqrt[3]{(\sqrt{1+x^2}-x)^2} + \sqrt{x^2+1}\sqrt[3]{\sqrt{x^2+1}-x}+1]}$$

at the point

$$\frac{1+\sqrt[3]{(\sqrt{1+x^2}-x)^2}}{\sqrt[3]{\sqrt{1+x^2}-x}}.$$

Combining this with the fact that the function

$$\theta \mapsto \frac{1+\theta}{\theta + \sqrt{1+x^2}}$$

is increasing and the function

$$\theta \mapsto \frac{\theta + \sqrt{1+r^2}}{\theta + \sqrt{1+x^2}}$$

is decreasing, we establish from Theorem 2 the following best and sharp double inequalities:

$$\frac{3x}{2+\sqrt{1+x^2}} < \operatorname{arcsinh} x \le \frac{[(2+\sqrt{1+r^2})(\operatorname{arcsinh} r)/r]x}{2+\sqrt{1+x^2}} \tag{8}$$

and

$$\begin{aligned} &\frac{4x\sqrt[3]{\sqrt{x^2+1}-x}\{[\sqrt[3]{(\sqrt{1+x^2}-x)^2}+1]^2 - \sqrt[3]{(\sqrt{1+x^2}-x)^2}\}}{[\sqrt[3]{(\sqrt{1+x^2}-x)^2}+1]^2[\sqrt[3]{(\sqrt{1+x^2}-x)^2} + \sqrt{x^2+1}\sqrt[3]{\sqrt{x^2+1}-x}+1]} \\ &\quad < \operatorname{arcsinh} x < \frac{(\sqrt{r^2+1}-1)(\operatorname{arcsinh} r)x}{(\sqrt{r^2+1}\operatorname{arcsinh} r - r) + (r - \operatorname{arcsinh} r)\sqrt{x^2+1}} \end{aligned} \tag{9}$$

for $0 < x < r$.

The famous software Mathematica 7.0 shows that double inequality (9) is better than (8).

Remark 4 By a similar approach to that presented in the next section, we can procure similar monotonicity results and inequalities for the inverse hyperbolic cosine and other inverse trigonometric functions. For more information on this topic, please refer to [2, 4–10] and closely related references therein.

Remark 5 We note that Shafer-type inequalities from [11] were applied recently in [12] for obtaining upper and lower bounds on the Gaussian Q-function.

Proofs of theorems

Now we are in a position to elementarily prove our theorems.

Proof of Theorem 1 Direct differentiation yields

$$
\begin{aligned}
f_\theta'(x) &= \frac{1}{x^2}\left(\theta + \frac{1}{\sqrt{x^2+1}}\right)\left[\frac{x(\theta/\sqrt{x^2+1}+1)}{\theta+1/\sqrt{x^2+1}} - \operatorname{arcsinh} x\right] \\
&\triangleq \frac{1}{x^2}\left(\theta + \frac{1}{\sqrt{x^2+1}}\right)h_\theta(x)
\end{aligned}
$$

and

$$
h_\theta'(x) = \frac{x^2(2-\theta^2+\theta\sqrt{x^2+1})}{\sqrt{x^2+1}(\theta\sqrt{x^2+1}+1)^2} \triangleq \frac{x^2 q_x(\theta)}{\sqrt{x^2+1}(\theta\sqrt{x^2+1}+1)^2}.
$$

The function $q_x(\theta)$ has two zeros

$$
\theta_1(x) = \frac{\sqrt{1+x^2}-\sqrt{9+x^2}}{2}
$$

and

$$
\theta_2(x) = \frac{\sqrt{1+x^2}+\sqrt{9+x^2}}{2}.
$$

They are strictly increasing and have the bounds $-1 \le \theta_1(x) < 0$ and $\theta_2(x) \ge 2$ on $(0,\infty)$. As a result, under the condition $\theta \notin (-1,0)$,

1. when $\theta \le -1$, the function $q_x(\theta)$ and the derivative $h_\theta'(x)$ are negative, and so the function $h_\theta(x)$ is strictly decreasing on $(0,\infty)$;
2. when $0 \le \theta \le 2$, the function $q_x(\theta)$ and the derivative $h_\theta'(x)$ are positive, and so the function $h_\theta(x)$ is strictly increasing on $(0,\infty)$;
3. when $\theta > 2$, the function $q_x(\theta)$ and the derivative $h_\theta'(x)$ have the unique zero $x = \frac{\sqrt{\theta^4-5\theta^2+4}}{\theta}$ which is the unique minimum point of $h_\theta(x)$.

Furthermore, since $\lim_{x\to\infty} h_\theta(x) = \infty$ for $\theta \ge 0$ and $\lim_{x\to 0^+} h_\theta(x) = 0$, it follows that

1. when $\theta \le -1$, the function $h_\theta(x)$ is negative, and so the derivative $f_\theta'(x)$ is positive, that is, the function $f_\theta(x)$ is strictly increasing on $(0,\infty)$;
2. when $0 \le \theta \le 2$, the function $h_\theta(x)$ is positive, and so the derivative $f_\theta'(x)$ is also positive, accordingly, the function $f_\theta(x)$ is strictly increasing on $(0,\infty)$;
3. when $\theta > 2$, the function $h_\theta(x)$ and the derivative $f_\theta'(x)$ have a unique zero as a solution of the equation

   $$
   \frac{x\sqrt{x^2+1}-\ln(x+\sqrt{x^2+1})}{\sqrt{x^2+1}\ln(x+\sqrt{x^2+1})-x} = \theta,
   $$

   which is the unique minimum point of the function $f_\theta(x)$ on $(0,\infty)$.

On the other hand, when $\theta \in (-1,0)$, we have

$$
\left[x^2 f_\theta'(x)\right]' = \frac{x^2[\sqrt{x^2+1}+(\operatorname{arcsinh} x)/x-\theta]}{(x^2+1)^{3/2}} > 0
$$

which means that the function $x^2 f'_\theta(x)$ is strictly increasing on $(0,\infty)$. From the limit $\lim_{x\to 0^+}[x^2 f'_\theta(x)] = 0$, it is derived that the function $x^2 f'_\theta(x)$ is positive. Hence the function $f_\theta(x)$ is strictly increasing on $(0,\infty)$. The proof of Theorem 1 is complete. □

Proof of Theorem 2 Since $\lim_{x\to 0^+} f_\theta(x) = 1+\theta$, by Theorem 1, it is easy to see that $1+\theta < f_\theta(x) \le f_\theta(r)$ on $(0,r]$ for $-1<\theta\le 2$. Inequality (4) is thus proved.

For $\theta > 2$, the minimum point $x_0 \in (0,\infty)$ satisfies

$$\operatorname{arcsinh} x_0 = \frac{x_0(\theta/\sqrt{x_0^2+1}+1)}{\theta+1/\sqrt{x_0^2+1}}.$$

Therefore, the minimum of the function $f_\theta(x)$ on $(0,\infty)$ equals

$$\begin{aligned}\frac{(\theta+\sqrt{1+x_0^2})\operatorname{arcsinh} x_0}{x_0} &= \frac{x_0(\theta/\sqrt{x_0^2+1}+1)}{\theta+1/\sqrt{x_0^2+1}} \cdot \frac{(\theta+\sqrt{1+x_0^2})}{x_0} \\ &= \frac{(\theta/\sqrt{x_0^2+1}+1)(\theta+\sqrt{1+x_0^2})}{\theta+1/\sqrt{x_0^2+1}} \\ &= \frac{(\theta+\sqrt{1+x_0^2})^2}{\theta\sqrt{x_0^2+1}+1} \ge 4\left(1-\frac{1}{\theta^2}\right).\end{aligned}$$

From this, it is obtained that

$$4\left(1-\frac{1}{\theta^2}\right) \le f_\theta(x) \le \max\left\{\lim_{x\to 0^+} f_\theta(x), f_\theta(r)\right\}$$

for $x \in (0,r]$, which implies inequality (5). The proof of Theorem 2 is thus completed. □

Remark 6 This paper is a slightly revised version of the preprint [13].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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References

1. Zhu, L: New inequalities of Shafer-Fink type for arc-hyperbolic sine. J. Inequal. Appl. **2008**, Article ID 368275 (2008). doi:10.1155/2008/368275
2. Qi, F, Luo, Q-M, Guo, B-N: A simple proof of Oppenheim's double inequality relating to the cosine and sine functions. J. Math. Inequal. **6**(4), 645-654 (2012). doi:10.7153/jmi-06-63
3. Qi, F, Niu, D-W, Guo, B-N: Refinements, generalizations, and applications of Jordan's inequality and related problems. J. Inequal. Appl. **2009**, Article ID 271923 (2009). doi:10.1155/2009/271923
4. Guo, B-N, Luo, Q-M, Qi, F: Sharpening and generalizations of Shafer-Fink's double inequality for the arc sine function. Filomat **27**(2), 261-265 (2013). doi:10.2298/FIL1302261G
5. Guo, B-N, Qi, F: Sharpening and generalizations of Carlson's inequality for the arc cosine function. Hacet. J. Math. Stat. **39**(3), 403-409 (2010)
6. Pan, W, Zhu, L: Generalizations of Shafer-Fink-type inequalities for the arc sine function. J. Inequal. Appl. **2009**, Article ID 705317 (2009). doi:10.1155/2009/705317
7. Qi, F, Guo, B-N: Sharpening and generalizations of Shafer's inequality for the arc sine function. Integral Transforms Spec. Funct. **23**(2), 129-134 (2012). doi:10.1080/10652469.2011.564578
8. Qi, F, Zhang, S-Q, Guo, B-N: Sharpening and generalizations of Shafer's inequality for the arc tangent function. J. Inequal. Appl. **2009**, Article ID 930294 (2009). doi:10.1155/2009/930294
9. Zhang, S-Q, Guo, B-N: Monotonicity results and inequalities for the inverse hyperbolic sine. Chin. Q. J. Math. **24**(3), 384-388 (2009)
10. Zhao, J-L, Wei, C-F, Guo, B-N, Qi, F: Sharpening and generalizations of Carlson's double inequality for the arc cosine function. Hacet. J. Math. Stat. **41**(2), 201-209 (2012)
11. Malešević, BJ: An application of λ-method on Shafer-Fink's inequality. Publ. Elektroteh. Fak. Univ. Beogr., Mat. **8**, 90-92 (1997)
12. de Abreu, GTF: Arbitrarily tight upper and lower bounds on the Gaussian *q*-function and related functions. In: 2009 IEEE International Conference on Communications (ICC 2009), vols. 1-8, pp. 1944-1949. Dresden, Germany, June 14-18 (2009). doi:10.1109/ICC.2009.5198762
13. Qi, F, Guo, B-N: Monotonicity results and bounds for the inverse hyperbolic sine. arXiv:0903.1676

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