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Bounds for the second Hankel determinant of certain univalent functions

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Dedicated to Professor Hari M Srivastava

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Abstract

The estimates for the second Hankel determinant $a_2a_4 - a_3^2$ of the analytic function $f(z) = z + a_2z^2 + a_3z^3 + \cdots$, for which either $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ is subordinate to a certain analytic function, are investigated. The estimates for the Hankel determinant for two other classes are also obtained. In particular, the estimates for the Hankel determinant of strongly starlike, parabolic starlike and lemniscate starlike functions are obtained.

MSC: 30C45; 30C80

1 Introduction

Let $\mathcal{A}$ denote the class of all analytic functions

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots \tag{1}$$

defined on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. The *Hankel determinants* $H_q(n)$ ($n = 1, 2, \ldots, q = 1, 2, \ldots$) of the function f are defined by

$$H_q(n) := \begin{bmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{bmatrix} \quad (a_1 = 1).$$

Hankel determinants are useful, for example, in showing that a function of bounded characteristic in $\mathbb{D}$, *i.e.*, a function which is a ratio of two bounded analytic functions with its Laurent series around the origin having integral coefficients, is rational [1]. For the use of Hankel determinants in the study of meromorphic functions, see [2], and various properties of these determinants can be found in [3, Chapter 4]. In 1966, Pommerenke [4] investigated the Hankel determinant of areally mean p-valent functions, univalent functions as well as of starlike functions. In [5], he proved that the Hankel determinants of univalent functions satisfy

$$\left|H_q(n)\right| < Kn^{-(\frac{1}{2}+\beta)q+\frac{3}{2}} \quad (n = 1, 2, \ldots, q = 2, 3, \ldots),$$

where $\beta > 1/4000$ and K depends only on q. Later, Hayman [6] proved that $|H_2(n)| < An^{1/2}$ ($n = 1, 2, \ldots; A$ an absolute constant) for areally mean univalent functions. In [7–9], the estimates for the Hankel determinant of areally mean p-valent functions were investigated. ElHosh obtained bounds for Hankel determinants of univalent functions with a positive Hayman index α [10] and of k-fold symmetric and close-to-convex functions [11]. For bounds on the Hankel determinants of close-to-convex functions, see [12–14]. Noor studied the Hankel determinant of Bazilevic functions in [15] and of functions with bounded boundary rotation in [16–19]. In the recent years, several authors have investigated bounds for the Hankel determinant of functions belonging to various subclasses of univalent and multivalent functions [20–27]. The Hankel determinant $H_2(1) = a_3 - a_2^2$ is the well-known Fekete-Szegö functional. For results related to this functional, see [28, 29]. The second Hankel determinant $H_2(2)$ is given by $H_2(2) = a_2a_4 - a_3^2$.

An analytic function f is *subordinate* to an analytic function g, written $f(z) \prec g(z)$, if there is an analytic function $w : \mathbb{D} \to \mathbb{D}$ with $w(0) = 0$ satisfying $f(z) = g(w(z))$. Ma and Minda [30] unified various subclasses of starlike ($\mathcal{S}^*$) and convex functions ($\mathcal{C}$) by requiring that either of the quantity $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ is subordinate to a function φ with a positive real part in the unit disk $\mathbb{D}$, $\varphi(0) = 1$, $\varphi'(0) > 0$, φ maps $\mathbb{D}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. He obtained distortion, growth and covering estimates as well as bounds for the initial coefficients of the unified classes.

The bounds for the second Hankel determinant $H_2(2) = a_2a_4 - a_3^2$ are obtained for functions belonging to these subclasses of Ma-Minda starlike and convex functions in Section 2. In Section 3, the problem is investigated for two other related classes defined by subordination. In proving our results, we do not assume the univalence or starlikeness of φ as they were required only in obtaining the distortion, growth estimates and the convolution theorems. The classes introduced by subordination naturally include several well-known classes of univalent functions and the results for some of these special classes are indicated as corollaries.

Let $\mathcal{P}$ be the class of *functions with positive real part* consisting of all analytic functions $p : \mathbb{D} \to \mathbb{C}$ satisfying $p(0) = 1$ and $\operatorname{Re} p(z) > 0$. We need the following results about the functions belonging to the class $\mathcal{P}$.

Lemma 1 [31] *If the function $p \in \mathcal{P}$ is given by the series*

$$p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots, \tag{2}$$

then the following sharp estimate holds:

$$|c_n| \leq 2 \quad (n = 1, 2, \ldots). \tag{3}$$

Lemma 2 [32] *If the function $p \in \mathcal{P}$ is given by the series* (2), *then*

$$2c_2 = c_1^2 + x(4 - c_1^2), \tag{4}$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z, \tag{5}$$

for some x, z with $|x| \leq 1$ and $|z| \leq 1$.

2 Second Hankel determinant of Ma-Minda starlike/convex functions

Subclasses of starlike functions are characterized by the quantity $zf'(z)/f(z)$ lying in some domain in the right half-plane. For example, f is strongly starlike of order β if $zf'(z)/f(z)$ lies in a sector $|\arg w| < \beta\pi/2$, while it is starlike of order α if $zf'(z)/f(z)$ lies in the half-plane $\operatorname{Re} w > \alpha$. The various subclasses of starlike functions were unified by subordination in [30]. The following definition of the class of Ma-Minda starlike functions is the same as the one in [30] except for the omission of starlikeness assumption of φ.

Definition 1 Let $\varphi : \mathbb{D} \to \mathbb{C}$ be analytic, and let the Maclaurin series of φ be given by

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1, B_2 \in \mathbb{R}, B_1 > 0). \tag{6}$$

The class $\mathcal{S}^*(\varphi)$ of *Ma-Minda starlike functions with respect to* φ consists of functions $f \in \mathcal{A}$ satisfying the subordination

$$\frac{zf'(z)}{f(z)} \prec \varphi(z).$$

For the function φ given by $\varphi_\alpha(z) := (1 + (1 - 2\alpha)z)/(1 - z)$, $0 < \alpha \leq 1$, the class $\mathcal{S}^*(\alpha) := \mathcal{S}^*(\varphi_\alpha)$ is the well-known class of starlike functions of order α. Let

$$\varphi_{PAR}(z) := 1 + \frac{2}{\pi^2}\left(\log\frac{1 + \sqrt{z}}{1 - \sqrt{z}}\right)^2.$$

Then the class

$$\mathcal{S}_P^* := \mathcal{S}^*(\varphi_{PAR}) = \left\{f \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|\right\}$$

is the *parabolic starlike* functions introduced by Rønning [33]. For a survey of parabolic starlike functions and the related class of uniformly convex functions, see [34]. For $0 < \beta \leq 1$, the class

$$\mathcal{S}_\beta^* := \mathcal{S}^*\left(\left(\frac{1 + z}{1 - z}\right)^\beta\right) = \left\{f \in \mathcal{A} : \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\beta\pi}{2}\right\}$$

is the familiar class of *strongly starlike functions of order* β. The class

$$\mathcal{S}_L^* := \mathcal{S}^*(\sqrt{1 + z}) = \left\{f \in \mathcal{A} : \left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 1\right\}$$

is the class of *lemniscate starlike* functions studied in [35].

Theorem 1 *Let the function* $f \in \mathcal{S}^*(\varphi)$ *be given by* (1).

1. *If* B_1, B_2 *and* B_3 *satisfy the conditions*

$$|B_2| \leq B_1, \qquad \left|4B_1B_3 - B_1^4 - 3B_2^2\right| - 3B_1^2 \leq 0,$$

then the second Hankel determinant satisfies

$$\left|a_2a_4 - a_3^2\right| \le \frac{B_1^2}{4}.$$

2. *If* B_1, B_2 *and* B_3 *satisfy the conditions*

$$|B_2| \ge B_1, \qquad \left|4B_1B_3 - B_1^4 - 3B_2^2\right| - B_1|B_2| - 2B_1^2 \ge 0,$$

or the conditions

$$|B_2| \le B_1, \qquad \left|4B_1B_3 - B_1^4 - 3B_2^2\right| - 3B_1^2 \ge 0,$$

then the second Hankel determinant satisfies

$$\left|a_2a_4 - a_3^2\right| \le \frac{1}{12}\left|4B_1B_3 - B_1^4 - 3B_2^2\right|.$$

3. *If* B_1, B_2 *and* B_3 *satisfy the conditions*

$$|B_2| > B_1, \qquad \left|4B_1B_3 - B_1^4 - 3B_2^2\right| - B_1|B_2| - 2B_1^2 \le 0,$$

then the second Hankel determinant satisfies

$$\left|a_2a_4 - a_3^2\right| \le \frac{B_1^2}{12}\left(\frac{3|4B_1B_3 - B_1^4 - 3B_2^2| - 4B_1|B_2| + 4B_1^2 - B_2^2}{|4B_1B_3 - B_1^4 - 3B_2^2| - 2B_1|B_2| - B_1^2}\right).$$

Proof Since $f \in \mathcal{S}^*(\varphi)$, there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ in $\mathbb{D}$ such that

$$\frac{zf'(z)}{f(z)} = \varphi\bigl(w(z)\bigr). \tag{7}$$

Define the functions p_1 by

$$p_1(z) := \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + \cdots,$$

or, equivalently,

$$w(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2}\left(c_1z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \cdots\right). \tag{8}$$

Then p_1 is analytic in $\mathbb{D}$ with $p_1(0) = 1$ and has a positive real part in $\mathbb{D}$. By using (8) together with (6), it is evident that

$$\varphi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) = 1 + \frac{1}{2}B_1c_1z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right)z^2 + \cdots. \tag{9}$$

Since

$$\frac{zf'(z)}{f(z)} = 1 + a_2z + \left(-a_2^2 + 2a_3\right)z^2 + \left(3a_4 - 3a_2a_3 + a_2^3\right)z^3 + \cdots, \tag{10}$$

it follows by (7), (9) and (10) that

$$\begin{aligned} a_2 &= \frac{B_1 c_1}{2}, \\ a_3 &= \frac{1}{8}\big[\big(B_1^2 - B_1 + B_2\big)c_1^2 + 2B_1 c_2\big], \\ a_4 &= \frac{1}{48}\big[\big(-4B_2 + 2B_1 + B_1^3 - 3B_1^2 + 3B_1B_2 + 2B_3\big)c_1^3 \\ &\quad + 2\big(3B_1^2 - 4B_1 + 4B_2\big)c_1c_2 + 8B_1c_3\big]. \end{aligned}$$

Therefore

$$a_2a_4 - a_3^2 = \frac{B_1}{96}\left[c_1^4\left(-\frac{B_1^3}{2} + \frac{B_1}{2} - B_2 + 2B_3 - \frac{3B_2^2}{2B_1}\right) + 2c_2c_1^2(B_2 - B_1) + 8B_1c_1c_3 - 6B_1c_2^2\right].$$

Let

$$\begin{aligned} &d_1 = 8B_1, \qquad d_2 = 2(B_2 - B_1), \\ &d_3 = -6B_1, \qquad d_4 = -\frac{B_1^3}{2} + \frac{B_1}{2} - B_2 + 2B_3 - \frac{3B_2^2}{2B_1}, \\ &T = \frac{B_1}{96}. \end{aligned} \tag{11}$$

Then

$$\big|a_2a_4 - a_3^2\big| = T\big|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4\big|. \tag{12}$$

Since the function $p(e^{i\theta}z)$ ($\theta \in \mathbb{R}$) is in the class $\mathcal{P}$ for any $p \in \mathcal{P}$, there is no loss of generality in assuming $c_1 > 0$. Write $c_1 = c$, $c \in [0,2]$. Substituting the values of c_2 and c_3 respectively from (4) and (5) in (12), we obtain

$$\begin{aligned} \big|a_2a_4 - a_3^2\big| = \frac{T}{4}\big|&c^4(d_1 + 2d_2 + d_3 + 4d_4) + 2xc^2\big(4 - c^2\big)(d_1 + d_2 + d_3) \\ &+ \big(4 - c^2\big)x^2\big(-d_1c^2 + d_3\big(4 - c^2\big)\big) + 2d_1c\big(4 - c^2\big)\big(1 - |x|^2\big)z\big|. \end{aligned}$$

Replacing $|x|$ by μ and substituting the values of d_1, d_2, d_3 and d_4 from (11) yield

$$\begin{aligned} \big|a_2a_4 - a_3^2\big| &\le \frac{T}{4}\left[c^4\left|-2B_1^3 + 8B_3 - 6\frac{B_2^2}{B_1}\right| + 4|B_2|\mu c^2\big(4 - c^2\big)\right. \\ &\qquad \left. + \mu^2\big(4 - c^2\big)\big(2B_1c^2 + 24B_1\big) + 16B_1c\big(4 - c^2\big)\big(1 - \mu^2\big)\right] \\ &= T\left[\frac{c^4}{4}\left|-2B_1^3 + 8B_3 - 6\frac{B_2^2}{B_1}\right| + 4B_1c\big(4 - c^2\big) + |B_2|\big(4 - c^2\big)\mu c^2\right. \\ &\qquad \left. + \frac{B_1}{2}\mu^2\big(4 - c^2\big)(c - 6)(c - 2)\right] \\ &\equiv F(c, \mu). \end{aligned} \tag{13}$$

Note that for $(c,\mu) \in [0,2] \times [0,1]$, differentiating $F(c,\mu)$ in (13) partially with respect to μ yields

$$\frac{\partial F}{\partial \mu} = T\big[|B_2|\big(4-c^2\big) + B_1\mu\big(4-c^2\big)(c-2)(c-6)\big]. \tag{14}$$

Then, for $0 < \mu < 1$ and for any fixed c with $0 < c < 2$, it is clear from (14) that $\frac{\partial F}{\partial \mu} > 0$, that is, $F(c,\mu)$ is an increasing function of μ. Hence, for fixed $c \in [0,2]$, the maximum of $F(c,\mu)$ occurs at $\mu = 1$, and

$$\max F(c,\mu) = F(c,1) \equiv G(c).$$

Also note that

$$G(c) = \frac{B_1}{96}\left[\frac{c^4}{4}\left(\left|-2B_1^3 + 8B_3 - 6\frac{B_2^2}{B_1}\right| - 4|B_2| - 2B_1\right) + 4c^2\big(|B_2| - B_1\big) + 24B_1\right].$$

Let

$$\begin{aligned} P &= \frac{1}{4}\left(\left|-2B_1^3 + 8B_3 - 6\frac{B_2^2}{B_1}\right| - 4|B_2| - 2B_1\right), \\ Q &= 4\big(|B_2| - B_1\big), \\ R &= 24B_1. \end{aligned} \tag{15}$$

Since

$$\max_{0\le t\le 4}\big(Pt^2 + Qt + R\big) = \begin{cases} R, & Q \le 0, P \le -\frac{Q}{4}; \\ 16P + 4Q + R, & Q \ge 0, P \ge -\frac{Q}{8} \text{ or } Q \le 0, P \ge -\frac{Q}{4}; \\ \frac{4PR-Q^2}{4P}, & Q > 0, P \le -\frac{Q}{8}, \end{cases} \tag{16}$$

we have

$$\big|a_2a_4 - a_3^2\big| \le \frac{B_1}{96}\begin{cases} R, & Q \le 0, P \le -\frac{Q}{4}; \\ 16P + 4Q + R, & Q \ge 0, P \ge -\frac{Q}{8} \text{ or } Q \le 0, P \ge -\frac{Q}{4}; \\ \frac{4PR-Q^2}{4P}, & Q > 0, P \le -\frac{Q}{8}, \end{cases}$$

where P, Q, R are given by (15). □

Remark 1 When $B_1 = B_2 = B_3 = 2$, Theorem 1 reduces to [24, Theorem 3.1].

Corollary 1

1. *If $f \in \mathcal{S}^*(\alpha)$, then* $|a_2a_4 - a_3^2| \le (1-\alpha)^2$.
2. *If $f \in \mathcal{S}^*_L$, then* $|a_2a_4 - a_3^2| \le 1/16 = 0.0625$.
3. *If $f \in \mathcal{S}^*_P$, then* $|a_2a_4 - a_3^2| \le 16/\pi^4 \approx 0.164255$.
4. *If $f \in \mathcal{S}^*_\beta$, then* $|a_2a_4 - a_3^2| \le \beta^2$.

Definition 2 Let $\varphi : \mathbb{D} \to \mathbb{C}$ be analytic, and let $\varphi(z)$ be given as in (6). The class $\mathcal{C}(\varphi)$ of *Ma-Minda convex functions with respect to* φ consists of functions f satisfying the subordination

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z).$$

Theorem 2 *Let the function* $f \in \mathcal{C}(\varphi)$ *be given by* (1).

1. *If* B_1, B_2 *and* B_3 *satisfy the conditions*

$$B_1^2 + 4|B_2| - 2B_1 \leq 0, \qquad \left|6B_1B_3 + B_1^2B_2 - B_1^4 - 4B_2^2\right| - 4B_1^2 \leq 0,$$

then the second Hankel determinant satisfies

$$\left|a_2a_4 - a_3^2\right| \leq \frac{B_1^2}{36}.$$

2. *If* B_1, B_2 *and* B_3 *satisfy the conditions*

$$B_1^2 + 4|B_2| - 2B_1 \geq 0, \qquad 2\left|6B_1B_3 + B_1^2B_2 - B_1^4 - 4B_2^2\right| - B_1^3 - 4B_1|B_2| - 6B_1^2 \geq 0,$$

or the conditions

$$B_1^2 + 4|B_2| - 2B_1 \leq 0, \qquad \left|6B_1B_3 + B_1^2B_2 - B_1^4 - 4B_2^2\right| - 4B_1^2 \geq 0,$$

then the second Hankel determinant satisfies

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{144}\left|6B_1B_3 + B_1^2B_2 - B_1^4 - 4B_2^2\right|.$$

3. *If* B_1, B_2 *and* B_3 *satisfy the conditions*

$$B_1^2 + 4|B_2| - 2B_1 > 0, \qquad 2\left|6B_1B_3 + B_1^2B_2 - B_1^4 - 4B_2^2\right| - B_1^3 - 4B_1|B_2| - 6B_1^2 \leq 0,$$

then the second Hankel determinant satisfies

$$\left|a_2a_4 - a_3^2\right| \leq \frac{B_1^2}{576}\left(\frac{\begin{array}{c}16|6B_1B_3 + B_1^2B_2 - B_1^4 - 4B_2^2| - 12B_1^3 - 48B_1|B_2| \\ - 36B_1^2 - B_1^4 - 8B_1^2|B_2| - 16B_2^2\end{array}}{|6B_1B_3 + B_1^2B_2 - B_1^4 - 4B_2^2| - B_1^3 - 4B_1|B_2| - 2B_1^2}\right).$$

Proof Since $f \in \mathcal{C}(\varphi)$, there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ in $\mathbb{D}$ such that

$$1 + \frac{zf''(z)}{f'(z)} = \varphi\bigl(w(z)\bigr). \tag{17}$$

Since

$$1 + \frac{zf''(z)}{f'(z)} = 1 + 2a_2z + \bigl(-4a_2^2 + 6a_3\bigr)z^2 + \bigl(8a_2^3 - 18a_2a_3 + 12a_4\bigr)z^3 + \cdots, \tag{18}$$

equations (9), (17) and (18) yield

$$\begin{aligned}
a_2 &= \frac{B_1 c_1}{4}, \\
a_3 &= \frac{1}{24}\bigl[\bigl(B_1^2 - B_1 + B_2\bigr)c_1^2 + 2B_1 c_2\bigr], \\
a_4 &= \frac{1}{192}\bigl[\bigl(-4B_2 + 2B_1 + B_1^3 - 3B_1^2 + 3B_1B_2 + 2B_3\bigr)c_1^3 \\
&\quad + 2\bigl(3B_1^2 - 4B_1 + 4B_2\bigr)c_1c_2 + 8B_1c_3\bigr].
\end{aligned}$$

Therefore

$$\begin{aligned}
a_2a_4 - a_3^2 = \frac{B_1}{768}\biggl[c_1^4\biggl(-\frac{4}{3}B_2 + \frac{2}{3}B_1 - \frac{1}{3}B_1^3 - \frac{1}{3}B_1^2 + \frac{1}{3}B_1B_2 + 2B_3 - \frac{4}{3}\frac{B_2^2}{B_1}\biggr) \\
+ \frac{2}{3}c_2c_1^2\bigl(B_1^2 - 4B_1 + 4B_2\bigr) + 8B_1c_1c_3 - \frac{16}{3}B_1c_2^2\biggr].
\end{aligned}$$

By writing

$$\begin{aligned}
&d_1 = 8B_1, \qquad d_2 = \frac{2}{3}\bigl(B_1^2 - 4B_1 + 4B_2\bigr), \\
&d_3 = -\frac{16}{3}B_1, \qquad d_4 = -\frac{4}{3}B_2 + \frac{2}{3}B_1 - \frac{1}{3}B_1^3 - \frac{1}{3}B_1^2 + \frac{1}{3}B_1B_2 + 2B_3 - \frac{4}{3}\frac{B_2^2}{B_1}, \\
&T = \frac{B_1}{768},
\end{aligned} \tag{19}$$

we have

$$\bigl|a_2a_4 - a_3^2\bigr| = T\bigl|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4\bigr|. \tag{20}$$

Similar as in Theorems 1, it follows from (4) and (5) that

$$\begin{aligned}
\bigl|a_2a_4 - a_3^2\bigr| = \frac{T}{4}\bigl|c^4(d_1 + 2d_2 + d_3 + 4d_4) + 2xc^2\bigl(4 - c^2\bigr)(d_1 + d_2 + d_3) \\
+ \bigl(4 - c^2\bigr)x^2\bigl(-d_1c^2 + d_3\bigl(4 - c^2\bigr)\bigr) + 2d_1c\bigl(4 - c^2\bigr)\bigl(1 - |x|^2\bigr)z\bigr|.
\end{aligned}$$

Replacing $|x|$ by μ and then substituting the values of d_1, d_2, d_3 and d_4 from (19) yield

$$\begin{aligned}
\bigl|a_2a_4 - a_3^2\bigr| &\le \frac{T}{4}\biggl[c^4\biggl|-\frac{4}{3}B_1^3 + \frac{4}{3}B_1B_2 + 8B_3 - \frac{16}{3}\frac{B_2^2}{B_1}\biggr| \\
&\quad + 2\mu c^2\bigl(4 - c^2\bigr)\biggl(\frac{2}{3}B_1^2 + \frac{8}{3}|B_2|\biggr) \\
&\quad + \mu^2\bigl(4 - c^2\bigr)\biggl(\frac{8}{3}B_1c^2 + \frac{64}{3}B_1\biggr) + 16B_1c\bigl(4 - c^2\bigr)\bigl(1 - \mu^2\bigr)\biggr] \\
&= T\biggl[\frac{c^4}{3}\biggl|-B_1^3 + B_1B_2 + 6B_3 - 4\frac{B_2^2}{B_1}\biggr| + 4B_1c\bigl(4 - c^2\bigr) \\
&\quad + \frac{1}{3}\mu c^2\bigl(4 - c^2\bigr)\bigl(B_1^2 + 4|B_2|\bigr)
\end{aligned}$$

$$\begin{aligned} &+ \frac{2B_1}{3}\mu^2\left(4-c^2\right)(c-4)(c-2)\Bigg] \\ &\equiv F(c,\mu). \end{aligned} \tag{21}$$

Again, differentiating $F(c,\mu)$ in (21) partially with respect to μ yields

$$\frac{\partial F}{\partial \mu} = T\left[\frac{c^2}{3}\left(4-c^2\right)\left(B_1^2+4|B_2|\right)+\frac{4B_1}{3}\mu\left(4-c^2\right)(c-4)(c-2)\right]. \tag{22}$$

It is clear from (22) that $\frac{\partial F}{\partial \mu} > 0$. Thus $F(c,\mu)$ is an increasing function of μ for $0 < \mu < 1$ and for any fixed c with $0 < c < 2$. So, the maximum of $F(c,\mu)$ occurs at $\mu = 1$ and

$$\max F(c,\mu) = F(c,1) \equiv G(c).$$

Note that

$$\begin{aligned} G(c) = T\Bigg[&\frac{c^4}{3}\left(\left|-B_1^3+B_1B_2+6B_3-4\frac{B_2^2}{B_1}\right|-B_1^2-4|B_2|-2B_1\right) \\ &+\frac{4}{3}c^2\left(B_1^2+4|B_2|-2B_1\right)+\frac{64}{3}B_1\Bigg]. \end{aligned}$$

Let

$$\begin{aligned} &P = \frac{1}{3}\left(\left|-B_1^3+B_1B_2+6B_3-4\frac{B_2^2}{B_1}\right|-B_1^2-4|B_2|-2B_1\right), \\ &Q = \frac{4}{3}\left(B_1^2+4|B_2|-2B_1\right), \\ &R = \frac{64}{3}B_1. \end{aligned} \tag{23}$$

By using (16), we have

$$\left|a_2a_4-a_3^2\right| \le \frac{B_1}{768}\begin{cases} R, & Q \le 0, P \le -\frac{Q}{4}; \\ 16P+4Q+R, & Q \ge 0, P \ge -\frac{Q}{8} \text{ or } Q \le 0, P \ge -\frac{Q}{4}; \\ \frac{4PR-Q^2}{4P}, & Q > 0, P \le -\frac{Q}{8}, \end{cases}$$

where P, Q, R are given in (23). □

Remark 2 For the choice of $\varphi(z) = (1+z)/(1-z)$, Theorem 2 reduces to [24, Theorem 3.2].

3 Further results on the second Hankel determinant

Definition 3 Let $\varphi : \mathbb{D} \to \mathbb{C}$ be analytic, and let $\varphi(z)$ be as given in (6). Let $0 \le \gamma \le 1$ and $\tau \in \mathbb{C} \setminus \{0\}$. A function $f \in \mathcal{A}$ is in the class $\mathcal{R}_\gamma^\tau(\varphi)$ if it satisfies the following subordination:

$$1 + \frac{1}{\tau}\left(f'(z) + \gamma z f''(z) - 1\right) \prec \varphi(z).$$

Theorem 3 *Let $0 \leq \gamma \leq 1$, $\tau \in \mathbb{C} \setminus \{0\}$, and let the function f as in* (1) *be in the class $\mathcal{R}_{\gamma}^{\tau}(\varphi)$. Also, let*

$$p = \frac{8}{9}\frac{(1+\gamma)(1+3\gamma)}{(1+2\gamma)^2}.$$

1. *If B_1, B_2 and B_3 satisfy the conditions*

$$2|B_2|(1-p) + B_1(1-2p) \leq 0, \qquad \left|B_1B_3 - pB_2^2\right| - pB_1^2 \leq 0,$$

then the second Hankel determinant satisfies

$$\left|a_2a_4 - a_3^2\right| \leq \frac{|\tau|^2 B_1^2}{9(1+2\gamma)^2}.$$

2. *If B_1, B_2 and B_3 satisfy the conditions*

$$2|B_2|(1-p) + B_1(1-2p) \geq 0, \qquad 2\left|B_1B_3 - pB_2^2\right| - 2(1-p)B_1|B_2| - B_1 \geq 0,$$

or the conditions

$$2|B_2|(1-p) + B_1(1-2p) \leq 0, \qquad \left|B_1B_3 - pB_2^2\right| - B_1^2 \geq 0,$$

then the second Hankel determinant satisfies

$$\left|a_2a_4 - a_3^2\right| \leq \frac{|\tau|^2}{8(1+\gamma)(1+3\gamma)}\left|B_3B_1 - pB_2^2\right|.$$

3. *If B_1, B_2 and B_3 satisfy the conditions*

$$2|B_2|(1-p) + B_1(1-2p) > 0, \qquad 2\left|B_1B_3 - pB_2^2\right| - 2(1-p)B_1|B_2| - B_1^2 \leq 0,$$

then the second Hankel determinant satisfies

$$\left|a_2a_4 - a_3^2\right| \leq \frac{|\tau|^2 B_1^2}{32(1+\gamma)(1+3\gamma)} \times \left(\frac{4p|B_3B_1 - pB_2^2| - 4(1-p)B_1[|B_2|(3-2p) + B_1] - 4B_2^2(1-p)^2 - B_1^2(1-2p)^2}{|B_3B_1 - pB_2^2| - (1-p)B_1(2|B_2| + B_1)}\right).$$

Proof For $f \in \mathcal{R}_{\gamma}^{\tau}(\varphi)$, there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ in $\mathbb{D}$ such that

$$1 + \frac{1}{\tau}\left(f'(z) + \gamma z f''(z) - 1\right) = \varphi\left(w(z)\right). \tag{24}$$

Since f has the Maclaurin series given by (1), a computation shows that

$$\begin{aligned} &1+\frac{1}{\tau}\big(f'(z)+\gamma z f''(z)-1\big) \\ &\quad = 1+\frac{2a_2(1+\gamma)}{\tau}z+\frac{3a_3(1+2\gamma)}{\tau}z^2+\frac{4a_4(1+3\gamma)}{\tau}z^3+\cdots. \end{aligned} \tag{25}$$

It follows from (24), (9) and (25) that

$$\begin{aligned} a_2 &= \frac{\tau B_1 c_1}{4(1+\gamma)}, \\ a_3 &= \frac{\tau B_1}{12(1+2\gamma)}\left[2c_2+c_1^2\left(\frac{B_2}{B_1}-1\right)\right], \\ a_4 &= \frac{\tau}{32(1+3\gamma)}\left[B_1\big(4c_3-4c_1c_2+c_1^3\big)+2B_2c_1\big(2c_2-c_1^2\big)+B_3c_1^3\right]. \end{aligned}$$

Therefore

$$\begin{aligned} &a_2a_4-a_3^2 \\ &\quad = \frac{\tau^2 B_1 c_1}{128(1+\gamma)(1+3\gamma)}\left[B_1\big(4c_3-4c_1c_2+c_1^3\big)+2B_2c_1\big(2c_2-c_1^2\big)+B_3c_1^3\right] \\ &\qquad - \frac{\tau^2 B_1^2}{144(1+2\gamma)^2}\left[4c_2^2+c_1^4\left(\frac{B_2}{B_1}-1\right)^2+4c_2c_1^2\left(\frac{B_2}{B_1}-1\right)\right] \\ &\quad = \frac{\tau^2 B_1^2}{128(1+\gamma)(1+3\gamma)}\left\{\left[\big(4c_1c_3-4c_1^2c_2+c_1^4\big)+\frac{2B_2c_1^2}{B_1}\big(2c_2-c_1^2\big)+\frac{B_3}{B_1}c_1^4\right]\right. \\ &\qquad \left.-\frac{8}{9}\frac{(1+\gamma)(1+3\gamma)}{(1+2\gamma)^2}\left[4c_2^2+c_1^4\left(\frac{B_2}{B_1}-1\right)^2+4c_2c_1^2\left(\frac{B_2}{B_1}-1\right)\right]\right\}, \end{aligned}$$

which yields

$$\begin{aligned} \big|a_2a_4-a_3^2\big| = T\left|4c_1c_3+c_1^4\left[1-2\frac{B_2}{B_1}-p\left(\frac{B_2}{B_1}-1\right)^2+\frac{B_3}{B_1}\right]-4pc_2^2\right. \\ \left. -4c_1^2c_2\left[1-\frac{B_2}{B_1}+p\left(\frac{B_2}{B_1}-1\right)\right]\right|, \end{aligned} \tag{26}$$

where

$$T=\frac{|\tau|^2B_1^2}{128(1+\gamma)(1+3\gamma)} \quad \text{and} \quad p=\frac{8}{9}\frac{(1+\gamma)(1+3\gamma)}{(1+2\gamma)^2}.$$

It can be easily verified that $p \in [\frac{64}{81}, \frac{8}{9}]$ for $0 \le \gamma \le 1$.

Let

$$\begin{aligned} &d_1 = 4, \qquad d_2 = -4\left[1-\frac{B_2}{B_1}+p\left(\frac{B_2}{B_1}-1\right)\right], \\ &d_3 = -4p, \qquad d_4 = 1-2\frac{B_2}{B_1}-p\left(\frac{B_2}{B_1}-1\right)^2+\frac{B_3}{B_1}. \end{aligned} \tag{27}$$

Then (26) becomes

$$\left|a_2a_4 - a_3^2\right| = T\left|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4\right|. \tag{28}$$

It follows that

$$\begin{aligned}\left|a_2a_4 - a_3^2\right| = \frac{T}{4}\big|&c^4(d_1 + 2d_2 + d_3 + 4d_4) + 2xc^2\left(4 - c^2\right)(d_1 + d_2 + d_3)\\ &+ \left(4 - c^2\right)x^2\left(-d_1c^2 + d_3\left(4 - c^2\right)\right) + 2d_1c\left(4 - c^2\right)\left(1 - |x|^2\right)z\big|.\end{aligned}$$

Application of the triangle inequality, replacement of $|x|$ by μ and substituting the values of d_1, d_2, d_3 and d_4 from (27) yield

$$\begin{aligned}\left|a_2a_4 - a_3^2\right| &\le \frac{T}{4}\left[4c^4\left|\frac{B_3}{B_1} - p\frac{B_2^2}{B_1^2}\right| + 8\left|\frac{B_2}{B_1}\right|\mu c^2\left(4 - c^2\right)(1 - p)\right.\\ &\quad\left. + \left(4 - c^2\right)\mu^2\left(4c^2 + 4p\left(4 - c^2\right)\right) + 8c\left(4 - c^2\right)\left(1 - \mu^2\right)\right]\\ &= T\left[c^4\left|\frac{B_3}{B_1} - p\frac{B_2^2}{B_1^2}\right| + 2c\left(4 - c^2\right) + 2\mu\left|\frac{B_2}{B_1}\right|c^2\left(4 - c^2\right)(1 - p)\right.\\ &\quad\left. + \mu^2\left(4 - c^2\right)(1 - p)(c - \alpha)(c - \beta)\right]\\ &\equiv F(c, \mu),\end{aligned} \tag{29}$$

where $\alpha = 2$, $\beta = 2p/(1 - p) > 2$.

Similarly as in the previous proofs, it can be shown that $F(c, \mu)$ is an increasing function of μ for $0 < \mu < 1$. So, for fixed $c \in [0, 2]$, let

$$\max F(c, \mu) = F(c, 1) \equiv G(c),$$

which is

$$\begin{aligned}G(c) = T\Bigg\{&c^4\left[\left|\frac{B_3}{B_1} - p\frac{B_2^2}{B_1^2}\right| - (1 - p)\left(2\left|\frac{B_2}{B_1}\right| + 1\right)\right]\\ &+ 4c^2\left[2\left|\frac{B_2}{B_1}\right|(1 - p) + 1 - 2p\right] + 16p\Bigg\}.\end{aligned}$$

Let

$$\begin{aligned}P &= \left|\frac{B_3}{B_1} - p\frac{B_2^2}{B_1^2}\right| - (1 - p)\left(2\left|\frac{B_2}{B_1}\right| + 1\right),\\ Q &= 4\left[2\left|\frac{B_2}{B_1}\right|(1 - p) + 1 - 2p\right],\\ R &= 16p.\end{aligned} \tag{30}$$

Using (16), we have

$$\left|a_2a_4 - a_3^2\right| \le T \begin{cases} R, & Q \le 0, P \le -\frac{Q}{4}; \\ 16P + 4Q + R, & Q \ge 0, P \ge -\frac{Q}{8} \text{ or } Q \le 0, P \ge -\frac{Q}{4}; \\ \frac{4PR - Q^2}{4P}, & Q > 0, P \le -\frac{Q}{8}, \end{cases}$$

where P, Q, R are given in (30). □

Remark 3 For the choice $\varphi(z) := (1 + Az)/(1 + Bz)$ with $-1 \le B < A \le 1$, Theorem 3 reduces to [36, Theorem 2.1].

Definition 4 Let $\varphi : \mathbb{D} \to \mathbb{C}$ be analytic, and let $\varphi(z)$ be as given in (6). For a fixed real number α, the function $f \in \mathcal{A}$ is in the class $\mathcal{G}_\alpha(\varphi)$ if it satisfies the following subordination:

$$(1 - \alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z).$$

Al-Amiri and Reade [37] introduced the class $\mathcal{G}_\alpha := \mathcal{G}_\alpha((1 + z)/(1 - z))$ and they showed that $\mathcal{G}_\alpha \subset \mathcal{S}$ for $\alpha < 0$. Univalence of the functions in the class $\mathcal{G}_\alpha$ was also investigated in [38, 39]. Singh *et al.* also obtained the bound for the second Hankel determinant of functions in $\mathcal{G}_\alpha$. The following theorem provides a bound for the second Hankel determinant of the functions in the class $\mathcal{G}_\alpha(\varphi)$.

Theorem 4 *Let the function f given by* (1) *be in the class $\mathcal{G}_\alpha(\varphi)$, $0 \le \alpha \le 1$. Also, let*

$$p = \frac{8}{9}\frac{(1 + 2\alpha)}{(1 + \alpha)}.$$

1. *If B_1, B_2 and B_3 satisfy the conditions*

$$B_1^2\alpha(3 - 2p) + 2|B_2|(1 + \alpha - p) + B_1(1 + \alpha - 2p) \le 0,$$
$$\left|B_1^4\alpha(2\alpha - 1 - p\alpha) + \alpha B_1^2B_2(3 - 2p) + (\alpha + 1)B_1B_3 - pB_2^2\right| - pB_1^2 \le 0,$$

then the second Hankel determinant satisfies

$$\left|a_2a_4 - a_3^2\right| \le \frac{B_1^2}{9(1 + \alpha)^2}.$$

2. *If B_1, B_2 and B_3 satisfy the conditions*

$$B_1^2\alpha(3 - 2p) + 2|B_2|(1 + \alpha - p) + B_1(1 + \alpha - 2p) \ge 0,$$
$$2\left|B_1^4\alpha(2\alpha - 1 - p\alpha) + \alpha B_1^2B_2(3 - 2p) + (\alpha + 1)B_1B_3 - pB_2^2\right| - B_1^3\alpha(3 - 2p) - 2(1 + \alpha - p)B_1|B_2| - (\alpha + 1)B_1^2 \ge 0,$$

or

$$B_1^2\alpha(3 - 2p) + 2|B_2|(1 + \alpha - p) + B_1(1 + \alpha - 2p) \le 0,$$
$$\left|B_1^4\alpha(2\alpha - 1 - p\alpha) + \alpha B_1^2B_2(3 - 2p) + (\alpha + 1)B_1B_3 - pB_2^2\right| - pB_1^2 \ge 0,$$

then the second Hankel determinant satisfies

$$\left|a_2a_4-a_3^2\right| \le \frac{|B_1^4\alpha(2\alpha-1-p\alpha)+\alpha B_1^2B_2(3-2p)+(\alpha+1)B_1B_3-pB_2^2|}{8(1+\alpha)(1+2\alpha)}.$$

3. *If B_1, B_2 and B_3 satisfy the conditions*

$$\begin{aligned}&B_1^2\alpha(3-2p)+2|B_2|(1+\alpha-p)+B_1(1+\alpha-2p)>0,\\&2\left|B_1^4\alpha(2\alpha-1-p\alpha)+\alpha B_1^2B_2(3-2p)+(\alpha+1)B_1B_3-pB_2^2\right|-B_1^3\alpha(3-2p)\\&\quad-2(1+\alpha-p)B_1|B_2|-(\alpha+1)B_1^2\le 0,\end{aligned}$$

then the second Hankel determinant satisfies

$$\begin{aligned}&\left|a_2a_4-a_3^2\right|\\&\quad\le\frac{B_1^2}{32(1+\alpha)(1+2\alpha)}\\&\qquad\times\left[4p-\frac{[B_1^2\alpha(3-2p)+2|B_2|(1+\alpha-p)+B_1(1+\alpha-2p)]^2}{\begin{array}{c}|B_1^4\alpha(2\alpha-1-p\alpha)+\alpha B_1^2B_2(3-2p)+(\alpha+1)B_1B_3-pB_2^2|\\-B_1^3\alpha(3-2p)-(1+\alpha-p)B_1(2|B_2|+B_1)\end{array}}\right].\end{aligned}$$

Proof For $f\in\mathcal{G}_\alpha(\varphi)$, a calculation shows that

$$\begin{aligned}&\left|a_2a_4-a_3^2\right|\\&\quad=T\Bigg|4(1+\alpha)B_1c_1c_3+c_1^4\bigg[-3\alpha B_1^2+\alpha(2\alpha-1)B_1^3+B_1(1+\alpha)+3\alpha B_1B_2\\&\qquad+(1+\alpha)(B_3-2B_2)-p\frac{(\alpha B_1^2-B_1+B_2)^2}{B_1}\bigg]-4pB_1c_2^2\\&\qquad+2c_1^2c_2\big[-2(1+\alpha)B_1+3\alpha B_1^2+2(1+\alpha)B_2-2p\big(\alpha B_1^2-B_1+B_2\big)\big]\Bigg|,\end{aligned}\tag{31}$$

where

$$T=\frac{B_1}{128(1+\alpha)(1+2\alpha)}\quad\text{and}\quad p=\frac{8}{9}\frac{(1+2\alpha)}{(1+\alpha)}.$$

It can be easily verified that for $0\le\alpha\le1$, $p\in[\frac{8}{9},\frac{4}{3}]$. Let

$$\begin{aligned}&d_1=4(1+\alpha)B_1,\\&d_2=2\big[-2(1+\alpha)B_1+3\alpha B_1^2+2(1+\alpha)B_2-2p\big(\alpha B_1^2-B_1+B_2\big)\big],\\&d_3=-4pB_1,\\&d_4=-3\alpha B_1^2+\alpha(2\alpha-1)B_1^3+B_1(1+\alpha)+3\alpha B_1B_2\\&\qquad+(1+\alpha)(B_3-2B_2)-p\frac{(\alpha B_1^2-B_1+B_2)^2}{B_1}.\end{aligned}\tag{32}$$

Then

$$\left|a_2a_4 - a_3^2\right| = T\left|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4\right|. \tag{33}$$

Similarly as in earlier theorems, it follows that

$$\begin{aligned}
\left|a_2a_4 - a_3^2\right| &= \frac{T}{4}\left|c^4(d_1 + 2d_2 + d_3 + 4d_4) + 2xc^2\left(4 - c^2\right)(d_1 + d_2 + d_3)\right. \\
&\quad \left. + \left(4 - c^2\right)x^2\left(-d_1c^2 + d_3\left(4 - c^2\right)\right) + 2d_1c\left(4 - c^2\right)\left(1 - |x|^2\right)z\right| \\
&\le T\left[c^4\left|B_1^3\alpha(2\alpha - 1 - p\alpha) + \alpha B_1B_2(3 - 2p)\right.\right. \\
&\quad \left. + (\alpha + 1)B_3 - p\frac{B_2^2}{B_1}\right| + \mu c^2\left(4 - c^2\right)\left[B_1^2\alpha(3 - 2p)\right. \\
&\quad \left. + 2|B_2|(1 + \alpha - p)\right] + 2c\left(4 - c^2\right)B_1(1 + \alpha) \\
&\quad \left. + \mu^2\left(4 - c^2\right)B_1(1 + \alpha - p)(c - 2)\left(c - \frac{2p}{1 + \alpha - p}\right)\right] \\
&\equiv F(c, \mu),
\end{aligned} \tag{34}$$

and for fixed $c \in [0, 2]$, $\max F(c, \mu) = F(c, 1) \equiv G(c)$ with

$$\begin{aligned}
G(c) &= T\left[c^4\left[\left|B_1^3\alpha(2\alpha - 1 - p\alpha) + \alpha B_1B_2(3 - 2p) + (\alpha + 1)B_3 - p\frac{B_2^2}{B_1}\right|\right.\right. \\
&\quad \left. - B_1^2\alpha(3 - 2p) - (1 + \alpha - p)(2|B_2| + B_1)\right] + 4c^2\left[B_1^2\alpha(3 - 2p)\right. \\
&\quad \left.\left. + 2|B_2|(1 + \alpha - p) + B_1(1 + \alpha - 2p)\right] + 16pB_1\right].
\end{aligned}$$

Let

$$\begin{aligned}
P &= \left|B_1^3\alpha(2\alpha - 1 - p\alpha) + \alpha B_1B_2(3 - 2p) + (\alpha + 1)B_3 - p\frac{B_2^2}{B_1}\right| \\
&\quad - B_1^2\alpha(3 - 2p) - (1 + \alpha - p)\left(2|B_2| + B_1\right), \\
Q &= 4\left[B_1^2\alpha(3 - 2p) + 2|B_2|(1 + \alpha - p) + B_1(1 + \alpha - 2p)\right], \\
R &= 16pB_1.
\end{aligned} \tag{35}$$

By using (16), we have

$$\left|a_2a_4 - a_3^2\right| \le T\begin{cases} R, & Q \le 0, P \le -\frac{Q}{4}; \\ 16P + 4Q + R, & Q \ge 0, P \ge -\frac{Q}{8} \text{ or } Q \le 0, P \ge -\frac{Q}{4}; \\ \frac{4PR - Q^2}{4P}, & Q > 0, P \le -\frac{Q}{8}, \end{cases}$$

where P, Q, R are given in (35). □

Remark 4 For $\alpha = 1$, Theorem 4 reduces to Theorem 2. For $0 \leq \alpha < 1$, let $\varphi(z) := (1 + (1 - 2\alpha)z)/(1 - z)$. For this function φ, $B_1 = B_2 = B_3 = 2(1 - \alpha)$. In this case, Theorem 4 reduces to [40, Theorem 3.1].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results and they read and approved the final manuscript.

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