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Weighted composition followed and proceeded by differentiation operators from $Q_k(p,q)$ spaces to Bloch-type spaces

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Abstract

In this paper, we investigate boundedness and compactness of the weighted composition followed and proceeded by differentiation operators from $Q_k(p,q)$ spaces to Bloch-type spaces and little Bloch-type spaces. Some sufficient and necessary conditions for the boundedness and compactness of these operators are obtained.

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Keywords: $Q_k(p,q)$ spaces; Bloch-type spaces; weighted composition followed and proceeded by differentiation operators; boundedness; compactness

1 Introduction

Let Δ be an open unit disc in the complex plane, and let $H(\Delta)$ be the class of all analytic functions on Δ. The α-Bloch space B^α ($\alpha > 0$) is, by definition, the set of all function f in $H(\Delta)$ such that

$$\|f\|_{B^\alpha} = |f(0)| + \sup_{z\in\Delta}\left(1-|z|^2\right)^\alpha \left|f'(z)\right| < \infty.$$

Under the above norm, B^α is a Banach space. When $\alpha = 1$, $B^1 = B$ is the well-known Bloch space. Let B_0^α denote the subspace of B^α, for f

$$B_0^\alpha = \left\{f : \left(1-|z|^2\right)^\alpha \left|f'(z)\right| \to 0 \text{ as } |z| \to 1, f \in B^\alpha\right\}.$$

This space is called a little α-Bloch space.

Assume that μ is a positive continuous function on $[0,1)$, having the property that there exist positive numbers s and t, $0 < s < t$, and $\delta \in [0,1)$, such that

$$\begin{aligned} &\frac{\mu(r)}{(1-r)^s} \quad \text{is decreasing on } [\delta,1), \lim_{r\to1}\frac{\mu(r)}{(1-r)^s} = 0, \\ &\frac{\mu(r)}{(1-r)^t} \quad \text{is increasing on } [\delta,1), \lim_{r\to1}\frac{\mu(r)}{(1-r)^t} = \infty. \end{aligned}$$

Then μ is called a normal function (see [9]).

Denote (see, *e.g.*, [2, 4, 10])

$$B_\mu = \Big\{ f : \|f\|_{B_\mu} = \big|f(0)\big| + \sup_{z\in\Delta} \mu\big(|z|\big)\big|f'(z)\big| < \infty, f \in H(\Delta) \Big\}.$$

It is known that B_μ is a Banach space with the norm $\|\cdot\|_{B_\mu}$ (see [4]).

Let $B_{\mu,0}$ denote the subspace of B_μ, *i.e.*,

$$B_{\mu,0} = \big\{ f : \mu\big(|z|\big)\big|f'(z)\big| \to 0 \text{ as } |z| \to 1, f \in B_\mu \big\}.$$

This space is called a little Bloch-type space. When $\mu(r) = (1-r^2)^\alpha$, the induced space B_μ becomes the α-Bloch space B^α.

Throughout this paper, we assume that K is a right continuous and nonnegative nondecreasing function. For $0 < p < \infty$, $-2 < q < \infty$, we say that a function $f \in H(\Delta)$ belongs to the space $Q_k(p,q)$ (see, [11]), if

$$\|f\| = \left\{ \sup_{z\in\Delta} \int_\Delta \big|f'(z)\big|^p \big(1-|z|^2\big)^q K\big(g(z,a)\big)\, dA(z) \right\}^{\frac{1}{p}} < \infty,$$

where dA denotes the normalized Lebesgue area measure on Δ, $g(z,a)$ is the Green function with logarithmic singularity at a, that is, $g(z,a) = \log \frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ for $a \in \Delta$. When $K(x) = x^s$, $s \geq 0$, the space $Q_k(p,q)$ equals to $F(p,q,s)$, which is introduced by Zhao in [13]. Moreover (see [13]), we have that $F(p,q,s) = B^{\frac{q+2}{p}}$ and $F_0(p,q,s) = B_0^{\frac{q+2}{p}}$ for $s > 1$, $F(p,q,s) \subseteq B^{\frac{q+2}{p}}$ and $F_0(p,q,s) \subseteq B_0^{\frac{q+2}{p}}$ for $0 \leq s < 1$. When $p \geq 1$, $Q_k(p,q)$ is a Banach space with the norm

$$\|f\|_{Q_k(p,q)} = \big|f(0)\big| + \|f\|.$$

From [11], we know that $Q_k(p,q) \subseteq B^{\frac{q+2}{p}}$, $Q_k(p,q) = B^{\frac{q+2}{p}}$ if and only if

$$\int_0^1 K\left(\log\frac{1}{r}\right)\big(1-r^2\big)^{-2} r\, dr < \infty.$$

Moreover, $\|f\|_{B^{\frac{q+2}{p}}} \leq C\|f\|_{Q_k(p,q)}$ (see [11, Theorem 2.1]).

Throughout the paper, we assume that

$$\int_0^1 K\left(\log\frac{1}{r}\right)\big(1-r^2\big)^{q} r\, dr < \infty,$$

otherwise $Q_k(p,q)$ consists only of constant functions (see [11]).

Let φ be a nonconstant analytic self-map of Δ, and let ϕ be an analytic function in Δ. We define the linear operators

$$\phi C_\varphi D f = \phi\big(f' o \varphi\big) = \phi f'(\varphi) \quad \text{and} \quad \phi D C_\varphi f = \phi(f o \varphi)' = \phi f'(\varphi)\varphi', \quad \text{for } f \in H(\Delta).$$

They are called weighted composition followed and proceeded by differentiation operators respectively, where C_φ and D are composition and differentiation operators respectively. The boundedness and compactness of DC_φ on the Hardy spaces were investigated by Hibschweiler and Portnoy in [3] and by Ohno in [8]. In [6], Li and Stević studied the boundedness and compactness of the operator DC_φ on the α-Bloch spaces. In [7], Li and Stević studied the boundedness and compactness of the composition and differentiation operators between H^∞ and α-Bloch spaces. In [12], Yang studied the boundedness and compactness of the operator DC_φ (or $C_\varphi D$) from $Q_k(p,q)$ to the Bloch-type spaces.

In this paper, we investigate the operators ϕDC_φ and $\phi C_\varphi D$ from $Q_k(p,q)$ spaces to Bloch-type spaces and little Bloch-type spaces. Some sufficient and necessary conditions for the boundedness and compactness of these operators are given. Our results also generalize some known results in [12].

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A \approx B$ means that there is a positive constant C such that $\frac{B}{C} \le A \le CB$.

2 Statement of the main results

In this paper, we shall prove the following results.

Theorem 2.1 *Let φ be an analytic self-map of Δ, and let ϕ be an analytic function in Δ. Suppose that μ is normal, $p > 0$, $q > -2$, and K is a nonnegative nondecreasing function on $[0,\infty)$ such that*

$$\int_0^1 K\left(\log\frac{1}{r}\right)(1-r)^{\min\{-1,q\}}\left(\log\frac{1}{1-r}\right)^{\chi_{-1}(q)} r\,dr < \infty, \tag{2.1}$$

where $\chi_A(x)$ denote the characteristic function of the set A. Then $\phi DC_\varphi : Q_k(p,q) \to B_\mu$ is bounded if and only if

$$\sup_{z\in\Delta}\mu\big(|z|\big)\frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}} < \infty, \qquad \sup_{z\in\Delta}\mu\big(|z|\big)\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} < \infty. \tag{2.2}$$

Theorem 2.2 *Let φ be an analytic self-map of Δ, and let ϕ be an analytic function in Δ. Suppose that μ is normal, $p > 0$, $q > -2$, and K is a nonnegative nondecreasing function on $[0,\infty)$ such that (2.1) hold. Then $\phi DC_\varphi : Q_k(p,q) \to B_\mu$ is compact if and only if $\phi DC_\varphi : Q_k(p,q) \to B_\mu$ is bounded, and*

$$\begin{aligned}&\lim_{|\varphi(z)|\to 1}\mu\big(|z|\big)\frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}} = 0,\\ &\lim_{|\varphi(z)|\to 1}\mu\big(|z|\big)\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} = 0.\end{aligned} \tag{2.3}$$

Theorem 2.3 *Let φ be an analytic self-map of Δ, and let ϕ be an analytic function in Δ. Suppose that μ is normal, $p > 0$, $q > -2$, and K is a nonnegative nondecreasing function on $[0,\infty)$ such that (2.1) hold. Then $\phi DC_\varphi : Q_k(p,q) \to B_{\mu,0}$ is compact if and only if*

$$\lim_{|z|\to 1}\mu\big(|z|\big)\frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}} = 0, \qquad \lim_{|z|\to 1}\mu\big(|z|\big)\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} = 0. \tag{2.4}$$

From the above three theorems, we get the following

Corollary 2.4 *Let φ be an analytic self-map of Δ, and let ϕ be an analytic function in Δ. Then the following statements hold.*

(i) *$\phi DC_\varphi : B \to B$ is bounded if and only if*

$$\sup_{z\in\Delta}\left(1-|z|^2\right)\frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2} < \infty, \qquad \sup_{z\in\Delta}\left(1-|z|^2\right)\frac{|\phi(z)(\varphi''(z)) + \phi'(z)\varphi'(z)|}{1-|\varphi(z)|^2} < \infty.$$

(ii) *$\phi DC_\varphi : B \to B$ is compact if and only if $\phi DC_\varphi : B \to B$ is bounded, and*

$$\lim_{|\varphi(z)|\to 1}\left(1-|z|^2\right)\frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2} = 0,$$
$$\lim_{|\varphi(z)|\to 1}\left(1-|z|^2\right)\frac{|\phi(z)(\varphi''(z)) + \phi'(z)\varphi'(z)|}{1-|\varphi(z)|^2} = 0.$$

(iii) *$\phi DC_\varphi : B \to B_0$ is compact if and only if*

$$\lim_{|z|\to 1}\left(1-|z|^2\right)\frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^2} = 0, \qquad \lim_{|z|\to 1}\left(1-|z|^2\right)\frac{|\phi(z)(\varphi''(z)) + \phi'(z)\varphi'(z)|}{1-|\varphi(z)|^2} = 0.$$

Theorem 2.5 *Let φ be an analytic self-map of Δ, and let ϕ be an analytic function in Δ. Suppose that μ is normal, $p > 0$, $q > -2$, and K is a nonnegative nondecreasing function on $[0,\infty)$ such that (2.1) hold. Then the following statements hold.*

(i) *$\phi C_\varphi D : Q_k(p,q) \to B_\mu$ is bounded if and only if*

$$\sup_{z\in\Delta}\mu\left(|z|\right)\frac{|\phi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}} < \infty, \qquad \sup_{z\in\Delta}\mu\left(|z|\right)\frac{|\phi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} < \infty.$$

(ii) *$\phi C_\varphi D : Q_k(p,q) \to B_\mu$ is compact if and only if $\phi C_\varphi D : Q_k(p,q) \to B_\mu$ is bounded, and*

$$\lim_{|\varphi(z)|\to 1}\mu\left(|z|\right)\frac{|\phi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}} = 0, \qquad \lim_{|\varphi(z)|\to 1}\mu\left(|z|\right)\frac{|\phi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} = 0.$$

(iii) *$\phi C_\varphi D : Q_k(p,q) \to B_{\mu,0}$ is compact if and only if*

$$\lim_{|z|\to 1}\mu\left(|z|\right)\frac{|\phi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}} = 0, \qquad \lim_{|z|\to 1}\mu\left(|z|\right)\frac{|\phi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} = 0.$$

From Theorem 2.5, we get the following

Corollary 2.6 *Let φ be an analytic self-map of Δ, and let ϕ be an analytic function in Δ. Then the following statements hold.*

(i) *$\phi C_\varphi D : B \to B$ is bounded if and only if*

$$\sup_{z\in\Delta}\left(1-|z|^2\right)\frac{|\phi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^2} < \infty, \qquad \sup_{z\in\Delta}\left(1-|z|^2\right)\frac{|\phi'(z)|}{1-|\varphi(z)|^2} < \infty.$$

(ii) *$\phi C_\varphi D : B \to B$ is compact if and only if $\phi C_\varphi D : B \to B$ is bounded, and*

$$\lim_{|\varphi(z)|\to 1}\left(1-|z|^2\right)\frac{|\phi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^2} = 0, \qquad \lim_{|\varphi(z)|\to 1}\left(1-|z|^2\right)\frac{|\phi'(z)|}{1-|\varphi(z)|^2} = 0.$$

(iii) $\phi C_\varphi D : B \to B_0$ *is compact if and only if*

$$\lim_{|z|\to 1}\left(1-|z|^2\right)\frac{|\phi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^2} = 0, \qquad \lim_{|z|\to 1}\left(1-|z|^2\right)\frac{|\phi'(z)|}{1-|\varphi(z)|^2} = 0.$$

3 Proofs of the main results

In this section, we will prove our main results. For this purpose, we need some auxiliary results.

Lemma 3.1 *Let φ be an analytic self-map of Δ, ϕ be an analytic function in Δ. Suppose $p > 0$, $q > -2$. Then ϕDC_φ (or $\phi C_\varphi D$) : $Q_k(p,q) \to B_\mu$ is compact if and only if ϕDC_φ (or $\phi C_\varphi D$) : $Q_k(p,q) \to B_\mu$ is bounded and for any bounded sequence $\{f_n\}_{n\in N}$ in $Q_k(p,q)$ which converges to zero uniformly on compact subsets of Δ as $n \to \infty$, and $\|\phi DC_\varphi f_n\|_{B_\mu} \to 0$ (or $\|\phi C_\varphi D f_n\|_{B_\mu} \to 0$) as $n \to \infty$.*

Lemma 3.1 can be proved by standard way (see [1, Proposition 3.11]).

Lemma 3.2 *A closed set $\mathbb{K}$ of $B_{\mu,0}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in\mathbb{K}}\mu\left(|z|\right)\left|f'(z)\right| = 0. \tag{3.1}$$

Proof First of all, we suppose that $\mathbb{K}$ is compact and let $\varepsilon > 0$. By the definition of $B_{\mu,0}$, we can choose an $\frac{\varepsilon}{2}$-net which center at $f_1, f_2, \ldots, f_n$ in $\mathbb{K}$ respectively, and a positive number r $(0 < r < 1)$ such that $\mu(|z|)|f_i'(z)| < \frac{\varepsilon}{2}$, for $1 \le i \le n$ and $|z| > r$. If $f \in \mathbb{K}$, $\|f - f_i\|_{B_\mu} < \frac{\varepsilon}{2}$ for some f_i, so we have

$$\mu\left(|z|\right)\left|f'(z)\right| \le \|f - f_i\|_{B_\mu} + \mu\left(|z|\right)\left|f_i'(z)\right| < \varepsilon,$$

for $|z| > r$. This establishes (3.1). □

On the other hand, if $\mathbb{K}$ is a closed bounded set which satisfies (3.1) and $\{f_n\}$ is a sequence in $\mathbb{K}$, then by the Montel's theorem, there is a subsequence $\{f_{n_k}\}$ which converges uniformly on compact subsets of Δ to some analytic function f, and also $\{f_{n_k}'\}$ converges uniformly to f' on compact subsets of Δ. According to (3.1), for every $\varepsilon > 0$, there is an r, $0 < r < 1$, such that for all $g \in \mathbb{K}$, $\mu(|z|)|g'(z)| < \frac{\varepsilon}{2}$, if $|z| > r$. It follows that $\mu(|z|)|f'(z)| < \frac{\varepsilon}{2}$, if $|z| > r$. Since $\{f_{n_k}\}$ converges uniformly to f and $\{f_{n_k}'\}$ converges uniformly to f' on $|z| \le r$, it follows that $\lim_{k\to\infty}\sup\|f_{n_k} - f\|_{B_\mu} \le \varepsilon$, *i.e.*, $\lim_{k\to\infty}\|f_{n_k} - f\|_{B_\mu} = 0$, so that $\mathbb{K}$ is compact.

Lemma 3.3 ([14]) *Let $\alpha > 0$ and $f \in H(\Delta)$. Then we have*

$$\sup_{z\in\Delta}\left(1-|z|^2\right)^{\alpha}\left|f'(z)\right| \approx \left|f'(0)\right| + \sup_{z\in\Delta}\left(1-|z|^2\right)^{\alpha+1}\left|f''(z)\right|.$$

Proof of Theorem 2.1 First, suppose that the conditions in (2.2) hold. Then for any $z \in \Delta$ and $f \in Q_k(p,q)$, by use of the fact $\|f\|_{B^{\frac{q+2}{p}}} \le C\|f\|_{Q_k(p,q)}$ and Lemma 3.3, we have

$$\begin{aligned}
&\mu\left(|z|\right)\left|(\phi DC_\varphi f)'(z)\right| \\
&\quad = \mu\left(|z|\right)\left|\phi'(z)\varphi'(z)f'\left(\varphi(z)\right) + \phi(z)\left[f''\left(\varphi(z)\right)\left(\varphi'(z)\right)^2 + f'\left(\varphi(z)\right)\varphi''(z)\right]\right|
\end{aligned}$$

$$\begin{aligned}
&\le \mu\big(|z|\big)\big|\phi(z)f''\big(\varphi(z)\big)\big(\varphi'(z)\big)^2\big| + \mu\big(|z|\big)\big|\big[\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)\big]f'\big(\varphi(z)\big)\big| \\
&\le \mu\big(|z|\big)C\frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}}\|f\|_{B^{\frac{q+2}{p}}} + \mu\big(|z|\big)\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}}\|f\|_{B^{\frac{q+2}{p}}} \\
&\le \left\{\mu\big(|z|\big)C\frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}} + \mu\big(|z|\big)\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}}\right\}\|f\|_{Q_k(p,q)}. \qquad (3.2)
\end{aligned}$$

Taking the supremum in (3.2) for $z \in \Delta$, and employing (2.2), we deduce that

$$\phi DC_\varphi : Q_k(p,q) \to B_\mu$$

is bounded.

Conversely, suppose that $\phi DC_\varphi : Q_k(p,q) \to B_\mu$ is bounded. Then there exists a constant C such that $\|\phi DC_\varphi f\|_{B_\mu} \le C\|f\|_{Q_k(p,q)}$ for all $f \in Q_k(p,q)$. Taking the functions $f(z) \equiv z$, and $f(z) \equiv \frac{z^2}{2}$, which belong to $Q_k(p,q)$, we get

$$\sup_{z\in\Delta} \mu\big(|z|\big)\big|\phi(z)\varphi''(z) + \phi'(z)\varphi'(z)\big| < \infty \qquad (3.3)$$

and

$$\sup_{z\in\Delta} \mu\big(|z|\big)\big|\phi(z)\big(\varphi'(z)\big)^2 + \varphi(z)\big[\phi(z)\varphi''(z) + \phi'(z)\varphi'(z)\big]\big| < \infty. \qquad (3.4)$$

From (3.3), (3.4), and the boundedness of the function $\varphi(z)$, it follows that

$$\sup_{z\in\Delta} \mu\big(|z|\big)\big|\phi(z)\big(\varphi'(z)\big)^2\big| < \infty. \qquad (3.5)$$

For $w \in \Delta$, let

$$f_w(z) = \frac{1-|w|^2}{(1-\bar{w}z)^{\frac{q+2}{p}}},$$

by direct calculation, we get

$$f_w'(w) = \frac{q+2}{p}\frac{\bar{w}}{(1-|w|^2)^{\frac{q+2}{p}}}, \qquad f_w''(w) = \frac{q+2}{p}\frac{p+q+2}{p}\frac{\bar{w}^2}{(1-|w|^2)^{\frac{p+q+2}{p}}}.$$

From [5], we know that $f_w \in Q_k(p,q)$, for each $w \in \Delta$. Moreover, there is a positive constant C such that $\sup_{w\in\Delta}\|f_w\|_{Q_k(p,q)} \le C$. Hence, we have

$$\begin{aligned}
C\|\phi DC_\varphi\| &\ge \|\phi DC_\varphi f_{\varphi(z)}\|_{B_\mu} \\
&\ge -\frac{q+2}{p}\frac{p+q+2}{p}\mu\big(|z|\big)\frac{|\phi(z)(\varphi'(z))^2(\varphi(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}} \\
&\quad + \frac{q+2}{p}\mu\big(|z|\big)\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}}, \qquad (3.6)
\end{aligned}$$

for $z \in \Delta$. Therefore, we obtain

$$\begin{aligned}
&\mu(|z|)\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} \\
&\quad\le C\|\phi DC_\varphi\| + \frac{p+q+2}{p}\mu(|z|)\frac{|\phi(z)(\varphi'(z))^2(\varphi(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}}.
\end{aligned} \tag{3.7}$$

Next, for $w \in \Delta$, let

$$g_w(z) = \frac{(1-|w|^2)^2}{(1-\bar{w}z)^{\frac{p+q+2}{p}}} - \frac{p+q+2}{q+2}\frac{1-|w|^2}{(1-\bar{w}z)^{\frac{q+2}{p}}}.$$

Then from [5], we see that $g_w(z) \in Q_k(p,q)$ and $\sup_{w\in\Delta}\|g_w\|_{Q_k(p,q)} < \infty$. Since

$$g'_{\varphi(z)}(\varphi(z)) = 0, \qquad \left|g''_{\varphi(z)}(\varphi(z))\right| = \frac{p+q+2}{p}\frac{|\varphi(z)|^2}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}},$$

we have

$$\begin{aligned}
\infty &> C\|\phi DC_\varphi\| \ge \|\phi DC_\varphi g_{\varphi(z)}\|_{B_\mu} \\
&\ge \frac{p+q+2}{p}\mu(|z|)\frac{|\phi(z)(\varphi'(z))^2(\varphi(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}}.
\end{aligned} \tag{3.8}$$

Thus

$$\begin{aligned}
&\sup_{|\varphi(z)|>\frac{1}{2}}\mu(|z|)\frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}} \\
&\quad\le \sup_{|\varphi(z)|>\frac{1}{2}}4\mu(|z|)\frac{|\phi(z)(\varphi'(z))^2(\varphi(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}} \le C\|\phi DC_\varphi\| < \infty.
\end{aligned} \tag{3.9}$$

Inequality (3.5) gives

$$\sup_{|\varphi(z)|\le\frac{1}{2}}\mu(|z|)\frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}} \le \left(\frac{4}{3}\right)^{\frac{p+q+2}{p}}\sup_{|\varphi(z)|\le\frac{1}{2}}\mu(|z|)\left|\phi(z)(\varphi'(z))^2\right| < \infty. \tag{3.10}$$

Therefore, the first inequality in (2.2) follows from (3.9) and (3.10). From (3.7) and (3.8), we obtain

$$\sup_{z\in\Delta}\mu(|z|)\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} < \infty. \tag{3.11}$$

Inequalities (3.3) and (3.11) imply

$$\begin{aligned}
&\sup_{|\varphi(z)|>\frac{1}{2}}\mu(|z|)\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} \\
&\quad\le 2\sup_{|\varphi(z)|>\frac{1}{2}}\mu(|z|)\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} < \infty,
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
&\sup_{|\varphi(z)|\le\frac{1}{2}} \mu\big(|z|\big)\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} \\
&\quad\le \left(\frac{4}{3}\right)^{\frac{q+2}{p}} \sup_{|\varphi(z)|\le\frac{1}{2}} \mu\big(|z|\big)\big|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)\big| < \infty. \qquad (3.13)
\end{aligned}$$

Inequality (3.12) together with (3.13) implies the second inequality of (2.2). The proof of Theorem 2.1 is completed. □

Proof of Theorem 2.2 First, suppose that $\phi DC_\varphi : Q_k(p,q) \to B_\mu$ is bounded and (2.3) hold. Let $\{f_n\}_{n\in N}$ be a sequence in $Q_k(p,q)$ such that $\sup_{n\in N} \|f_n\|_{Q_k(p,q)} < \infty$, and f_n converges to 0 uniformly on compact subsets of Δ as $n \to \infty$. By the assumption, for any $\varepsilon > 0$, there exists a $\delta \in (0,1)$ such that

$$\mu\big(|z|\big)\frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}} < \varepsilon$$

and

$$\mu\big(|z|\big)\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} < \varepsilon$$

hold for $\delta < |\varphi(z)| < 1$. Since $\phi DC_\varphi : Q_k(p,q) \to B_\mu$ is bounded, it follows from the proof of Theorem 2.1 that

$$\begin{aligned}
M_1 &:= \sup_{z\in\Delta} \mu\big(|z|\big)\big|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)\big| < \infty, \\
M_2 &:= \sup_{z\in\Delta} \mu\big(|z|\big)\big|\phi(z)\big(\varphi'(z)\big)^2\big| < \infty.
\end{aligned}$$

Let $\mathbb{K} = \{z \in \Delta : |\varphi(z)| \le \delta\}$. Then we have

$$\begin{aligned}
\|\phi DC_\varphi f_n\|_{B_\mu} &= \sup_{z\in\Delta} \mu\big(|z|\big)\big|(\phi DC_\varphi f_n)'(z)\big| + \big|\phi(0)f_n'\big(\varphi(0)\big)\varphi'(0)\big| \\
&\le \sup_{z\in\Delta} \mu\big(|z|\big)\big|\phi(z)f_n''\big(\varphi(z)\big)\big(\varphi'(z)\big)^2\big| \\
&\quad + \sup_{z\in\Delta} \mu\big(|z|\big)\big|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)\big|\big|f_n'\big(\varphi(z)\big)\big| + \big|\phi(0)f_n'\big(\varphi(0)\big)\varphi'(0)\big| \\
&\le \sup_{z\in\mathbb{K}} \mu\big(|z|\big)\big|\phi(z)f_n''\big(\varphi(z)\big)\big(\varphi'(z)\big)^2\big| \\
&\quad + \sup_{z\in\mathbb{K}} \mu\big(|z|\big)\big|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)\big|\big|f_n'\big(\varphi(z)\big)\big| \\
&\quad + \sup_{z\in(\Delta-\mathbb{K})} \mu\big(|z|\big)\big|\phi(z)f_n''\big(\varphi(z)\big)\big(\varphi'(z)\big)^2\big| \\
&\quad + \sup_{z\in(\Delta-\mathbb{K})} \mu\big(|z|\big)\big|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)\big|\big|f_n'\big(\varphi(z)\big)\big| + \big|\phi(0)f_n'\big(\varphi(0)\big)\varphi'(0)\big| \\
&\le \sup_{z\in\mathbb{K}} \mu\big(|z|\big)\big|\phi(z)f_n''\big(\varphi(z)\big)\big(\varphi'(z)\big)^2\big|
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sup_{z\in\mathbb{K}} \mu(|z|)\left|\phi(z)\varphi''(z) + \phi'(z)\varphi'(z)\right|\left|f_n'(\varphi(z))\right| + \left|\phi(0)f_n'(\varphi(0))\varphi'(0)\right| \\
&\quad + C \sup_{z\in(\Delta-\mathbb{K})} \mu(|z|) \frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}} \|f_n\|_{Q_k(p,q)} \\
&\quad + \sup_{z\in(\Delta-\mathbb{K})} \mu(|z|) \frac{|\phi(z)\varphi''(z) + \phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} \|f_n\|_{Q_k(p,q)} \\
&\le M_2 \sup_{z\in\mathbb{K}} \left|f_n''(\varphi(z))\right| + M_1 \sup_{z\in\mathbb{K}} \left|f_n'(\varphi(z))\right| \\
&\quad + 2C\varepsilon \|f_n\|_{Q_k(p,q)} + \left|\phi(0)f_n'(\varphi(0))\varphi'(0)\right|. \qquad (3.14)
\end{aligned}
$$

From the fact that $f_n \to 0$ as $n \to \infty$ on compact subsets of Δ, and Cauchy's estimate, we conclude that $f_n' \to 0$ and $f_n'' \to 0$ as $n \to \infty$ on compact subsets of Δ. Letting $n \to \infty$ in (3.14) and using the fact that ε is an arbitrary positive number, we obtain $\lim_{n\to\infty} \|\phi DC_\varphi f_n\|_{B_\mu} = 0$. Applying Lemma 3.1, the result follows.

Conversely, suppose that $\phi DC_\varphi : Q_k(p,q) \to B_\mu$ is compact. Then it is clear that $\phi DC_\varphi : Q_k(p,q) \to B_\mu$ is bounded. Let $\{z_n\}$ be a sequence in Δ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. For $n \in N$, let

$$
f_n(z) = \frac{1-|\varphi(z_n)|^2}{(1-\overline{\varphi(z_n)}z)^{\frac{q+2}{p}}}.
$$

Then $\sup_{n\in N} \|f_n\|_{Q_k(p,q)} < \infty$ and f_n converges to 0 uniformly on compact subsets of Δ as $n \to \infty$. Since $\phi DC_\varphi : Q_k(p,q) \to B_\mu$ is compact, by Lemma 3.1, we have $\lim_{n\to\infty} \|\phi DC_\varphi \times f_n\|_{B_\mu} = 0$. On the other hand, from (3.6) we have

$$
\begin{aligned}
C\|\phi DC_\varphi f_n\|_{B_\mu} \ge{}& -\frac{q+2}{p}\frac{p+q+2}{p}\mu(|z_n|)\frac{|\phi(z_n)(\varphi'(z_n))^2(\varphi(z_n))^2|}{(1-|\varphi(z_n)|^2)^{\frac{p+q+2}{p}}} \\
&+ \frac{q+2}{p}\mu(|z_n|)\frac{|\phi(z_n)\varphi''(z_n) + \phi'(z_n)\varphi'(z_n)||\varphi(z_n)|}{(1-|\varphi(z_n)|^2)^{\frac{q+2}{p}}},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\lim_{|\varphi(z_n)|\to 1} \frac{p+q+2}{p}\mu(|z_n|)\frac{|\phi(z_n)(\varphi'(z_n))^2(\varphi(z_n))^2|}{(1-|\varphi(z_n)|^2)^{\frac{p+q+2}{p}}} \\
&\quad = \lim_{|\varphi(z_n)|\to 1} \mu(|z_n|)\frac{|\phi(z_n)\varphi''(z_n) + \phi'(z_n)\varphi'(z_n)||\varphi(z_n)|}{(1-|\varphi(z_n)|^2)^{\frac{q+2}{p}}}, \qquad (3.15)
\end{aligned}
$$

if one of these two limits exists.

Next, for $n \in N$, set

$$
g_n(z) = \frac{(1-|\varphi(z_n)|^2)^2}{(1-\overline{\varphi(z_n)}z)^{\frac{p+q+2}{p}}} - \frac{p+q+2}{q+2}\frac{1-|\varphi(z_n)|^2}{(1-\overline{\varphi(z_n)}z)^{\frac{q+2}{p}}}.
$$

Then $\{g_n\}_{n\in N}$ is a sequence in $Q_k(p,q)$. Notice that $g_n'(\varphi(z_n)) = 0$,

$$
\left|g_n''(\varphi(z_n))\right| = \frac{p+q+2}{p}\frac{|\varphi(z_n)|^2}{(1-|\varphi(z_n)|^2)^{\frac{p+q+2}{p}}}.
$$

And g_n converges to 0 uniformly on compact subsets of Δ as $n \to \infty$. Since $\phi DC_\varphi : Q_k(p,q) \to B_\mu$ is compact, we have $\lim_{n\to\infty} \|\phi DC_\varphi g_n\|_{B_\mu} = 0$. On the other hand, since

$$\|\phi DC_\varphi f_n\|_{B_\mu} \geq \frac{p+q+2}{p}\mu\big(|z_n|\big)\frac{|\phi(z_n)(\varphi'(z_n))^2(\varphi(z_n))^2|}{(1-|\varphi(z_n)|^2)^{\frac{p+q+2}{p}}},$$

we have

$$\begin{aligned}
&\lim_{|\varphi(z_n)|\to 1}\mu\big(|z_n|\big)\frac{|\phi(z_n)(\varphi'(z_n))^2|}{(1-|\varphi(z_n)|^2)^{\frac{p+q+2}{p}}} \\
&\quad = \lim_{|\varphi(z_n)|\to 1}\mu\big(|z_n|\big)\frac{|\phi(z_n)(\varphi'(z_n))^2(\varphi(z_n))^2|}{(1-|\varphi(z_n)|^2)^{\frac{p+q+2}{p}}} = 0.
\end{aligned} \tag{3.16}$$

From (3.15) and (3.16), we get

$$\lim_{|\varphi(z_n)|\to 1}\mu\big(|z_n|\big)\frac{|\phi(z_n)\varphi''(z_n)+\phi'(z_n)\varphi'(z_n)|}{(1-|\varphi(z_n)|^2)^{\frac{q+2}{p}}} = 0. \tag{3.17}$$

The proof of Theorem 2.2 is completed. □

Proof of Theorem 2.3 First, let $f \in Q_k(p,q)$. By the proof of Theorem 2.1, we have

$$\begin{aligned}
\mu\big(|z|\big)\big|(\phi DC_\varphi f)'(z)\big| \leq C\bigg\{&\mu\big(|z|\big)\bigg|\frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}} \\
&+\mu\big(|z|\big)\bigg|\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}}\bigg\}\|f\|_{Q_k(p,q)}.
\end{aligned} \tag{3.18}$$

Taking the supremum in (3.18) over all $f \in Q_k(p,q)$ such that $\|f\|_{Q_k(p,q)} \leq 1$, we can get

$$\lim_{|z|\to 1}\sup_{\|f\|_{Q_k(p,q)}\leq 1}\mu\big(|z|\big)\big|(\phi DC_\varphi f)'(z)\big| = 0.$$

By Lemma 3.2, we see that the operator $\phi DC_\varphi : Q_k(p,q) \to B_{\mu,0}$ is compact.

Conversely, suppose that $\phi DC_\varphi : Q_k(p,q) \to B_{\mu,0}$ is compact. By taking $f(z) \equiv z$ and using the boundedness of $\phi DC_\varphi : Q_k(p,q) \to B_{\mu,0}$, we get

$$\lim_{|z|\to 1}\mu\big(|z|\big)\big|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)\big| = 0. \tag{3.19}$$

From this, by taking the test function $f(z) \equiv \frac{z^2}{2}$ and using the boundedness of $\phi DC_\varphi : Q_k(p,q) \to B_{\mu,0}$, it follows that

$$\lim_{|z|\to 1}\mu\big(|z|\big)\big|\phi(z)\big(\varphi'(z)\big)^2\big| = 0. \tag{3.20}$$

In the following, we distinguish two cases:

First, we assume that $\|\varphi\|_\infty < 1$. From (3.19) and (3.20), we obtain

$$\begin{aligned}&\lim_{|z|\to 1}\mu(|z|)\frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}}\\&\quad\le\frac{1}{(1-\|\varphi\|_\infty)^{\frac{p+q+2}{p}}}\lim_{|z|\to 1}\mu(|z|)\left|\phi(z)(\varphi'(z))^2\right|=0\end{aligned}$$

and

$$\begin{aligned}&\lim_{|z|\to 1}\mu(|z|)\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}}\\&\quad\le\frac{1}{(1-\|\varphi\|_\infty)^{\frac{q+2}{p}}}\lim_{|z|\to 1}\mu(|z|)\left|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)\right|=0.\end{aligned}$$

So the result follows in this case.

Secondly, we assume that $\|\varphi\|_\infty = 1$. Let $\{\varphi(z_n)\}_{n\in N}$ be a sequence such that $\lim_{n\to\infty}|\varphi(z_n)| = 1$. From the compactness of $\phi DC_\varphi : Q_k(p,q)\to B_{\mu,0}$, we see that $\phi DC_\varphi : Q_k(p,q)\to B_\mu$ is compact. According to Theorem 2.2, we get

$$\lim_{|\varphi(z)|\to 1}\mu(|z|)\frac{|\phi(z)(\varphi'(z))^2|}{(1-|\varphi(z)|^2)^{\frac{p+q+2}{p}}}=0 \tag{3.21}$$

and

$$\lim_{|\varphi(z)|\to 1}\mu(|z|)\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}}=0. \tag{3.22}$$

For any $\varepsilon > 0$, from (3.19) and (3.22), there exists $r\in(0,1)$ such that

$$\mu(|z|)\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}}<\varepsilon,$$

for $r < |\varphi(z)| < 1$, and there exists $\sigma\in(0,1)$ such that

$$\mu(|z|)\left|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)\right|\le\varepsilon\left(1-r^2\right)^{\frac{q+2}{p}},$$

for $\sigma < |z| < 1$. Therefore, when $\sigma < |z| < 1$, and $r < |\varphi(z)| < 1$, we obtain

$$\mu(|z|)\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}}<\varepsilon. \tag{3.23}$$

On the other hand, if $\sigma < |z| < 1$, and $|\varphi(z)|\le r$, we have

$$\begin{aligned}&\mu(|z|)\frac{|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}}\\&\quad<\frac{1}{(1-r^2)^{\frac{q+2}{p}}}\mu(|z|)\left|\phi(z)\varphi''(z)+\phi'(z)\varphi'(z)\right|<\varepsilon.\end{aligned} \tag{3.24}$$

From (3.23) and (3.24), we get the second equality of (2.4). Similarly to the above arguments, by (3.20) and (3.21), we can get the first equality of (2.4). The proof of Theorem 2.3 is completed. □

Similarly to the proofs of Theorems 2.1-2.3, we can get the proofs of Corollary 2.4, Theorem 2.5 and Corollary 2.6. We omit the proofs.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors drafted the manuscript, read and approved the final manuscript.

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References
1. Cowen, CC, Maccluer, BD: Composition Operators on Spaces of Analytic Functions. Studies in Advanced Mathematics. CRC Press, Boca Raton (1995)
2. Fu, X, Zhu, X: Weighted composition operators on some weighted spaces in the unit ball. Abstr. Appl. Anal. **2008**, Article ID 605807 (2008)
3. Hibschweiler, RA, Portnoy, N: Composition followed by differentiation between Bergman and Hardy spaces. Rocky Mt. J. Math. **35**(3), 843-855 (2005)
4. Hu, ZJ, Wang, SS: Composition operators on Bloch-type spaces. Proc. R. Soc. Edinb. A **135**(6), 1229-1239 (2005)
5. Kotilainen, M: On composition operators in Q_k type spaces. J. Funct. Spaces Appl. **5**(2), 103-122 (2007)
6. Li, SX, Stevič, S: Composition followed by differentiation between Bloch type spaces. J. Comput. Anal. Appl. **9**(2), 195-205 (2007)
7. Li, SX, Stevič, S: Composition followed by differentiation between H^∞ and α-Bloch spaces. Houst. J. Math. **35**(1), 327-340 (2009)
8. Ohno, S: Products of composition and differentiation between Hardy spaces. Bull. Aust. Math. Soc. **73**(2), 235-243 (2006)
9. Shields, AL, Williams, DL: Bonded projections, duality, and multipliers in spaces of analytic functions. Trans. Am. Math. Soc. **162**, 287-302 (1971)
10. Stevič, S: Norm of weighted composition operators from Bloch space to H_μ^∞ on the unit ball. Ars Comb. **88**, 125-127 (2008)
11. Wulan, HS, Zhou, JZ: Q_k type spaces of analytic functions. J. Funct. Spaces Appl. **4**(1), 73-84 (2006)
12. Yang, WF: Products of composition and differentiation operators from $Q_k(p,q)$ to Bloch-type spaces. Abstr. Appl. Anal. **2009**, Article ID 741920 (2009)
13. Zhao, RH: On a general family of function spaces. Ann. Acad. Sci. Fenn., Ser. A 1 Math. **105**, 1-56 (1996)
14. Zhu, KH: Bloch type spaces of analytic functions. Rocky Mt. J. Math. **23**(3), 1143-1177 (1993)

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