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Strong convergence of a relaxed CQ algorithm for the split feasibility problem

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Abstract

The split feasibility problem (SFP) is finding a point in a given closed convex subset of a Hilbert space such that its image under a bounded linear operator belongs to a given closed convex subset of another Hilbert space. The most popular iterative method is Byrne's CQ algorithm. López *et al.* proposed a relaxed CQ algorithm for solving SFP where the two closed convex sets are both level sets of convex functions. This algorithm can be implemented easily since it computes projections onto half-spaces and has no need to know *a priori* the norm of the bounded linear operator. However, their algorithm has only weak convergence in the setting of infinite-dimensional Hilbert spaces. In this paper, we introduce a new relaxed CQ algorithm such that the strong convergence is guaranteed. Our result extends and improves the corresponding results of López *et al.* and some others.

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1 Introduction

The split feasibility problem (SFP) was proposed by Censer and Elfving in [1]. It can mathematically be formulated as the problem of finding a point x satisfying the property:

$$x \in C, \qquad Ax \in Q, \tag{1.1}$$

where A is a given $M \times N$ real matrix (where A^* is the transpose of A), C and Q are nonempty, closed and convex subsets in $\mathbb{R}^N$ and $\mathbb{R}^M$, respectively. This problem has received much attention [2] due to its applications in signal processing and image reconstruction, with particular progress in intensity-modulated radiation therapy [3–5], and many other applied fields.

We assume the SFP (1.1) is consistent, and let S be the solution set, *i.e.*,

$$S = \{x \in C : Ax \in Q\}.$$

It is easily seen that S is closed convex. Many iterative methods can be used to solve the SFP (1.1); see [6–16]. Byrne [6, 17] was among others the first to propose the so-called CQ algorithm, which is defined as follows:

$$x_{n+1} = P_C\bigl(x_n - \tau_n A^*(I - P_Q)Ax_n\bigr), \tag{1.2}$$

where $\tau_n \in (0, \frac{2}{\|A\|^2})$, and P_C and P_Q are the orthogonal projections onto the sets C and Q, respectively. Compared with Censer and Elfving' algorithm [1], the Byrne' algorithm is easily executed since it only deal with orthogonal projections with no need to compute matrix inverses.

The CQ algorithm (1.2) can be obtained from optimization. In fact, if we introduce the (convex) objective function

$$f(x) = \frac{1}{2}\left\|(I - P_Q)Ax\right\|^2, \tag{1.3}$$

and analyze the minimization problem

$$\min_{x\in C} f(x), \tag{1.4}$$

then the CQ algorithm (1.2) comes immediately as a special case of the gradient projection algorithm (GPA)(For more details about the GPA, the reader is referred to [18]). Since the convex objective $f(x)$ is differentiable, and has a Lipschitz gradient, which is given by

$$\nabla f(x) = A^*(I - P_Q)Ax, \tag{1.5}$$

the GPA for solving the minimization problem (1.4) generates a sequence (x_n) recursively as

$$x_{n+1} = P_C\big(x_n - \tau_n \nabla f(x_n)\big), \tag{1.6}$$

where the stepsize τ_n is chosen in the interval $(0, \frac{2}{L})$, where L is the Lipschitz constant of ∇f.

Observe that in algorithms (1.2) and (1.6) mentioned above, in order to implement the CQ algorithm, one has to compute the operator (matrix) norm $\|A\|$, which is in general not an easy work in practice. To overcome this difficulty, some authors proposed different adaptive choices of selecting the stepsize τ_n (see [6, 14, 19]). For instance, very recently López *et al.* introduced a new way of selecting the stepsize [19] as follows:

$$\tau_n := \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}, \quad 0 < \rho_n < 4. \tag{1.7}$$

The computation of a projection onto a general closed convex subset is generally difficult. To overcome this difficulty, Fukushima [20] suggested a so-called relaxed projection method to calculate the projection onto a level set of a convex function by computing a sequence of projections onto half-spaces containing the original level set. In the setting of finite-dimensional Hilbert spaces, this idea was followed by Yang [13], who introduced the relaxed CQ algorithms for solving SFP (1.1) where the closed convex subsets C and Q are level sets of convex functions.

Recently, for the purpose of generality, the SFP (1.1) is studied in a more general setting. For instance, Xu [12] and López *et al.* [19] considered the SFP (1.1) in infinite-dimensional Hilbert spaces (*i.e.*, the finite-dimensional Euclidean spaces $\mathbb{R}^N$ and $\mathbb{R}^M$ are replaced with general Hilbert spaces). Very recently, López *et al.* proposed a relaxed CQ algorithm with a new adaptive way of determining the stepsize sequence (τ_n) for solving the SFP (1.1) where the closed convex subsets C and Q are level sets of convex functions. This algorithm can be implemented easily since it computes projections onto half-spaces and has no need to know *a priori* the norm of the bounded linear operator. However, their algorithm has only weak convergence in the setting of infinite-dimensional Hilbert spaces. In this paper, we introduce a new relaxed CQ algorithm such that the strong convergence is guaranteed in infinite-dimensional Hilbert spaces. Our result extends and improves the corresponding results of López *et al.* and some others.

The rest of this paper is organized as follows. Some useful lemmas are listed in Section 2. In Section 3, the strong convergence of the new relaxed CQ algorithm of this paper is proved.

2 Preliminaries

Throughout the rest of this paper, we denote by H or K a Hilbert space, A is a bounded linear operator from H to K, and by I the identity operator on H or K. If $f : H \to \mathbb{R}$ is a differentiable function, then we denote by ∇f the gradient of the function f. We will also use the notations:

- $\to$ denotes strong convergence.
- $\rightharpoonup$ denotes weak convergence.
- $\omega_w(x_n) = \{x \mid \exists \{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup x\}$ denotes the weak *ω-limit* set of $\{x_n\}$.

Recall that a mapping $T : H \to H$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad x, y \in H.$$

$T : H \to H$ is said to be firmly nonexpansive if, for $x, y \in H$,

$$\|Tx - Ty\|^2 \le \|x - y\|^2 - \big\|(I - T)x - (I - T)y\big\|^2.$$

Recall that a function $f : H \to \mathbb{R}$ is called convex if

$$f\big(\lambda x + (1 - \lambda)y\big) \le \lambda f(x) + (1 - \lambda)f(y), \quad \forall \lambda \in (0, 1), \forall x, y \in H.$$

A differentiable function f is convex if and only if there holds the inequality:

$$f(z) \ge f(x) + \big\langle \nabla f(x), z - x\big\rangle, \quad \forall z \in H.$$

Recall that an element $g \in H$ is said to be a subgradient of $f : H \to \mathbb{R}$ at x if

$$f(z) \ge f(x) + \langle g, z - x\rangle, \quad \forall z \in H.$$

This relation is called the subdifferentiable inequality.

A function $f : H \to \mathbb{R}$ is said to be subdifferentiable at x, if it has at least one subgradient at x. The set of subgradients of f at the point x is called the subdifferentiable of f at x, and it is denoted by $\partial f(x)$. A function f is called subdifferentiable, if it is subdifferentiable at all $x \in H$. If a function f is differentiable and convex, then its gradient and subgradient coincide.

A function $f : H \to \mathbb{R}$ is said to be weakly lower semi-continuous (w-lsc) at x if $x_n \rightharpoonup x$ implies

$$f(x) \le \liminf_{n \to \infty} f(x_n).$$

We know that the orthogonal projection P_C from H onto a nonempty closed convex subset $C \subset H$ is a typical example of a firmly nonexpansive mapping, which is defined by

$$P_C x := \arg\min_{y \in C} \|x - y\|^2, \quad x \in H. \tag{2.1}$$

It is well known that P_C is characterized by the inequality (for $x \in H$)

$$\langle x - P_C x, y - P_C x \rangle \le 0, \quad \forall y \in C. \tag{2.2}$$

The following lemma is not hard to prove (see [17, 21]).

Lemma 2.1 *Let f be given as in* (1.3). *Then*
(i) *f is convex and differential.*
(ii) $\nabla f(x) = A^*(I - P_Q)Ax$, $x \in H$.
(iii) *f is w-lsc on H.*
(iv) *∇f is $\|A\|^2$-Lipschitz*: $\|\nabla f(x) - \nabla f(y)\| \le \|A\|^2 \|x - y\|$, $x, y \in H$.

The following are characterizations of firmly nanexpansive mappings (see [22]).

Lemma 2.2 *Let $T : H \to H$ be an operator. The following statements are equivalent.*
(i) *T is firmly nonexpansive.*
(ii) $\|Tx - Ty\|^2 \le \langle x - y, Tx - Ty \rangle$, $x, y \in H$.
(iii) *$I - T$ is firmly nonexpansive.*

Lemma 2.3 [23] *Assume (α_n) is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \le (1 - \gamma_n)\alpha_n + \gamma_n \sigma_n, \quad n = 0, 1, 2, \ldots,$$

where (γ_n) is a sequence in $(0,1)$, and (σ_n) is a sequence in $\mathbb{R}$ such that
(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$.
(ii) $\limsup_{n\to\infty} \sigma_n \le 0$, *or* $\sum_{n=0}^{\infty} |\gamma_n \sigma_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

3 Iterative Algorithm

In this section, we turn to consider a new relaxed CQ algorithm in the setting of infinite-dimensional Hilbert spaces for solving SFP (1.1) where the closed convex subsets C and Q are of the particular structure, *i.e.* level sets of convex functions given as follows:

$$C = \{x \in H : c(x) \le 0\} \quad \text{and} \quad Q = \{y \in K : q(y) \le 0\}, \tag{3.1}$$

where $c : H \to \mathbb{R}$ and $q : K \to \mathbb{R}$ are convex functions. We assume that both c and q are subdifferentiable on H and K, respectively, and that ∂c and ∂q are bounded operators (*i.e.*, bounded on bounded sets). By the way, we mention that every convex function defined on a finite-dimensional Hilbert space is subdifferentiable and its subdifferential operator is a bounded operator (see [24]).

Set

$$C_n = \{x \in H : c(x_n) \le \langle \xi_n, x_n - x \rangle\}, \tag{3.2}$$

where $\xi_n \in \partial c(x_n)$, and

$$Q_n = \{y \in K : q(Ax_n) \le \langle \zeta_n, Ax_n - y \rangle\}, \tag{3.3}$$

where $\zeta_n \in \partial q(Ax_n)$.

Obviously, C_n and Q_n are half-spaces and it is easy to verify that $C_n \supset C$ and $Q_n \supset Q$ hold for every $n \ge 0$ from the subdifferentiable inequality. In what follows, we define

$$f_n(x) = \frac{1}{2}\|(I - P_{Q_n})Ax\|^2, \quad n \ge 0,$$

where Q_n is given as in (3.3). We then have

$$\nabla f_n(x) = A^*(I - P_{Q_n})Ax.$$

Firstly, we recall the relaxed CQ algorithm of López *et al.* [19] for solving the SFP (1.1) where C and Q are given in (3.1) as follows.

Algorithm 3.1 Choose an arbitrary initial guess x_0. Assume x_n has been constructed. If $\nabla f_n(x_n) = 0$, then stop; otherwise, continue and construct x_{n+1} *via* the formula

$$x_{n+1} = P_{C_n}(x_n - \tau_n \nabla f_n(x_n)),$$

where C_n is given as (3.2), and

$$\tau_n = \frac{\rho_n f_n(x_n)}{\|\nabla f_n(x_n)\|^2}, \quad 0 < \rho_n < 4. \tag{3.4}$$

López *et al.* proved that under some certain conditions the sequence (x_n) generated by Algorithm 3.1 converges weakly to a solution of the SFP (1.1). Since the projections onto half-spaces C_n and Q_n have closed forms and τ_n is obtained adaptively *via* the formula (3.4) (no need to know *a priori* the norm of operator A), the above relaxed CQ algorithm 3.1 is implementable. But the weak convergence is its a weakness. To overcome this weakness, inspired by Algorithm 3.1, we will introduce a new relaxed CQ algorithm for solving the SFP (1.1) where C and Q are given in (3.1) so that the strong convergence is guaranteed.

It is well known that Halpern's algorithm has a strong convergence for finding a fixed point of a nonexpansive mapping [25, 26]. Then we are in a position to give our algorithm. The algorithm given below is referred to as a Halpern-type algorithm [27].

Algorithm 3.2 Let $u \in H$, and start an initial guess $x_0 \in H$ arbitrarily. Assume that the nth iterate x_n has been constructed. If $\nabla f_n(x_n) = 0$, then stop (x_n is a approximate solution of SFP (1.1)). Otherwise, continue and calculate the $(n+1)$th iterate x_{n+1} *via* the formula:

$$x_{n+1} = P_{C_n}[\alpha_n u + (1 - \alpha_n)(x_n - \tau_n \nabla f_n(x_n))], \tag{3.5}$$

where the sequence $(\alpha_n) \subset (0,1)$ and (τ_n) and (ρ_n) are given as in (3.4).

The convergence result of Algorithm 3.2 is stated in the next theorem.

Theorem 3.3 *Assume that* (α_n) *and* (ρ_n) *satisfy the assumptions*:
(a1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$.
(a2) $\inf_n \rho_n(4-\rho_n) > 0$.
Then the sequence (x_n) *generated by Algorithm* 3.2 *converges in norm to* $P_S u$.

Proof We may assume that the sequence (x_n) is infinite, that is, Algorithm 3.2 does not terminate in a finite number of iterations. Thus $\nabla f_n(x_n) \neq 0$ for all $n \geq 0$. Recall that S is the solution set of the SFP (1.1),

$$S = \{x \in C : Ax \in Q\}.$$

In the consistent case of the SFP (1.1), S is nonempty, closed and convex. Thus, the metric projection P_S is well-defined. We set $z = P_S u$. Since $z \in S \subset C_n$ and the projection operator P_{C_n} is nonexpansive, we obtain

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \left\|P_{C_n}\left[\alpha_n u + (1-\alpha_n)\left(x_n - \tau_n\nabla f_n(x_n)\right)\right] - z\right\|^2 \\ &\leq \left\|\alpha_n(u-z) + (1-\alpha_n)\left(x_n - \tau_n\nabla f_n(x_n) - z\right)\right\|^2 \\ &\leq (1-\alpha_n)\left\|x_n - \tau_n\nabla f_n(x_n) - z\right\|^2 + 2\alpha_n\langle u-z, x_{n+1}-z\rangle.\end{aligned}$$

Note that $I - P_{Q_n}$ is firmly nonexpansive and $\nabla f_n(z) = 0$, it is deduced from Lemma 2.2 that

$$\begin{aligned}\left\langle\nabla f_n(x_n), x_n - z\right\rangle &= \left\langle(I-P_{Q_n})Ax_n, Ax_n - Az\right\rangle \\ &\geq \left\|(I-P_{Q_n})Ax_n\right\|^2 \\ &= 2f_n(x_n),\end{aligned}$$

which implies that

$$\begin{aligned}\left\|x_n - \tau_n\nabla f_n(x_n) - z\right\|^2 &= \|x_n - z\|^2 + \left\|\tau_n\nabla f_n(x_n)\right\|^2 - 2\tau_n\left\langle\nabla f_n(x_n), x_n - z\right\rangle \\ &\leq \|x_n - z\|^2 + \tau_n^2\left\|\nabla f_n(x_n)\right\|^2 - 4\tau_n f_n(x_n) \\ &\leq \|x_n - z\|^2 - \rho_n(4-\rho_n)\frac{f_n^2(x_n)}{\|\nabla f_n(x_n)\|^2}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &\leq (1-\alpha_n)\left\|x_n - \tau_n\nabla f_n(x_n) - z\right\|^2 + 2\alpha_n\langle u-z, x_{n+1}-z\rangle \\ &\leq (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u-z, x_{n+1}-z\rangle \\ &\quad - (1-\alpha_n)\rho_n(4-\rho_n)\frac{f_n^2(x_n)}{\|\nabla f_n(x_n)\|^2}.\end{aligned} \tag{3.6}$$

Now we prove (x_n) is bounded. Indeed, we have from (3.6) that

$$\begin{aligned}\|x_{n+1}-z\|^2 &\leq (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u-z, x_{n+1}-z\rangle \\ &\leq (1-\alpha_n)\|x_n - z\|^2 + \frac{1}{4}\alpha_n\|x_{n+1}-z\|^2 + 4\alpha_n\|u-z\|^2,\end{aligned}$$

and consequently

$$\|x_{n+1}-z\|^2 \le \frac{1-\alpha_n}{1-\frac{1}{4}\alpha_n}\|x_n-z\|^2 + \frac{\frac{3}{4}\alpha_n}{1-\frac{1}{4}\alpha_n}\frac{16}{3}\|u-z\|^2.$$

It turns out that

$$\|x_{n+1}-z\| \le \max\left\{\|x_n-z\|, \frac{16}{3}\|u-z\|\right\},$$

and inductively

$$\|x_n-z\| \le \max\left\{\|x_0-z\|, \frac{16}{3}\|u-z\|\right\},$$

and this means that (x_n) is bounded. Since $\alpha_n \to 0$, with no loss of generality, we may assume that there is $\sigma > 0$ so that $\rho_n(4-\rho_n)(1-\alpha_n) \ge \sigma$ for all n. Setting $s_n = \|x_n - z\|^2$, from the last inequality of (3.6), we get the following inequality:

$$s_{n+1} - s_n + \alpha_n s_n + \frac{\sigma f_n^2(x_n)}{\|\nabla f_n(x_n)\|^2} \le 2\alpha_n\langle u-z, x_{n+1}-z\rangle. \tag{3.7}$$

Now, following an idea in [28], we prove $s_n \to 0$ by distinguishing two cases.

Case 1: (s_n) is eventually decreasing (*i.e.* there exists $k \ge 0$ such that $s_n > s_{n+1}$ holds for all $n \ge k$). In this case, (s_n) must be convergent, and from (3.7) it follows that

$$\frac{\sigma f_n^2(x_n)}{\|\nabla f_n(x_n)\|^2} \le M\alpha_n + (s_n - s_{n+1}), \tag{3.8}$$

where $M > 0$ is a constant such that $2\|x_{n+1}-z\|\|u-z\| \le M$ for all $n \in \mathbb{N}$. Using the condition (a1), we have from (3.8) that $\frac{f_n^2(x_n)}{\|\nabla f_n(x_n)\|^2} \to 0$. Thus, to verify that $f_n(x_n) \to 0$, it suffices to show that $(\nabla f(x_n))$ is bounded. In fact, it follows from Lemma 2.1 that (noting that $\nabla f_n(z) = 0$ due to $z \in S$)

$$\|\nabla f_n(x_n)\| = \|\nabla f_n(x_n) - \nabla f_n(z)\| \le \|A\|^2\|x_n - z\|.$$

This implies that $\|\nabla f_n(x_n)\|$ is bounded and it yields $f_n(x_n) \to 0$, namely $\|(I - P_{Q_n})Ax_n\| \to 0$.

Since ∂q is bounded on bounded sets, there exists a constant $\eta > 0$ such that $\|\zeta_n\| \le \eta$ for all $n \ge 0$. From (3.3) and the trivial fact that $P_{Q_n}(Ax_n) \in Q_n$, it follows that

$$q(Ax_n) \le \langle \zeta_n, Ax_n - P_{Q_n}(Ax_n)\rangle \le \eta\|(I - P_{Q_n})Ax_n\| \to 0. \tag{3.9}$$

If $x^* \in \omega_w(x_n)$, and (x_{n_k}) is a subsequence of (x_n) such that $x_{n_k} \rightharpoonup x^*$, then the *w-lsc* of q and (3.9) imply that

$$q(Ax^*) \le \liminf_{k\to\infty} q(Ax_{n_k}) \le 0.$$

It turns out that $Ax^* \in Q$. Next, we turn to prove $x^* \in C$. For convenience, we set $y_n := \alpha_n u + (1-\alpha_n)(x_n - \tau_n \nabla f_n(x_n))$. In fact, since the P_{C_n} is firmly nonexpansive, it concludes that

$$\|P_{C_n}x_n - P_{C_n}z\|^2 \le \|x_n - z\|^2 - \big\|(I - P_{C_n})x_n\big\|^2. \tag{3.10}$$

On the other hand, we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|P_{C_n}y_n - P_{C_n}z\|^2 \\ &= \|P_{C_n}y_n - P_{C_n}x_n + P_{C_n}x_n - P_{C_n}z\|^2 \\ &\le \|P_{C_n}x_n - P_{C_n}z\|^2 + 2\langle P_{C_n}y_n - P_{C_n}x_n, x_{n+1} - z\rangle,\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\|P_{C_n}y_n - P_{C_n}x_n\| &\le \|y_n - x_n\| \\ &= \big\|\alpha_n u + (1-\alpha_n)\big(x_n - \tau_n \nabla f_n(x_n)\big) - x_n\big\| \\ &\le \alpha_n\|u - x_n\| + \tau_n\big\|\nabla f_n(x_n)\big\|.\end{aligned} \tag{3.12}$$

Noting that (x_n) is bounded, we have from (3.10)-(3.12) that

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le \|x_n - z\|^2 - \big\|(I - P_{C_n})x_n\big\|^2 + 2\big(\alpha_n\|u - x_n\| + \tau_n\big\|\nabla f_n(x_n)\big\|\big)\|x_{n+1} - z\| \\ &\le \|x_n - z\|^2 - \big\|(I - P_{C_n})x_n\big\|^2 + \left(\alpha_n + \frac{f_n(x_n)}{\|\nabla f_n(x_n)\|}\right)M,\end{aligned} \tag{3.13}$$

where M is some positive constant. Clearly, from (3.13), it turns out that

$$\big\|(I - P_{C_n})x_n\big\|^2 \le s_n - s_{n+1} + \left(\alpha_n + \frac{f_n(x_n)}{\|\nabla f_n(x_n)\|}\right)M. \tag{3.14}$$

Thus, we assert that $\|(I - P_{C_n})x_n\| \to 0$ due to the fact that $s_n - s_{n+1} + (\alpha_n + \frac{f_n(x_n)}{\|\nabla f_n(x_n)\|})M \to 0$. Moreover, by the definition of C_n, we obtain

$$c(x_n) \le \big\langle \xi_n, x_n - P_{C_n}(x_n)\big\rangle \le \delta\|x_n - P_{C_n}x_n\| \to 0 \quad (n \to \infty),$$

where δ is a constant such that $\|\xi_n\| \le \delta$ for all $n \ge 0$. The w-lsc of c then implies that

$$c\big(x^*\big) \le \liminf_{k\to\infty} c(x_{n_k}) = 0.$$

Consequently, $x^* \in C$, and hence $\omega_w(x_n) \subset S$. Furthermore, due to (2.2), we get

$$\limsup_{n\to\infty}\langle u - z, x_n - z\rangle = \max_{w\in\omega_w(x_n)}\langle u - P_S u, w - P_S u\rangle \le 0. \tag{3.15}$$

Taking into account of (3.7), we have

$$s_{n+1} \le (1-\alpha_n)s_n + 2\alpha_n\langle u - z, x_{n+1} - z\rangle. \tag{3.16}$$

Applying Lemma 2.3 to (3.16), we obtain $s_n \to 0$.

Case 2: (s_n) is not eventually decreasing, that is, we can find an integer n_0 such that $s_{n_0} \le s_{n_0+1}$. Now we define

$$V_n := \{n_0 \le k \le n : s_k \le s_{k+1}\}, \quad n > n_0.$$

It is easy to see that V_n is nonempty and satisfies $V_n \subseteq V_{n+1}$. Let

$$\psi(n) := \max V_n, \quad n > n_0.$$

It is clear that $\psi(n) \to \infty$ as $n \to \infty$ (otherwise, (s_n) is eventually decreasing). It is also clear that $s_{\psi(n)} \le s_{\psi(n)+1}$ for all $n > n_0$. Moreover,

$$s_n \le s_{\psi(n)+1}, \quad n > n_0. \tag{3.17}$$

In fact, if $\psi(n) = n$, then the inequity (3.17) is trivial; if $\psi(n) < n$, from the definition of $\psi(n)$, there exists some $i \in \mathbb{N}$ such that $\psi(n) + i = n$, we deduce that

$$s_{\psi(n)+1} > s_{\psi(n)+2} > \cdots > s_{\psi(n)+i} = s_n,$$

and the inequity (3.17) holds again. Since $s_{\psi(n)} \le s_{\psi(n)+1}$ for all $n > n_0$, it follows from (3.8) that

$$\frac{\sigma f^2_{\psi(n)}(x_{\psi(n)})}{\|\nabla f_{\psi(n)}(x_{\psi(n)})\|^2} \le M\alpha_{\psi(n)} \to 0,$$

so that $f_{\psi(n)}(x_{\psi(n)}) \to 0$ as $n \to \infty$ (noting that $\|\nabla f_{\psi(n)}(x_{\psi(n)})\|$ is bounded). By the same argument to the proof in case 1, we have $\omega_w(x_{\psi(n)}) \subset S$. On the other hand, noting $s_{\psi(n)} \le s_{\psi(n)+1}$ again, we have from (3.12) and (3.14) that

$$\begin{aligned}\|x_{\psi(n)} - x_{\psi(n)+1}\| &\le \|x_{\psi(n)} - P_{C_{\psi(n)}}x_{\psi(n)}\| + \|P_{C_{\psi(n)}}x_{\psi(n)} - P_{C_{\psi(n)}}y_{\psi(n)}\| \\ &\le \sqrt{\alpha_{\psi(n)} + \frac{f_{\psi(n)}(x_{\psi(n)})}{\|\nabla f_{\psi(n)}(x_{\psi(n)})\|}} \\ &\quad \times \left(\sqrt{\alpha_{\psi(n)} + \frac{f_{\psi(n)}(x_{\psi(n)})}{\|\nabla f_{\psi(n)}(x_{\psi(n)})\|}} + 1\right)M,\end{aligned}$$

where M is a positive constant. Letting $n \to \infty$ yields that

$$\|x_{\psi(n)} - x_{\psi(n)+1}\| \to 0, \tag{3.18}$$

from which one can deduce that

$$\begin{aligned}\limsup_{n\to\infty}\langle u - z, x_{\psi(n)+1} - z\rangle &= \limsup_{n\to\infty}\langle u - z, x_{\psi(n)} - z\rangle \\ &= \max_{w\in\omega_w(x_\psi(n))}\langle u - P_S u, w - P_S u\rangle \le 0.\end{aligned} \tag{3.19}$$

Since $s_{\psi(n)} \le s_{\psi(n)+1}$, it follows from (3.7) that

$$s_\psi(n) \le 2\langle u - z, x_{\psi(n)+1} - z\rangle, \quad n > n_0. \tag{3.20}$$

Combining (3.19) and (3.20) yields

$$\limsup_{n\to\infty} s_{\psi(n)} \le 0, \tag{3.21}$$

and hence $s_{\psi(n)} \to 0$, which together with (3.18) implies that

$$\begin{aligned}\sqrt{s_{\psi(n)+1}} &\le \left\| (x_{\psi(n)} - z) + (x_{\psi(n)+1} - x_{\psi(n)}) \right\| \\ &\le \sqrt{s_{\psi(n)}} + \|x_{\psi(n)+1} - x_{\psi(n)}\| \to 0, \end{aligned} \tag{3.22}$$

which, together with (3.17), in turn implies that $s_n \to 0$, that is, $x_n \to z$. □

Remark 3.4 Since u can be chosen in H arbitrarily, one can compute the minimum-norm solution of SFP (1.1) where C and Q are given in (3.1) by taking $u = 0$ in Algorithm 3.2 whether $0 \in C$ or $0 \notin C$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors read and approved the final manuscript.

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