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A half-discrete Hilbert-type inequality with the non-monotone kernel and the best constant factor

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Abstract

By introducing two pairs of conjugate exponents and using the improved Euler-Maclaurin summation formula, we estimate the weight functions and obtain a half-discrete Hilbert-type inequality with the non-monotone kernel and the best constant factor. We also consider its equivalent forms.

MSC: 26D15

Keywords: Hilbert-type inequality; conjugate exponent; Hölder's inequality; best constant factor; equivalent form

1 Introduction

If $a_n, b_n \geq 0$, such that $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then we have the famous Hilbert's inequality as follows (cf. [1]):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left(\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}b_n^2\right)^{\frac{1}{2}}, \tag{1}$$

where the constant factor π is the best possible.

Under the same condition of (1), Xin *et al.* [2] gave the following inequality:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{|\ln(m/n)|}{m+n}a_m b_n < c_0\left(\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}b_n^2\right)^{\frac{1}{2}}, \tag{2}$$

where the constant factor $c_0 = 8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2} = 7.3277^+$ is the best possible. And Yang [3] gave the integral analogues of (2).

In 1934, Hardy *et al.* [1] established a few results on the half-discrete Hilbert-type inequalities with the non-homogeneous kernel (see Theorem 351). But they did not prove that the constant factors are the best possible. However, Yang [4] gave a result by introducing an interval variable and proved that the constant factor is the best possible. Recently, Yang *et al.* [5–9] gave some half-discrete Hilbert-type inequalities and their reverses with the monotone kernels and best constant factors.

Recently, Yang [10] gave the following half-discrete Hilbert-type inequality with the non-monotone kernel and the best constant factor 8:

$$\sum_{n=1}^{\infty}\int_{1}^{\infty}\frac{|\ln(x/n)|a_n}{\max\{x,n\}}f(x)\,dx < 8\left(\sum_{n=1}^{\infty}a_n^2\int_{1}^{\infty}f^2(x)\,dx\right)^{\frac{1}{2}}. \tag{3}$$

Obviously, for a half-discrete Hilbert-type inequality with the monotone kernel, it is easy to build the relating inequality by estimating the series form and the integral form of weight functions. However, for a half-discrete Hilbert-type inequality with the non-monotone kernel, it is much more difficult to prove.

In this paper, by using the way of weight functions, we give a new half-discrete Hilbert-type inequality with the non-monotone kernel as follows:

$$\sum_{n=1}^{\infty}a_n\int_{1}^{\infty}\frac{|\ln(\frac{x}{n})|f(x)}{x+n}\,dx < 8\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\left(\sum_{n=1}^{\infty}a_n^2\int_{1}^{\infty}f^2(x)\right)^{\frac{1}{2}}, \tag{4}$$

where the constant factor $8\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}$ is the best possible. The main objective of this paper is to build the best extension of (4) with parameters and equivalent forms.

2 Some lemmas

Lemma 2.1 *If $x_1 \in \mathbf{R}$, $n_1 \in \mathbf{Z}$ ($\mathbf{Z}$ is the set of non-negative integers), $[x_1] = n_1$, $\rho(y) = y - [y] - \frac{1}{2}$ ($y \in \mathbf{R}$) is the Bernoulli function of first order [11], then we have (cf. [10])*

$$\int_{n_1}^{x_1}\rho(y)\,dy = -\frac{\varepsilon_1}{8} \quad \bigl(\varepsilon_1 \in [0,1]\bigr). \tag{5}$$

Lemma 2.2 *If $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $f(x,y) := \frac{|\ln(\frac{x}{y})|}{x+y}(\frac{x}{y})^{\frac{1}{r}}$ ($x, y \in (0,\infty)$), $\mathbf{N}$ is the set of positive integers, define the weight functions as follows:*

$$\omega(n) := \int_{1}^{\infty}\frac{|\ln(\frac{x}{n})|}{x+n}\left(\frac{n}{x}\right)^{\frac{1}{s}}dx \quad (n \in \mathbf{N}), \tag{6}$$

$$\varpi(x) := \sum_{n=1}^{\infty}\frac{|\ln(\frac{x}{n})|}{x+n}\left(\frac{x}{n}\right)^{\frac{1}{r}} \quad \bigl(x \in [1,\infty)\bigr). \tag{7}$$

Then we have

$$\omega(n) < c_r := \sum_{k=0}^{\infty}(-1)^k\left[\frac{1}{(k+\frac{1}{r})^2} + \frac{1}{(k+\frac{1}{s})^2}\right], \quad \varpi(x) < c_r. \tag{8}$$

Proof Setting $u = \frac{x}{n}$ in (6), we have

$$\omega(n) = \int_{\frac{1}{n}}^{\infty}\frac{|\ln u|}{u+1}u^{-\frac{1}{s}}\,du < \int_{0}^{\infty}\frac{|\ln u|}{u+1}u^{-\frac{1}{s}}\,du = c_r.$$

Setting $u = \frac{y}{x}$, then it follows

$$\int_{0}^{1}f(x,y)\,dy = \int_{0}^{\frac{1}{x}}\frac{(-\ln u)}{u+1}u^{-\frac{1}{r}}\,du > s\int_{0}^{\frac{1}{x}}\frac{(-\ln u)}{\frac{1}{x}+1}\,du^{-\frac{1}{r}+1} = \frac{x^{\frac{1}{r}}}{x+1}\bigl(s\ln x + s^2\bigr). \tag{9}$$

For $1 \leq y < x, f(x,y) = -\frac{\ln(\frac{y}{x})}{x+y}(\frac{x}{y})^{\frac{1}{r}}$, we have

$$\begin{aligned} f'_y(x,y) &= x^{\frac{1}{r}}\left[-\frac{1}{(y+x)y^{1+\frac{1}{r}}} + \frac{\ln(\frac{y}{x})}{(y+x)^2 y^{\frac{1}{r}}} + \frac{\ln(\frac{y}{x})}{r(y+x)y^{1+\frac{1}{r}}}\right] \\ &= -x^{\frac{1}{r}}\left[\frac{1}{(y+x)y^{1+\frac{1}{r}}} + \frac{2x\ln(\frac{y}{x})}{(y-x)(y+x)^2 y^{\frac{1}{r}}} + \frac{2x\ln(\frac{y}{x})}{r(y-x)(y+x)y^{1+\frac{1}{r}}}\right] \\ &\quad + x^{\frac{1}{r}}\left[\frac{\ln(\frac{y}{x})}{(y-x)(y+x)y^{\frac{1}{r}}} + \frac{\ln(\frac{y}{x})}{r(y-x)y^{1+\frac{1}{r}}}\right]. \end{aligned}$$

For $y \geq x, f(x,y) = \frac{\ln(\frac{y}{x})}{x+y}(\frac{x}{y})^{\frac{1}{r}}$, we find

$$\begin{aligned} f'_y(x,y) &= x^{\frac{1}{r}}\left[\frac{1}{(y+x)y^{1+\frac{1}{r}}} - \frac{\ln(\frac{y}{x})}{(y+x)^2 y^{\frac{1}{r}}} - \frac{\ln(\frac{y}{x})}{r(y+x)y^{1+\frac{1}{r}}}\right] \\ &= x^{\frac{1}{r}}\left[\frac{1}{(y+x)y^{1+\frac{1}{r}}} + \frac{2x\ln(\frac{y}{x})}{(y-x)(y+x)^2 y^{\frac{1}{r}}} + \frac{2x\ln(\frac{y}{x})}{r(y-x)(y+x)y^{1+\frac{1}{r}}}\right] \\ &\quad - x^{\frac{1}{r}}\left[\frac{\ln(\frac{y}{x})}{(y-x)(y+x)y^{\frac{1}{r}}} + \frac{\ln(\frac{y}{x})}{r(y-x)y^{1+\frac{1}{r}}}\right]. \end{aligned}$$

Define two functions as follows:

$$g(y) = \begin{cases} g_1(y) = x^{\frac{1}{r}}[\frac{1}{(y+x)y^{1+\frac{1}{r}}} + \frac{2x\ln(\frac{y}{x})}{(y-x)(y+x)^2 y^{\frac{1}{r}}} + \frac{2x\ln(\frac{y}{x})}{r(y-x)(y+x)y^{1+\frac{1}{r}}}], & y < x; \\ g_2(y) = x^{\frac{1}{r}}[\frac{\ln(\frac{y}{x})}{(y-x)(y+x)y^{\frac{1}{r}}} + \frac{\ln(\frac{y}{x})}{r(y-x)y^{1+\frac{1}{r}}}], & y \geq x, \end{cases}$$

$$h(y) = \begin{cases} h_1(y) = x^{\frac{1}{r}}[\frac{\ln(\frac{y}{x})}{(y-x)(y+x)y^{\frac{1}{r}}} + \frac{\ln(\frac{y}{x})}{r(y-x)y^{1+\frac{1}{r}}}], & y < x; \\ h_2(y) = x^{\frac{1}{r}}[\frac{1}{(y+x)y^{1+\frac{1}{r}}} + \frac{2x\ln(\frac{y}{x})}{(y-x)(y+x)^2 y^{\frac{1}{r}}} + \frac{2x\ln(\frac{y}{x})}{r(y-x)(y+x)y^{1+\frac{1}{r}}}], & y \geq x. \end{cases}$$

Then we have $-f'_y(x,y) = g(y) - h(y)$. Setting $a = \frac{1}{2x^2}$, $b = -\frac{1}{2x^2}$, then $a - b = \frac{1}{x^2}$. Define two functions as follows:

$$\tilde{g}(y) = \begin{cases} g_1(y) - a, & y < x; \\ g_2(y), & y \geq x, \end{cases} \qquad \tilde{h}(y) = \begin{cases} h_1(y) - b, & y < x; \\ h_2(y), & y \geq x. \end{cases}$$

Since $g_1(x-0) - a = g_2(x)$, $h_1(x-0) - b = h_2(x)$, then both $\tilde{g}(y)$ and $\tilde{h}(y)$ $(y \in [1,\infty))$ are decreasing and continuous. Besides $y = x$, we have $(-1)^i \tilde{g}^{(i)}(y) \geq 0$, $(-1)^i \tilde{h}^{(i)}(y) \geq 0$ $(i = 0,1)$, and $\tilde{g}(\infty) = \tilde{h}(\infty) = 0$. By the improved Euler-Maclaurin summation formula (cf. [11], Theorem 2.2.2) and (5), for $\varepsilon_1 \in [0,1]$, $\varepsilon_i \in (0,1)$ $(i = 2,3)$, it follows

$$\begin{aligned} &-\int_1^\infty \rho(y) f'_y(x,y)\,dy \\ &\quad = \int_1^\infty \rho(y) g(y)\,dy - \int_1^\infty \rho(y) h(y)\,dy \\ &\quad = \int_1^\infty \rho(y)\tilde{g}(y)\,dy + a\int_1^x \rho(y)\,dy - \left[\int_1^\infty \rho(y)\tilde{h}(y)\,dy + b\int_1^x \rho(y)\,dy\right] \end{aligned}$$

$$= \int_1^\infty \rho(y)\tilde{g}(y)\,dy - \int_1^\infty \rho(y)\tilde{h}(y)\,dy + (a-b)\left[\int_1^{[x]} \rho(y)\,dy + \int_{[x]}^x \rho(y)\,dy\right]$$
$$= -\frac{\varepsilon_2}{8}\tilde{g}(1) + \frac{\varepsilon_3}{8}\tilde{h}(1) - \frac{\varepsilon_1}{8}(a-b) = -\frac{\varepsilon_2}{8}\bigl(g_1(1)-a\bigr) + \frac{\varepsilon_3}{8}\bigl(h_1(1)-b\bigr) - \frac{\varepsilon_1}{8}(a-b).$$

Since $g_1(1) - a \geq g_1(x-0) - a = g_2(x) > 0$, $h_1(1) - b \geq h_1(x-0) - b = h_2(x) > 0$, then we have

$$-\int_1^\infty \rho(y) f'_y(x,y)\,dy > -\frac{1}{8}\bigl(g_1(1)-a\bigr) - \frac{1}{8}(a-b)$$
$$= -\frac{x^{\frac{1}{r}}}{8(x+1)} - \frac{x^{1+\frac{1}{r}}\ln x}{4(x+1)^2(x-1)} - \frac{x^{1+\frac{1}{r}}\ln x}{4r(x+1)(x-1)} - \frac{1}{16x^2}. \tag{10}$$

By the improved Euler-Maclaurin summation formula [11], we have

$$\varpi(x) = \sum_{k=1}^\infty f(x,k) = \int_1^\infty f(x,y)\,dy + \frac{1}{2}f(x,1) + \int_1^\infty \rho(y) f'_y(x,y)\,dy$$
$$= \int_0^\infty f(x,y)\,dy - \left(\int_0^1 f(x,y)\,dy - \frac{1}{2}f(x,1) - \int_1^\infty \rho(y) f'_y(x,y)\,dy\right) = c_r - \theta(x),$$

where

$$\theta(x) := \int_0^1 f(x,y)\,dy - \frac{1}{2}f(x,1) - \int_1^\infty \rho(y) f'_y(x,y)\,dy.$$

Since $-\frac{1}{2}f(x,1) = -\frac{x^{\frac{1}{r}}\ln x}{2(x+1)}$, in view of (9), (10), (i) for $1 \leq x < 2$, $-\frac{\ln x}{x-1} \geq -1$, we have

$$\theta(x) > \frac{x^{\frac{1}{r}}}{x+1}\bigl(s\ln x + s^2\bigr) - \frac{x^{\frac{1}{r}}\ln x}{2(x+1)} - \frac{x^{\frac{1}{r}}}{8(x+1)}$$
$$- \frac{x^{1+\frac{1}{r}}\ln x}{4(x+1)^2(x-1)} - \frac{x^{1+\frac{1}{r}}\ln x}{4r(x+1)(x-1)} - \frac{1}{16x^2}$$
$$\geq \frac{x^{\frac{1}{r}}\ln x}{(x+1)}\left(s - \frac{1}{2}\right) + \frac{x^{\frac{1}{r}}}{x+1}\left[s^2 - \frac{1}{8} - \frac{x}{4(x+1)} - \frac{x}{4r} - \frac{x+1}{16x^{2+\frac{1}{r}}}\right]$$
$$> \frac{x^{\frac{1}{r}}}{(x+1)}\left(1 - \frac{1}{8} - \frac{1}{4} - \frac{1}{2} - \frac{1}{8}\right) = 0;$$

(ii) for $x \geq 2$, $-\frac{1}{x-1} \geq -\frac{2}{x}$, we have

$$\theta(x) \geq \frac{x^{\frac{1}{r}}\ln x}{(x+1)}\left[s - \frac{1}{2} - \frac{1}{2(x+1)} - \frac{1}{2r}\right] + \frac{x^{\frac{1}{r}}}{x+1}\left(s^2 - \frac{1}{8} - \frac{x+1}{16x^{2+\frac{1}{r}}}\right)$$
$$> \frac{x^{\frac{1}{r}}\ln x}{(x+1)}\left(s - \frac{1}{2} - \frac{1}{6} - \frac{1}{2r}\right) = \frac{(6s+2r-3)x^{\frac{1}{r}}\ln x}{6r(x+1)} > 0.$$

Hence, for $x \geq 1$, we have $\theta(x) > 0$, it follows $\varpi(x) < c_r$. The lemma is proved. □

Lemma 2.3 *As the assumption of Lemma 2.2, if* $0 < \varepsilon < \frac{p}{r}$, $R = (\frac{1}{r} - \frac{\varepsilon}{p})^{-1}$, $S = (\frac{1}{s} + \frac{\varepsilon}{p})^{-1}$, *then we have*

$$\bar{I} := \sum_{n=1}^{\infty} n^{\frac{1}{S}-\varepsilon-1} \int_1^{\infty} \frac{|\ln(\frac{x}{n})|}{x+n} x^{\frac{1}{R}-1}\,dx \geq \frac{c_R}{\varepsilon} - O(1). \tag{11}$$

Proof It is obvious that $R > 1$, $\frac{1}{R} + \frac{1}{S} = 1$. Setting $u = \frac{x}{n}$, we have

$$\begin{aligned}
\bar{I} &= \sum_{n=1}^{\infty} n^{-\varepsilon-1} \int_{\frac{1}{n}}^{\infty} \frac{|\ln u|}{u+1} u^{\frac{1}{R}-1}\,du \\
&= \sum_{n=1}^{\infty} n^{-\varepsilon-1} \biggl[\int_0^{\infty} \frac{|\ln u|}{u+1} u^{\frac{1}{R}-1}\,du + \int_0^{\frac{1}{n}} \frac{\ln u}{u+1} u^{\frac{1}{R}-1}\,du \biggr] \\
&> c_R \int_1^{\infty} x^{-\varepsilon-1}\,dx + R \sum_{n=1}^{\infty} n^{-\varepsilon-1} \int_0^{\frac{1}{n}} \ln u\,du^{\frac{1}{R}} \\
&= \frac{c_R}{\varepsilon} - \Biggl[R \sum_{n=1}^{\infty} \frac{\ln n}{n^{1+\varepsilon+\frac{1}{R}}} + R^2 \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon+\frac{1}{R}}} \Biggr] \\
&= \frac{c_R}{\varepsilon} - O(1) \quad \bigl(\varepsilon \to 0^+\bigr).
\end{aligned}$$

The lemma is proved. □

Lemma 2.4 *If* $p, r > 1$, $\frac{1}{r} + \frac{1}{s} = \frac{1}{p} + \frac{1}{q} = 1$, $a_n \geq 0$, $f(x)$ *is a non-negative measurable function, then we have*

$$J_1 := \sum_{n=1}^{\infty} n^{\frac{p}{s}-1} \biggl[\int_1^{\infty} \frac{|\ln(\frac{x}{n})| f(x)}{x+n}\,dx \biggr]^p \leq c_r^p \int_1^{\infty} x^{\frac{p}{s}-1} f^p(x)\,dx, \tag{12}$$

$$J_2 := \int_1^{\infty} x^{\frac{q}{r}-1} \Biggl[\sum_{n=1}^{\infty} \frac{|\ln(\frac{x}{n})| a_n}{x+n} \Biggr]^q dx \leq c_r^q \sum_{n=1}^{\infty} n^{\frac{q}{r}-1} a_n^q. \tag{13}$$

Proof By Hölder's inequality [12], in view of (6) and (7), we have

$$\begin{aligned}
&\biggl[\int_1^{\infty} \frac{|\ln(\frac{x}{n})| f(x)}{x+n}\,dx \biggr]^p \\
&\quad = \Biggl\{ \int_1^{\infty} \frac{|\ln(\frac{x}{n})|}{x+n} \biggl[\frac{x^{\frac{1}{sq}}}{n^{\frac{1}{rp}}} f(x) \biggr] \biggl[\frac{n^{\frac{1}{rp}}}{x^{\frac{1}{sq}}} \biggr] dx \Biggr\}^p \\
&\quad \leq \int_1^{\infty} \frac{|\ln(\frac{x}{n})|}{x+n} \biggl(\frac{x}{n} \biggr)^{\frac{1}{r}} x^{\frac{p}{s}-1} f^p(x)\,dx \biggl[\int_1^{\infty} \frac{|\ln(\frac{x}{n})|}{x+n} \biggl(\frac{n}{x} \biggr)^{\frac{1}{s}} n^{\frac{q}{r}-1}\,dx \biggr]^{p-1} \\
&\quad = \int_1^{\infty} \frac{|\ln(\frac{x}{n})|}{x+n} \biggl(\frac{x}{n} \biggr)^{\frac{1}{r}} x^{\frac{p}{s}-1} f^p(x)\,dx \bigl[n^{\frac{q}{r}-1} \omega(n) \bigr]^{p-1} \\
&\quad \leq n^{-\frac{p}{s}+1} c_r^{p-1} \int_1^{\infty} \frac{|\ln(\frac{x}{n})|}{x+n} \biggl(\frac{x}{n} \biggr)^{\frac{1}{r}} x^{\frac{p}{s}-1} f^p(x)\,dx,
\end{aligned}$$

$$J_1 \leq c_r^{p-1} \sum_{n=1}^{\infty} \int_1^{\infty} \frac{|\ln(\frac{x}{n})|}{x+n} \biggl(\frac{x}{n} \biggr)^{\frac{1}{r}} x^{\frac{p}{s}-1} f^p(x)\,dx$$

$$= c_r^{p-1} \int_1^\infty \left[\sum_{n=1}^\infty \frac{|\ln(\frac{x}{n})|}{x+n} \left(\frac{x}{n}\right)^{\frac{1}{r}} \right] x^{\frac{p}{s}-1} f^p(x)\,dx$$

$$= c_r^{p-1} \int_1^\infty \varpi(x) x^{\frac{p}{s}-1} f^p(x)\,dx \le c_r^p \int_1^\infty x^{\frac{p}{s}-1} f^p(x)\,dx.$$

Hence we have (12). Still by Hölder's inequality [12], (6) and (7), we have

$$\left[\sum_{n=1}^\infty \frac{|\ln(\frac{x}{n})| a_n}{x+n} \right]^q = \left\{ \sum_{n=1}^\infty \frac{|\ln(\frac{x}{n})|}{x+n} \left[\frac{x^{\frac{1}{sq}}}{n^{\frac{1}{rp}}} \right] \left[\frac{n^{\frac{1}{rp}}}{x^{\frac{1}{sq}}} a_n \right] \right\}^q$$

$$\le \left[\varpi(x) \sum_{n=1}^\infty x^{\frac{p}{s}-1} \right]^{q-1} \sum_{n=1}^\infty \frac{|\ln(\frac{x}{n})|}{x+n} \left(\frac{x}{n}\right)^{\frac{1}{r}} n^{\frac{q}{r}-1} a_n^q$$

$$\le x^{-\frac{q}{r}+1} c_r^{q-1} \sum_{n=1}^\infty \frac{|\ln(\frac{x}{n})|}{x+n} \left(\frac{x}{n}\right)^{\frac{1}{r}} n^{\frac{q}{r}-1} a_n^q,$$

$$J_2 \le c_r^{q-1} \int_1^\infty \left[\sum_{n=1}^\infty \frac{|\ln(\frac{x}{n})|}{x+n} \left(\frac{x}{n}\right)^{\frac{1}{r}} n^{\frac{q}{r}-1} a_n^q \right] dx$$

$$= c_r^{q-1} \sum_{n=1}^\infty \left[\int_1^\infty \frac{|\ln(\frac{x}{n})|}{x+n} \left(\frac{x}{n}\right)^{\frac{1}{r}} dx \right] n^{\frac{q}{r}-1} a_n^q$$

$$= c_r^{q-1} \sum_{n=1}^\infty \omega(n) n^{\frac{q}{r}-1} a_n^q \le c_r^q \sum_{n=1}^\infty n^{\frac{q}{r}-1} a_n^q.$$

Then we have (13). The lemma is proved. □

3 Main results and applications

Theorem 3.1 *If* $p, r > 1$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{1}{s} = 1$, $a_n, f(x) \ge 0$ *such that* $0 < \int_1^\infty x^{\frac{p}{s}-1} f^p(x)\,dx < \infty$, $\sum_{n=1}^\infty n^{\frac{q}{r}-1} a_n^q < \infty$, *then we have the following equivalent inequalities:*

$$I := \sum_{n=1}^\infty a_n \int_1^\infty \frac{|\ln(\frac{x}{n})| f(x)}{x+n}\,dx = \int_1^\infty f(x) \sum_{n=1}^\infty \frac{|\ln(\frac{x}{n})| a_n}{x+n}\,dx$$

$$< c_r \left\{ \int_1^\infty x^{\frac{p}{s}-1} f^p(x)\,dx \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty n^{\frac{q}{r}-1} a_n^q \right\}^{\frac{1}{q}}, \tag{14}$$

$$J_1 = \sum_{n=1}^\infty n^{\frac{p}{s}-1} \left[\int_1^\infty \frac{|\ln(\frac{x}{n})| f(x)}{x+n}\,dx \right]^p < c_r^p \int_1^\infty x^{\frac{p}{s}-1} f^p(x)\,dx, \tag{15}$$

$$J_2 = \int_1^\infty x^{\frac{q}{r}-1} \left[\sum_{n=1}^\infty \frac{|\ln(\frac{x}{n})| a_n}{x+n} \right]^q dx < c_r^q \sum_{n=1}^\infty n^{\frac{q}{r}-1} a_n^q, \tag{16}$$

where the constant factors $c_r = \sum_{k=0}^\infty (-1)^k [\frac{1}{(k+\frac{1}{r})^2} + \frac{1}{(k+\frac{1}{s})^2}]$, c_r^p, c_r^q *are the best possible.*

Proof By the Lebesgue term-by-term integration theorem [13], there are two kinds of representation in (14). By the conditions of Theorem 3.1, (12) takes the form of a strict inequality, and we have (15). By Hölder's inequality [12], we have

$$I=\sum_{n=1}^{\infty}\left[n^{\frac{1}{s}-\frac{1}{p}}\int_1^{\infty}\frac{|\ln(\frac{x}{n})|f(x)}{x+n}\,dx\right]\left[n^{-\frac{1}{s}+\frac{1}{p}}a_n\right]\le J_1^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{\frac{q}{r}-1}a_n^q\right\}^{\frac{1}{q}}. \tag{17}$$

By (15), we have (14). On the other hand, suppose that (14) is valid. Setting $a_n := n^{\frac{p}{s}-1}[\int_1^{\infty}\frac{|\ln(\frac{x}{n})|f(x)}{x+n}\,dx]^{p-1}$, $n\in\mathbf{N}$, then it follows $J_1=\sum_{n=1}^{\infty}n^{\frac{q}{r}-1}a_n^q$. By (12), we have $J<\infty$. If $J=0$, then (15) is obvious value; if $0<J<\infty$, then by (14), we obtain

$$\begin{aligned}&\sum_{n=1}^{\infty}n^{\frac{q}{r}-1}a_n^q=J_1=I<c_r\left\{\int_1^{\infty}x^{\frac{p}{s}-1}f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{\frac{q}{r}-1}a_n^q\right\}^{\frac{1}{q}},\\&J_1^{\frac{1}{p}}=\left\{\sum_{n=1}^{\infty}n^{\frac{q}{r}-1}a_n^q\right\}^{\frac{1}{p}}<c_r\left\{\int_1^{\infty}x^{\frac{p}{s}-1}f^p(x)\,dx\right\}^{\frac{1}{p}}.\end{aligned} \tag{18}$$

Hence we have (15), which is equivalent to (14).

By Hölder's inequality [12], we have

$$I=\int_1^{\infty}\left[x^{\frac{1}{r}-\frac{1}{q}}f(x)\right]\left[x^{-\frac{1}{r}+\frac{1}{q}}\sum_{n=1}^{\infty}\frac{|\ln(\frac{x}{n})|a_n}{x+n}\right]dx\le\left\{\int_1^{\infty}x^{\frac{p}{s}-1}f^p(x)\,dx\right\}^{\frac{1}{p}}J_2^{\frac{1}{q}}. \tag{19}$$

By (16), we have (14). On the other hand, suppose that (14) is valid. Setting $f(x) := x^{\frac{q}{r}-1}[\sum_{n=1}^{\infty}\frac{|\ln(\frac{x}{n})|a_n}{x+n}]^{q-1}$, $x\in[1,\infty)$, then it follows $J_2=\int_1^{\infty}x^{\frac{p}{s}-1}f^p(x)\,dx$. By (13), we have $J_2<\infty$. If $J_2=0$, then (16) is obvious value; if $0<J_2<\infty$, then by (14), we obtain

$$\begin{aligned}&\int_1^{\infty}x^{\frac{p}{s}-1}f^p(x)\,dx=J_2=I<c_r\left\{\int_1^{\infty}x^{\frac{p}{s}-1}f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{\frac{q}{r}-1}a_n^q\right\}^{\frac{1}{q}},\\&J_2^{\frac{1}{q}}=\left\{\int_1^{\infty}x^{\frac{p}{s}-1}f^p(x)\,dx\right\}^{\frac{1}{q}}<c_r\left\{\sum_{n=1}^{\infty}n^{\frac{q}{r}-1}a_n^q\right\}^{\frac{1}{q}}.\end{aligned} \tag{20}$$

Hence we have (16), which is equivalent to (14). Therefore (14), (15) and (16) are equivalent.

If the constant factor c_r in (14) is not best possible, then there exists a positive number K, with $0<K<c_r$, such that (14) is still valid if we replace c_r by K. For $0<\varepsilon<\varepsilon_0$, setting $\tilde{f}(x)=n^{\frac{1}{r}-\frac{\varepsilon}{p}-1}$, $\tilde{a}_n=n^{\frac{1}{s}-\frac{\varepsilon}{q}-1}$ $(n\in N)$, we have

$$\begin{aligned}\tilde{I}&=\sum_{n=1}^{\infty}\tilde{a}_n\int_1^{\infty}\frac{|\ln(\frac{x}{n})|\tilde{f}(x)}{x+n}\,dx<K\left\{\int_1^{\infty}x^{\frac{p}{s}-1}\tilde{f}^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{\frac{q}{r}-1}\tilde{a}_n^q\right\}^{\frac{1}{q}}\\&=K\left(\int_1^{\infty}x^{-1-\varepsilon}\,dx\right)^{\frac{1}{p}}\left(1+\sum_{n=2}^{\infty}n^{-1-\varepsilon}\right)^{\frac{1}{q}}\\&<K\left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}}\left(1+\int_1^{\infty}x^{-1-\varepsilon}\,dx\right)^{\frac{1}{q}}=\frac{K}{\varepsilon}(\varepsilon+1)^{\frac{1}{q}}.\end{aligned} \tag{21}$$

By (11) and (21), we have $c_R-\varepsilon O(1)<K(\varepsilon+1)^{\frac{1}{q}}$ and for $\varepsilon\to0^+$, by Fatou lemma [13], we have $c_r\le\lim_{\varepsilon\to0^+}(c_R-\varepsilon O(1))\le K$. This is a contradiction. Hence we can conclude that

the constant c_r in (14) is the best possible. If the constant factors in (15) and (16) are not the best possible, then we can imply a contradiction that the constant factor in (14) is not the best possible by (17) and (19). The theorem is proved. □

Remark For $p = q = r = s = 2$, (14) reduces to (4). Inequality (4) is a new basic half-discrete Hilbert-type inequality with the non-monotone kernel.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
DX carried out the study, and wrote the manuscript. BY participated in the design of the study, and reformed the manuscript. All authors read and approved the final manuscript.

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References
1. Hardy, GH, Littlewood, JE, Polya, G: Inequalities. Cambridge University Press, Cambridge (1952)
2. Xin, D, Yang, B: A basic Hilbert-type inequality. J. Math. (Wuhan) **30**(3), 554-560 (2010)
3. Yang, B: On a basic Hilbert-type integral inequality and extensions. Coll. Math. J. **24**(1), 87-92 (2008)
4. Yang, B: A mixed Hilbert-type inequality with a best constant factor. Int. J. Pure Appl. Math. **20**(3), 319-328 (2005)
5. Yang, B: On a half-discrete Hilbert-type inequality. J. Shandou Univ., Nat. Sci. **26**(4), 5-10 (2011)
6. Yang, B: A half-discrete Hilbert-type inequality with a homogeneous kernel of positive degree. J. Zhanjiang Norm. Coll. **32**(3), 5-9 (2011)
7. Yang, B: A half-discrete reverse Hilbert-type inequality with non-homogeneous kernel. J. Hunan Inst. Sci. Technol., Nat. Sci. **24**(3), 1-4 (2011)
8. Yang, B: On a half-discrete reverse Hilbert-type inequality with a non-homogeneous kernel. J. Inn. Mong. Norm. Univ., Nat. Sci. **40**(5), 433-436 (2011)
9. Zhong, W: A mixed Hilbert-type inequality and its equivalent forms. J. Guangdong Univ. Educ. **31**(5), 18-22 (2011)
10. Yang, B: A half-discrete Hilbert-type inequality with the non-monotone kernel. J. Jilin Univ., Sci. **49**(1), 87-92 (2012)
11. Yang, B: The Norm of Operator and Hilbert-Type Inequalities. Science Press, Beijing (2009)
12. Kuang, J: Applied Inequalities. Shangdong Science Press, Jinan (2004)
13. Kuang, J: Introduction to Real Analysis. Hunan Education Press, Changsha (1996)

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