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On the growth of solutions of second order complex differential equation with meromorphic coefficients

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Abstract

We consider the differential equation $f'' + Af' + Bf = 0$ where $A(z)$ and $B(z) \not\equiv 0$ are mero-morphic functions. Assume that $A(z)$ belongs to the Edrei-Fuchs class and $B(z)$ has a deficient value ∞, if $f \not\equiv 0$ is a meromorphic solution of the equation, then f must have infinite order.

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1 Introduction and main results

In this article, we shall consider the second order linear differential equation

$$f'' + A(z)f' + B(z)f = 0, \tag{1.1}$$

where $A(z)$ and $B(z) \not\equiv 0$ are meromorphic functions. We use the standard notations of value distribution theory of meromorphic function (see [1,2]). In particular, for a meromorphic function $f(z)$, we use the notation $\rho(f)$ and $\mu(f)$ to denote its order and lower order, respectively and for a closed domain D in C, we use $n(D, f = a) = n(D, \frac{1}{f-a})$, if $a \neq \infty$; and $n(D, f = a) = n(D, f)$, if $a = \infty$ to denote the number of zeros for $f - a$ in D, with due count of multiplicities.

It is well known that if $A(z)$ is entire and $B(z)$ is transcendental entire and f_1, f_2 are two linearly independent solutions of Equation (1.1), then at least one of f_1, f_2 must have infinite order. However, there are some equations of the form (1.1) that possess a solution $f \not\equiv 0$ of finite order; for example, $f(z) = e^z$ satisfies $f'' + e^{-z} f' - (e^{-z} + 1)f = 0$. Thus, the main problem is that what conditions on $A(z)$ and $B(z)$ can guarantee that every solution $f \not\equiv 0$ of the Equation (1.1) has infinite order? There has been much work on this subject (cf. [3-8]). Furthermore, we also mention that if $A(z)$ is entire with finite order having a finite deficient value, and $B(z)$ is transcendental entire with $\mu(B) < \frac{1}{2}$, then every solution $f \not\equiv 0$ of the Equation (1.1) has infinite order [8].

It seems that there are few work done on the Equation (1.1), where $A(z)$ and $B(z)$ are mero-morphic functions. It would be interesting to get some relations between the Equation (1.1) and some deep results in value distribution theory of meromorphic functions. To this end, we note that when the zeros and poles of a meromorphic function distributed near some curves, Edrei and Fuchs proved that the number of deficient values can not be infinite. To relate the result of Edrei and Fuchs with the Equation (1.1), we first make some preparations.

In this following we use the notation $\Omega(\theta_1, \theta_2, r) = \{z: \theta_1 < \arg z < \theta_2, |z| < r\}$ and $\bar{\Omega}(\theta_1, \theta_2, r) = \{z : \theta_1 \le \arg z \le \theta_2, |z| \le r\}$.

Definition. Let $f(z)$ be a meromorphic function in the finite complex plane C of order $0 < \rho(f) < \infty$. A ray $\arg z = \theta$ staring from the origin is called a zero-pole accumulation ray of $f(z)$, if for any given real number $\varepsilon > 0$, the following equality holds

$$\overline{\lim_{r\to\infty}} \frac{\log n\{\bar{\Omega}(\theta-\varepsilon, \theta+\varepsilon, r), f=0\} + \log n\{\bar{\Omega}(\theta-\varepsilon, \theta+\varepsilon, r), f=\infty\}}{\log r} = \rho(f). \quad (1.2)$$

The following result which is a weaker form of the Edrei-Fuchs Theorem [9,10] will be used later.

Theorem A. [[11], Theorem 3.10] Let $f(z)$ be a meromorphic function in the complex plane C of order $0 < \rho(f) < +\infty$. Assume that $f(z)$ has q zero-pole accumulation rays and p deficient values other than 0 and ∞, then $p \le q$.

For simplicity, we shall call the inequality $p \le q$ in Theorem A the Edrei-Fuchs inequality. It is easy to see that the Edrei-Fuchs inequality is sharp. In the following, we shall say that a meromorphic function $f(z) \in EF$, called it Edrei-Fuchs Class, if $f(z)$ satisfies the conditions of Theorem A with $p = q \ge 1$, that is, $f(z)$ is of finite and positive order and has p zero-pole accumulation rays and p non-zero finite deficient values.

The main result in this article is based on the class EF. Now, we are able to state our result as follows.

Theorem. Let $A(z) \in EF$ be a meromorphic function and let $B(z)$ be a transcendental meromorphic function having a deficient value ∞. If $f \not\equiv 0$ is a meromorphic solution of Equation (1.1), then $\rho(f) = \infty$.

As our result depends largely on the EF class, we give some examples below from which we can see the EF class contains many familiar functions.

Example 1. The first example can be constructed as follows.

$$A(z) = \frac{ae^z + b}{ce^z + d}, \; a, \; b, \; c, \; d \in C\backslash\{0\}, \; ad - bc \neq 0.$$

Clearly, $\rho(A) = 1$, and e^z has two deficient values 0 and ∞. So $A(z)$ has $p = 2$ deficient values a/c and b/d. On the other hand, for every complex number $\beta \in C \setminus \{0\}$ and given constant $\varepsilon > 0$, all the zeros, except for finitely many number of them, of $e^z - \beta$ are in the angular region $\Omega_1 = \{z : \frac{\pi}{2} - \varepsilon < \arg z < \frac{\pi}{2} + \varepsilon\}$ and $\Omega_2 = \{z : -\frac{\pi}{2} - \varepsilon < \arg z < -\frac{\pi}{2} + \varepsilon\}$. Hence, $A(z)$ has $q = 2$ zero-pole accumulation rays $\arg z = -\frac{\pi}{2}, \frac{\pi}{2}$. So $p = q = 2$ and $A(z) \in EF$.

Clearly, if $A \in EF$, then $1/A \in EF$ and $aA \in EF$ for $a \in C \setminus \{0\}$. Similarly, for any $\alpha \in C \setminus \{0\}$, we get

$$A(\alpha z) = \frac{ae^{\alpha z} + b}{ce^{\alpha z} + d} \in EF, \quad a, \; b, \; c, \; d \in C\backslash\{0\}, \quad ad - bc \neq 0.$$

In this case, $A(\alpha z)$ has $q = 2$ zero-pole accumulation rays $\arg z = -\frac{\pi}{2} - \arg\alpha, \; \frac{\pi}{2} - \arg\alpha$. Especially, we have

$$\tan z = \frac{1}{-i}\frac{e^{2iz}-1}{e^{2iz}+1} \in EF, \quad \frac{e^z-1}{e^z+1} \in EF.$$

Remark 1. Let $A(z)$ in (1.1) be defined as

$$A(z) = \frac{ae^{P(z)}+b}{ce^{P(z)}+d}, \quad a,\ b,\ c,\ d \in C\backslash\{0\}, \quad ad-bc \neq 0,$$

where $P(z)$ is a non-constant polynomial. In this case of the degree of $P(z)$ is bigger than 1, then $A(z) \notin EF$. But, we can see in the proof of the main theorem that if $B(z)$ is a meromorphic function having deficient value ∞ and $f \not\equiv 0$ is a meromorphic solution of Equation (1.1), then $\rho(f) = \infty$.

A little bit more complicated example can be constructed as follows.

Example 2. Let p be a positive integer and set

$$A^*(z) = \frac{J_{\frac{1}{p}}\left(\frac{2z^{\frac{p}{2}}}{p}\right)}{J_{-\frac{1}{p}}\left(\frac{2z^{\frac{p}{2}}}{p}\right)},$$

where

$$J_{\frac{1}{p}}\left(\frac{2z^{\frac{p}{2}}}{p}\right) = \frac{1}{p^{\frac{1}{p}}z^{\frac{1}{2}}}\sum_{k=0}^{\infty}\frac{(-1)^k z^{pk+1}}{p^{2k}k!\Gamma(\frac{1}{p}+k+1)};$$
$$J_{-\frac{1}{p}}\left(\frac{2z^{\frac{p}{2}}}{p}\right) = \frac{1}{z^{\frac{1}{2}}}\sum_{k=0}^{\infty}\frac{(-1)^k z^{pk}}{p^{2k}k!\Gamma(-\frac{1}{p}+k+1)}.$$

Then we know that (see [[12], Chap 7]), $\rho(A^*) = \frac{p}{2}$, $A^*(z)$ has p deficient values

$$a_k =: e^{\frac{(2k+1)\pi i}{p}}, \quad (k = 0, 1, \ldots, p-1),$$

and p Borel directions

$$\theta_k =: \frac{2k\pi}{p}, \quad (k = 0, 1, \ldots, p-1).$$

Hence, we can take two distinct complex numbers b, c, such that $b, c \neq a_k, \infty$ for all $k = 0, 1, \ldots, p-1$ and let

$$A(z) = \frac{A^*(z)-b}{A^*(z)-c}.$$

It can be seen that $A(z)$ has p deficient values $\frac{a_k-b}{a_k-c}$ and p zero-pole accumulation rays θ_k, $k = 0, 1, \ldots, p-1$. Hence $A(z) \in EF$.

In the end of this section, we give two easy examples of the Equation (1.1) which satisfy our theorem.

Example 3. Let $f(z) = e^{\sin z}$, then $\rho(f) = \infty$ and $f(z)$ satisfies the following equation

$$f'' + (\tan z)f' - (\cos^2 z)f = 0.$$

Example 4. Let $f(z) = e^{e^z + z}$, then $\rho(f) = \infty$ and $f(z)$ satisfies the following equation

$$f'' + \left(\frac{e^z - 1}{e^z + 1}\right)f' - (e^{2z} + 4e^z)f = 0.$$

Furthermore, we also point out that if $A(z) \in EF$ and $B(z)$ has no deficient value ∞, our theorem is in general false. The counterexample can be constructed as follows.

Example 5. Let $f(z) = e^z$, then $\rho(f) = 1$ and $f(z)$ satisfies the following equation

$$f'' + \left(\frac{e^z - 1}{e^z + 1}\right)f' - \frac{2e^z}{e^z + 1}f = 0.$$

In this case, $B(z) = -\frac{2e^z}{e^z+1}$ has only two deficient values 0 and -2, because 0 and -2 are Picard values of $B(z)$.

The article is organized as the following: in Section 2, we shall give and prove some lemmas. In Section 3, we give the proof of Theorem. In Section 4, we give some further results.

2 Lemmas

In this article, for a measurable set $E \subset [0, \infty)$, we define the Lebesgue measure of E by $m(E)$ and the logarithmic measure of $E \subset [1, \infty)$ by $m_l(E) = \int_E \frac{dt}{t}$. We also define the upper and lower logarithmic density of $E \subset [1, \infty)$, respectively, by

$$\overline{\text{logdens}}E = \overline{\lim_{r\to\infty}} \frac{m_l(E \cap [0, r])}{\log r}, \quad \textit{and} \quad \underline{\log\text{dens}}E = \underline{\lim_{r\to\infty}} \frac{m_l(E \cap [0, r])}{\log r}.$$

We need serval lemmas to prove our theorem.

Lemma 2.1.[13] Let $w(z)$ be a transcendental meromorphic function of finite order, then there exits a set $E \subset [0, \infty)$ that has finite linear measure, such that for all z satisfying $|z| \notin E$ and for all integers k, j $(k > j)$, we have

$$\left|w^{(k)}(z)/w^{(j)}(z)\right| \leq |z|^{(k-j)(\rho(w)+\varepsilon)}. \tag{2.1}$$

Lemma 2.2.[11] Let $A(z)$ be a meromorphic function with $\rho(A) < +\infty$. Then, for any given real constants $c > 0$ and $H > \rho(A)$, there exists a set $E \subset (0, \infty)$ such that $\underline{\log\text{dens}}E \geq 1 - \frac{\rho(A)}{H}$, where

$$E = \{t | T(te^c, A) \leq e^k T(t, A)\}, \tag{2.2}$$

and $k = cH$.

Lemma 2.3.[7] Let $T(r) > 1$ be a nonconstant increasing function in $(0, +\infty)$ of finite order ρ, i.e.

$$\overline{\lim_{r\to\infty}} \frac{\log T(r)}{\log r} = \rho < \infty.$$

For any η such that $0 \leq \eta < \rho$, if $\rho > 0$, and $\eta = 0$ if $\rho = 0$, define

$$E(\eta) = \{r \geq 1 \ : \ r^{\eta} < T(r)\}. \tag{2.3}$$

Then $\overline{\text{logdens}} E(\eta) > 0$.

Lemma 2.4. Let $A(z)$ be a meromorphic function of order $0 < \rho(A) < \infty$ having ρ finite deficient values, $a_1, a_2, \ldots, a_p (p \geq 1)$ and let $B(z)$ be a meromorphic function with finite order having a deficient value ∞. Suppose that $\beta > 1$ and $0 < \eta < \rho(A)$ are two constants. Then there exists a sequence $\{t_n\}$ such that

$$\lim_{n\to\infty} \frac{t_n^{\eta}}{T(t_n, A)} = 0. \tag{2.4}$$

Moreover, for every sufficiently large n, there is a set $F_n \subset [t_n, (\beta+1)t_n]$ with $m(F_n) \leq \frac{(\beta-1)t_n}{4}$ such that, for all $R \in [t_n, \beta t_n] \setminus F_n$, the arguments θ sets $E_{\nu}(R), (\nu = 1, 2, \ldots, p)$ and $E_{\infty}(R)$ satisfying the following inequalities

$$m(E_{\nu}(R)) =: m\left(\left\{\theta \in [0, 2\pi) \mid \log \frac{1}{|A(Re^{i\theta}) - a_{\nu}|} \geq \frac{\delta_0}{4} T(R, A)\right\}\right) \geq M_1 > 0 \tag{2.5}$$

and

$$m(E_{\infty}(R)) =: m\left(\left\{\theta \in [0, 2\pi) \mid \log |B(Re^{i\theta})| \geq \frac{\delta_1}{4} T(R,\ B)\right\}\right) \geq M_2 > 0 \tag{2.6}$$

where M_1, M_2 are two positive constants depending only on A, B, $\delta_0 = \min_{1\leq \nu \leq p} \delta(a_{\nu}, A)$, $\delta_1 = \delta(\infty, B)$, β and η.

Proof. For any given constant η and for $\eta < \eta_1 < \rho(A)$, applying Lemma 2.3 to $A(z)$ with $T(r, A)$, we see that

$$h := \overline{\text{logdens}} E(\eta_1) := \overline{\text{logdens}} \{r \geq 1 : r^{\eta_1} < T(r, A)\} > 0. \tag{2.7}$$

Let $\beta > 1$ be given and let $c = \log 2(\beta+2)$, $H_0 = \frac{1}{h}\rho(A) + 1 > \rho(A)$. Applying Lemma 2.2 to $A(z)$, we deduce that there exists a set $E = E(\beta, \eta) \subset (0, \infty)$ such that

$$\underline{\text{logdens}} E \geq 1 - \frac{\rho(A)}{H_0}, \tag{2.8}$$

where $E = \{t \mid T((2\beta+4)t, A) \leq (2\beta+4)^{H_0} T(t, A)\}$. Set $E_1 = E(\eta_1) \cap E$. Then by simple computation we get

$$\overline{\text{logdens}} E_1 \geq \overline{\text{logdens}} E(\eta_1) - \frac{\rho(A)}{H_0} > 0.$$

Hence, we can choose a sequence $\{t_n\}$ such that $t_n \in E_1$ and (2.4) holds.

Now we consider all the zeros and poles of $A(z) - a_{\nu}$ in $|z| \leq (\beta + 1)t_n$, $(\nu = 1, 2, \ldots p)$

$$x_1^{(\nu)},\ x_2^{(\nu)},\ \ldots,\ x_{\nu_n}^{(\nu)}; \quad \gamma_1, \gamma_2, \ldots, \gamma_{l_n}$$

where $\nu_n = n((\beta + 1)t_n, A - a_{\nu})$, and $l_n = n((\beta + 1)t_n, A)$. At the same time, we let

$$\xi_1, \xi_2, \ldots, \xi_{s_n}; \quad \eta_1,\ \eta_2,\ \ldots,\ \eta_{q_n}$$

be all the zeros and poles of $B(z)$ in $|z| \le (\beta + 1)t_n$, respectively, where $s_n = n((\beta+1)t_n, \frac{1}{B})$ and $q_n = n((\beta + 1)t_n, B)$. By the Boutroux-Cartan theorem, if $|z| = r \in [t_n, \beta t_n]$ and $z \notin (\gamma^{(1)})_n$ we have

$$\prod_{j=1}^{v_n} \left|z - x_j^{(v)}\right| > \left(\frac{L}{e}\right)^{v_n}, \prod_{j=1}^{l_n} \left|z - y_j\right| > \left(\frac{L}{e}\right)^{l_n}; \tag{2.9}$$

$$\prod_{j=1}^{s_n} \left|z - \xi_j\right| > \left(\frac{L}{e}\right)^{s_n}, \prod_{j=1}^{q_n} \left|z - \eta_j\right| > \left(\frac{L}{e}\right)^{q_n}, \tag{2.10}$$

where $(\gamma^{(1)})_n \subset \{z\colon |z| \le (\beta + 1)t_n\}$ are some disks with the sum of total radius not exceeding $2L$ where $L = \frac{(\beta-1)t_n}{16}$. For every integer n, let $F_n = \{|z|\colon z \in (\gamma^{(1)})_n\}$ then $m(F_n) \le \frac{(\beta-1)t_n}{4}$. Hence, for all $R \in [t_n, \beta t_n] \setminus F_n$, we easily see that $\{z\colon |z| = R\} \cap \left(\gamma^{(1)}\right)_n = \emptyset$.

It follows from (2.9) and the Poisson-Jensen formula, for every $1 \le v \le p$, we have

$$\begin{aligned}
&\log \frac{1}{\left|A(Re^{i\theta}) - a_v\right|} \le \\
&\frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{1}{A((\beta+1)t_n e^{i\varphi}) - a_v}\right| \frac{((\beta+1)t_n)^2 - R^2}{((\beta+1)t_n)^2 - 2(\beta+1)t_n R\cos(\theta-\varphi) + R^2} d\varphi \\
&\qquad + \sum_{j=1}^{v_n} \log\left|\frac{((\beta+1)t_n)^2 - \bar{x}_j^v Re^{i\theta}}{(\beta+1)t_n(Re^{i\theta} - x_j^v)}\right| + \sum_{j=1}^{l_n} \log\left|\frac{((\beta+1)t_n)^2 - \bar{y}_j Re^{i\theta}}{(\beta+1)t_n(Re^{i\theta} - y_j)}\right|.
\end{aligned}$$

So, for all $n \ge N_0$, we get

$$\begin{aligned}
\log \frac{1}{\left|A(Re^{i\theta}) - a_v\right|} &\le \frac{(\beta+1)t_n + R}{(\beta+1)t_n - R} m\left((\beta+1)t_n, \frac{1}{A - a_v}\right) + (v_n + l_n)\log\frac{(2\beta+1)et_n}{L} \\
&\le (2\beta+1)m\left((\beta+1)\, t_n, \frac{1}{A - a_v}\right) \\
&+ (v_n + l_n)\log\frac{16e(2\beta+1)}{\beta-1}(2\beta+1)m\left((\beta+1)t_n, \frac{1}{A - a_v}\right) \\
&+ \left(\frac{N((\beta+2)t_n, A = a_v)}{\log\frac{\beta+2}{\beta+1}} + \frac{N((\beta+2)t_n, A)}{\log\frac{\beta+2}{\beta+1}}\right)\log\frac{16e(2\beta+1)}{\beta-1} \\
&\le (2\beta+1)T\left((\beta+1)t_n, \frac{1}{A - a_v}\right) \\
&+ \left\{T\left((\beta+2)t_n, \frac{1}{A - a_v}\right) + T((\beta+2)t_n,\ A)\right\}\frac{\log\frac{16e(2\beta+1)}{\beta-1}}{\log\frac{\beta+2}{\beta+1}} \\
&\le \left\{(2\beta+1) + \frac{2\log\frac{16e(2\beta+1)}{\beta-1}}{\log\frac{\beta+2}{\beta+1}}\right\} T((2\beta+4)t_n,\ A) \\
&\le (2\beta+4)^{H_0}\left\{(2\beta+1) + \frac{2\log\frac{16e(2\beta+1)}{\beta-1}}{\log\frac{\beta+2}{\beta+1}}\right\} T(t_n,\ A).
\end{aligned}$$

Denote $\delta_0 = \min_{1\le v\le p} \delta(a_v, A)$ and

$$E_v(R) = \{\theta \in [0, 2\pi) \mid \log \frac{1}{|A(Re^{i\theta}) - a_v|} \ge \frac{\delta_0}{4} T(R, A)\}.$$

There exists a constant $N_1 > N_0$ such that for all $n > N_1$, we have

$$\begin{aligned}\frac{\delta_0}{2} T(R, A) &< \frac{1}{2\pi} \int_0^{2\pi} \log \frac{1}{|A(Re^{i\theta}) - a_v|} d\theta \\ &\le \frac{1}{2\pi} \int_{E_v(R)} \log \frac{1}{|A(Re^{i\theta}) - a_v|} d\theta + \frac{\delta_0}{4} T(R, A).\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\delta_0}{4} T(R, A) &\le \frac{1}{2\pi} \int_{E_v(R)} \log \frac{1}{|A(Re^{i\theta}) - a_v|} d\theta \\ &\le \frac{1}{2\pi} (2\beta + 4)^{H_0} \left\{ (2\beta + 1) + \frac{2 \log \frac{16e(2\beta+1)}{\beta-1}}{\log \frac{\beta+2}{\beta+1}} \right\} T(R,\ A) m(E_v(R)).\end{aligned}$$

So

$$M_1 = \frac{\delta_0}{4} \left\{ \frac{1}{2\pi} (2\beta + 4)^{H_0} \left[(2\beta + 1) + \frac{2 \log \frac{16e(2\beta+1)}{\beta-1}}{\log \frac{\beta+2}{\beta+1}} \right] \right\}^{-1} \le m(E_v(R)). \qquad (2.11)$$

This gives (2.5). Similarly, set $\delta_1 = \delta(\infty, B)$ and

$$E_\infty(R) = \{\theta \in [0, 2\pi) \mid \log |B(Re^{i\theta})| \ge \frac{\delta_1}{4} T(R,\ B)\}. \qquad (2.12)$$

From (2.10), (2.12) and the Poisson-Jensen formula, we get

$$M_2 = \frac{\delta_1}{4} \left\{ \frac{1}{2\pi} (2\beta + 4)^{H_0} \left[(2\beta + 1) + \frac{2 \log \frac{16e(2\beta+1)}{\beta-1}}{\log \frac{\beta+2}{\beta+1}} \right] \right\}^{-1} \le m(E_\infty(R)). \qquad (2.13)$$

This gives (2.6) and the proof of Lemma 2.4 is completed.

Lemma 2.5. [[11], Lemma 3.13] Let $f(z)$ be a meromorphic function of order $0 < \rho(f) < \infty$ satisfying

$$\overline{\lim_{r\to\infty}} \frac{\log n\{\bar{\Omega}(-\theta, \theta,\ r), f = X\}}{\log r} \le \lambda < \rho(f), \quad X = 0,\ \infty, \quad 0 < \theta \le \pi.$$

Suppose for any given constant ε, $0 < \varepsilon < \theta$, there exists a sequence $\{R_n\}$ such that

$$m(E_n) =: m \left\{ z :\ \log \frac{1}{|f(z) - a|} \ge N_n,\ |z| = R_n,\ -\theta + \varepsilon \le \arg z \le \theta - \varepsilon \right\} \ge \alpha R_n,$$

where $\alpha \ge \frac{\varepsilon}{2}$ is a constant and $a \ne 0, \infty$ is a complex number and $N_n > 0$ is a real number such that for any given constant $\eta_0 > 0$, and $R_{n1} \le R_n \le R_{n2}$, $R_{n1} \to \infty$,

$$\lim_{n\to\infty}\left\{\left(\frac{R_{n2}}{R_{n1}}\right)^{6+2(\lambda+\eta_0)+\dfrac{3\pi}{\theta}} R_{n2}^{\lambda+2\eta_0}\log R_{n2}\right\}N_n^{-1}=0. \tag{2.14}$$

Furthermore, if $z\in\bar{\Omega}(-\theta+\varepsilon,\ \theta-\varepsilon,R_{n1},R_{n2})=\{z\ :\ R_{n1}\le|z|\le R_{n2},\ -\theta+\varepsilon\le\arg z\le\theta-\varepsilon\}$ and $z\notin(\gamma^{(2)})_n$, then

$$\log\frac{1}{\left|f(z)-a\right|}\ge\frac{H(\alpha,\varepsilon,\theta)}{L(\theta)\log\frac{R_{n2}}{R_{n1}}+J(\alpha,\varepsilon,\theta)}\left(\frac{R_{n1}}{R_{n2}}\right)^{6+\dfrac{3\pi}{\theta}}N_n \tag{2.15}$$

holds for every sufficiently large n where $(\gamma^{(2)})_n$ are some disks with the sum of total radius not exceeding $\frac{1}{8}\varepsilon R_{n1}, H(\alpha,\ \varepsilon,\ \theta)>0$ and $0<J(\alpha,\ \varepsilon,\ \theta)<+\infty$ are two constants depending only on α, ε, θ, and $0<L(\theta)<+\infty$ is a constants depending only on θ.

In the following, we will give the basic property of EF class which is key to the proof of our theorem.

Lemma 2.6 Let $A(z)\in EF$, then for any given $\varepsilon>0$ (sufficiently small) and $\beta>1$, when n is sufficiently large, there exists a sequence of angular regions $\bar{\Omega}(\theta_{k_\nu}+2\varepsilon,\ \theta_{k_\nu+1}-2\varepsilon,\ t_n,\ \beta t_n)$, $n=1, 2, 3 \ldots$, $\nu=1, 2, \ldots p$ such that for every $1\le\nu\le p$, the following inequalities

$$\log\frac{1}{\left|A(z)-a_\nu\right|}>\log\frac{4}{d} \tag{2.16}$$

holds for $z\in\bar{\Omega}(\theta_{k_\nu}+2\varepsilon,\ \theta_{k_\nu+1}-2\varepsilon,\ t_n,\ \beta t_n)\backslash\bigcup_{\nu=1}^{p}(\gamma_\nu)_n$, where $\bigcup_{\nu=1}^{p}(\gamma_\nu)_n$ is defined by Lemma 2.5 with the sum of total radius not exceeding $\frac{p}{8}\varepsilon t_n$ and t_n, βt_n are defined by Lemma 2.4 and $d=\min_{1\le\nu\ne\nu'\le p}\{|a_\nu-a_{\nu'}|\}$ and a_ν are deficient values of $A(z)$.

Proof. let $\beta>1$ be fixed and for any given constant ε with,

$$0<\varepsilon<\min\left\{\frac{\omega}{2},\frac{\beta-1}{2}\right\}, \tag{2.17}$$

where $\omega=\min_{1\le k\le\nu}(\theta_{k+1}-\theta_k)$. From (1.2), we get

$$\varlimsup_{r\to\infty}\frac{\log n\{\Omega(\theta_k+\varepsilon,\theta_{k+1}-\varepsilon,r),A=X\}}{\log r}\le\lambda<\rho(A),\quad X=0,\ \infty.$$

Now let η_0 be fixed such that $0<\eta_0<\frac{1}{6}(\rho(A)-\lambda)$. Applying Lemma 2.4 to $A(z)$ with $\eta=\lambda+4\eta_0$ and suppose that $[t_n,\ \beta t_n]$, $E_\nu(R_n)$, F_n are defined in Lemma 2.4 which satisfy the conclusions (2.4) and (2.5) of Lemma 2.4 and

$$t_n\in\{t\mid T((2\beta+4)t,A)\le(2\beta+4)^{H_0}T(t,A)\}. \tag{2.18}$$

and choose $R_n\in[t_n,\ \beta t_n]\setminus F_n$ for every sufficiently large n.

Without loss of generality, let $0 < \varepsilon < \frac{M_1}{8p}$, for every $1 \le \nu \le p$, there exists a set $E_\nu(R_n) \cap [\theta_{k_\nu} + 2\varepsilon, \theta_{k_\nu+1} - 2\varepsilon](1 \le k_\nu \le p)$ such that

$$m(E_\nu(R_n) \bigcap [\theta_{k_\nu} + 2\varepsilon, \theta_{k_\nu+1} - 2\varepsilon]) \ge \frac{M_1}{2p}. \tag{2.19}$$

Furthermore, we also have

$$\varlimsup_{r\to\infty} \frac{\log n\{\Omega(\theta_{k_\nu} + \varepsilon, \theta_{k_\nu+1} - \varepsilon, r), A = X\}}{\log r} \le \lambda < \rho(A), \quad X = 0,\ \infty.$$

Set $N_n = \frac{1}{4}T(R_n,\ A), \alpha = \frac{M_1}{2p}$, $R_{n1} = t_n$, $R_{n2} = \beta t_n$, and using Lemma 2.5 for $A(z)$, we have

$$\begin{aligned}\left\{\left(\frac{R_{n2}}{R_{n1}}\right)^{6+2(\lambda+\eta_0)+\frac{3\pi}{\theta}} R_{n2}^{\lambda+2\eta_0} \log R_{n2}\right\} N_n^{-1} &= \frac{\beta^{6+2(\lambda+\eta_0)+\frac{3\pi}{\theta_{k_\nu}}} \beta^{\lambda+2\eta_0} t_n^{\lambda+2\eta_0} (\log\beta + \log t_n)}{\frac{1}{4}T(R_n, A)} \\ &\le 8\beta^{6+2(\lambda+\eta_0)+\frac{3\pi}{\theta_{k_\nu}}+\lambda+2\eta_0} \frac{t_n^{\lambda+3\eta_0}}{T(R_n, A)}, n \ge n_1.\end{aligned}$$

Note that,

$$\lim_{r\to\infty} \frac{t_n^{\lambda+3\eta_0}}{T(R_n, A)} \le \lim_{n\to\infty} \frac{t_n^{\lambda+3\eta_0}}{T(t_n, A)} = 0.$$

Therefore, if we let $d = \min\limits_{1\le\nu\ne\nu'\le p} \{|a_\nu - a_{\nu'}|\}$, it follows from Lemma 2.5 that, for $z \in \bar{\Omega}(\theta_{k_\nu} + 2\varepsilon, \theta_{k_\nu+1} - 2\varepsilon, t_n, \beta t_n)\backslash(\gamma_\nu)_n$ we have

$$\log\frac{1}{|A(z) - a|} \ge H(\alpha,\ \varepsilon, \beta,\ \delta_0,\ \theta_{k_\nu})T(R_n,\ A) > \log\frac{4}{d} \tag{2.20}$$

where $H(\alpha,\ \varepsilon, \beta,\ \delta_0,\ \theta_{k_\nu}) > 0$ is a constant not depending on n, and $\bigcup\limits_{\nu=1}^{p}(\gamma_\nu)_n$ are some disks with the sum of total radius not exceeding $\frac{p}{8}\varepsilon t_n$. Thus, if $z \notin \bigcup\limits_{\nu=1}^{p}(\gamma_\nu)_n$ and $z \in \bar{\Omega}(\theta_{k_\nu} + 2\varepsilon,\ \theta_{k_\nu+1} - 2\varepsilon,\ t_n,\ \beta t_n)$, then (2.20) gives (2.16). Obviously, there is a unique deficient value a_ν corresponding to every angular region $\bar{\Omega}(\theta_{k_\nu} + 2\varepsilon,\ \theta_{k_\nu+1} - 2\varepsilon,\ t_n,\ \beta t_n)$ for n sufficiently large, otherwise this gives a contradiction to (2.16). The proof of Lemma 2.6 is completed.

Remark 2. It can be seen from Lemma 2.6 that if $A \in EF$, then for any given $\varepsilon > 0, \beta > 1$, there exists a sequence of angular regions $\bar{\Omega}(\theta_{k_\nu} + 2\varepsilon,\ \theta_{k_\nu+1} - 2\varepsilon,\ t_n,\ \beta t_n)$, $(\nu = 1, 2, ... p)$ such that in every angular region, $A(z)$ is close to a deficient value in a uniform way except for those points in some disks with sum of total radii not exceeding $\frac{p}{8}\varepsilon t_n$. This means that the measure of the the set of values $\theta \in [0, 2\pi]$ such that the ray $\arg z = \theta$ meets the exceptional disks in the angular regions $\bar{\Omega}(\theta_{k_\nu} + 2\varepsilon,\ \theta_{k_\nu+1} - 2\varepsilon,\ t_n,\ \beta t_n)$, $(\nu = 1, 2, ... p)$ is at most $\frac{p}{8}\varepsilon$.

3 Proof of theorem

Suppose that $A(z)$ has p non-zero finite deficient values, $a_1, a_2, \ldots, a_p$ with deficiency $\delta(a_v, A) > 0$, $1 \le v \le p$ and has p zero-pole accumulation rays, $0 \le \theta_1 < \theta_2 < \ldots < \theta_p < \theta_1 + 2\pi$. From the Equation (1.1), we get

$$|B(z)| \le \left|\frac{f''(z)}{f(z)}\right| + |A(z)| \left|\frac{f'(z)}{f(z)}\right|. \tag{3.1}$$

If $\rho(B) = \infty$, using the standard lemma on the logarithmic derivative in (1.1), we have

$$T(r,\ B) \le T(r, A) + 2N\left(r, \frac{1}{f}\right) + 2N(r, f) + O(\log r).$$

According to the assumption, $\rho(A) < \infty$, we immediately get a contradiction. Hence $\rho(f) = \infty$ in the case $\rho(B) = \infty$. Now the rest of proof should be devoted to the case $\rho(B) < \infty$.

It is easy to see that, the Equation (1.1) can not have any nonzero rational solution by (3.1), (2.6) and $A(z) \in EF$. So now we assume that $f \not\equiv 0$ is a transcendental meromorphic solution of Equation (1.1) with $\rho(f) < +\infty$. We shall seek a contradiction.

Applying Lemma 2.1 to $f(z)$, there exists a set $E_1 \subset [0, \infty]$ with $m(E_1) < \infty$ such that

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \ \le\ |z|^{(2\rho(f)+\varepsilon)}, \quad k = 1, 2, \tag{3.2}$$

holds for $|z| \notin E_1 \cup [0,1]$. It follows from Lemma 2.4 that, there exists a sequence of closed intervals $\{[t_n, \beta t_n]\}$ with $t_n \to \infty$, $t_{n+1} > \beta t_n$ and a set $F_n \subset [t_n, (\beta + 1)t_n]$ with $m(F_n) \le \frac{(\beta-1)t_n}{4}$ and a sequence $R_n \in [t_n, \beta t_n] \setminus F_n$ such that (2.5) and (2.6) simultaneously hold.

Let $\omega = \min\limits_{1\le k\le v}(\theta_{k+1} - \theta_k)$ and $0 < \varepsilon_0 < \min\left\{\frac{M_1}{8p}, \frac{M_2}{8p}, \frac{\omega}{2}, \frac{\beta-1}{2p}\right\}$. According to Lemma 2.6, we

choose $R_n^* \in [t_n, \beta t_n] \setminus (F_n \cup E_1 \cup [0, 1])$ such that for every $n \ge n_0$

$$\{z : |z| = R_n^*\} \cap \left(\bigcup_{v=1}^{p} (\gamma_v)_n\right) = \emptyset, \tag{3.3}$$

where $\cup_{v=1}^{p}(\gamma_v)_n$ are some disks with the sum of total radius not exceeding $\frac{p}{8}\varepsilon_0 t_n < \frac{\beta-1}{16} t_n$. Hence, from Lemma 2.6 and (2.16), the following inequalities

$$\log\frac{1}{|A(R_n^* e^{i\varphi}) - a_v|} > \log\frac{4}{d}, \quad v = 1, 2, \ldots, p \tag{3.4}$$

holds for $n \ge n_1 > n_0$ and $R_n^* e^{i\varphi} \in \cup_{v=1}^{p} \bar{\Omega}\left(\theta_{k_v} + 2\varepsilon_0,\ \theta_{k_v+1} - 2\varepsilon_0,\ t_n,\ \beta t_n\right)$.

On the other hand, from Lemma 2.4, for the sequence $\{R_n^*\}$, the following equality

$$m(E_\infty(R_n^*)) =: m\left(\left\{\theta \in [0, 2\pi) \mid \log|B(R_n^* e^{i\theta})| \ge \frac{\delta_1}{4} T(R_n^*,\ B)\right\}\right) \ge M_2 > 0 \tag{3.5}$$

also holds for sufficiently large n. Hence, there exists a set $E_\infty(R_n^*) \cap [\theta_{k_{v_0}} + 2\varepsilon_0,\ \theta_{k_{v_0}+1} - 2\varepsilon_0]\ (1 \le k_{v_0} \le p)$ such that

$$m\left(E_{\infty}(R_n^*) \cap [\theta_{k_{v_0}} + 2\varepsilon_0,\ \theta_{k_{v_0}+1} - 2\varepsilon_0]\right) \geq \frac{M_2}{2p}. \tag{3.6}$$

Now for sufficiently large n, we choose $\varphi_n \in E_{\infty}\left(R_n^*\right)$ such that (3.4) and (3.5) hold. From (3.1) to (3.5) we get

$$\left|B(R_n^* e^{i\varphi_n})\right| \leq (R_n^*)^{2(\rho(f)+\varepsilon_0)}\left(1 + \frac{d}{4} + \left|a_{v_0}\right|\right).$$

So

$$\frac{\delta_1}{4} T(R_n^*,\ B) \leq 2(\rho(f) + \varepsilon_0)\log R_n^* + \log\left(1 + \frac{d}{4} + \left|a_{v_0}\right|\right). \tag{3.7}$$

From (3.7), it implies that $B(z)$ is a rational function. This gives a contradiction. The proof of the theorem is completed.

4 Some further results

Although, Example 5 implies that our theorem is general false for $B(z)$ has no deficient value ∞.

However, our theorem also holds if we give some conditions on $B(z)$.

Now let $B(z)$ be a transcendental meromorphic function which its form is defined below

$$B(z) = \frac{a(z)e^{P(z)} + b(z)}{c(z)e^{Q(z)} + d(z)}, \tag{4.1}$$

where $P(z) = \alpha z^n + \ldots$ is a polynomial with degree of $n \geq 1$, $Q(z) = \beta z^m + \ldots$ is also a polynomial with degree $m \geq 0$ ($\alpha, \beta \in C$, $|\alpha| + |\beta| \neq 0$); $a(z) \not\equiv 0$, $b(z)$, $c(z)$ and $d(z)$ are entire functions with

$$\max\{\rho(a), \rho(b), \rho(c), \rho(d)\} < n. \tag{4.2}$$

Now, we are able to state the theorem as follows.

Theorem 4.1. Let $A(z) \in EF$ be a meromorphic function and let $B(z)$ be a transcendental meromorphic function defined by (4.1) and (4.2) satisfying one of the following conditions:

(1) $m \neq n$;

(2) $m = n$, arg $\alpha \neq$ arg β;

(3) $m = n$, $\beta = c\alpha$, $c \in (0, 1)$. If $f \not\equiv 0$ is a meromorphic solution of Equation (1.1), then $\rho(f) = \infty$.

Proof of Theorem 4.1. To prove this theorem, we only need to use Remark 2 and the following Lemma 4.1 and the same methods as the proof of main theorem. Hence, we shall omit its proofs.

Lemma 4.1.[14] Suppose that $P(z) = (\xi + i\eta)z^n + \cdots$ (ξ,η are real numbers, $|\xi| + |\eta| \neq 0$) is a polynomial with degree $n \geq 1$, and suppose that $a(z) \not\equiv 0$ is an entire function with $\rho(a) < n$. Set $g(z) = a(z)e^{P(z)}$, $z = re^{i\theta}$, $\delta(P, \theta) = \xi \cos n\theta - \eta \sin n\theta$. Then for any given $\varepsilon > 0$, there exits a set $H_1 \subset [0, 2\pi)$ that has the linear measure zero, such that for any $\theta \in [0, 2\pi) \setminus (H_1 \cup H_2)$ there is $R = R(\theta) > 0$ such that for $|z| = r > R$, we have

(i) if $\delta(P, \theta) > 0$, then

$$\exp\{(1-\varepsilon)\delta(P, \theta)r^n\} < \left|g(re^{i\theta})\right| < \exp\{(1+\varepsilon)\delta(P, \theta)r^n\};$$

(ii) if $\delta(P, \theta) < 0$, then

$$\exp\{(1+\varepsilon)\delta(P, \theta)r^n\} < \left|g(re^{i\theta})\right| < \exp\{(1-\varepsilon)\delta(P, \theta)r^n\},$$

where $H_2 = \{\theta \in [0, 2\pi): \delta(P, \theta) = 0\}$ is a finite set.

It is easy to see that, if $a \in C \setminus \{0\}$, $b, c, d \in C$ and $|c| + |d| \neq 0$ in (4.1), we can obtain the particular situation of Theorem 4.1. But if $m = n$, $\beta = c\alpha$, $c \in [1, \infty)$ in Theorem 4.1, then the conclusion is in general false (see Example 5). Another counterexample can be constructed as follows.

Example 6. Let $f(z) = e^z - 1$, then $f(z)$ satisfies the following equation

$$f'' - \left(\frac{e^z - 1}{e^z + 1}\right)f' - \frac{2e^z}{e^{2z} - 1}f = 0.$$

Furthermore, if $P(z)$ has the degree $n = 0$ in (4.1), the conclusion is also in general false. The counterexample can be easily constructed as follows.

Example 7. Let $f(z) = e^z$, then $f(z)$ satisfies the following equation

$$f'' - \left(\frac{e^z - 1}{e^z + 1}\right)f' - \frac{2}{e^z + 1}f = 0.$$

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