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Research Article

Analysis of Vector Quantizers Using Transformed Codebooks with Application to Feedback-Based Multiple Antenna Systems

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Transformed codebooks are obtained by a transformation of a given codebook to best match the statistical environment at hand. The procedure, though suboptimal, has recently been suggested for feedback of channel state information (CSI) in multiple antenna systems with correlated channels because of their simplicity and effectiveness. In this paper, we first consider the general distortion analysis of vector quantizers with transformed codebooks. Bounds on the average system distortion of this class of quantizers are provided. It exposes the effects of two kinds of suboptimality introduced by the transformed codebook, namely, the loss caused by suboptimal point density and the loss caused by mismatched Voronoi shape. We then focus our attention on the application of the proposed general framework to providing capacity analysis of a feedback-based MISO system over spatially correlated fading channels. In particular, with capacity loss as an objective function, upper and lower bounds on the average distortion of MISO systems with transformed codebooks are provided and compared to that of the optimal channel quantizers. The expressions are examined to provide interesting insights in the high and low SNR regime. Numerical and simulation results are presented which confirm the tightness of the distortion bounds.

1. INTRODUCTION

This paper considers multiple antenna systems when partial channel state information (CSI) is available at the transmitter from the receiver through a finite-rate feedback link. Recently, several interesting papers have appeared, proposing design algorithms, as well as analytically quantifying the performance of finite-rate feedback multiple antenna systems [1–18]. We briefly discuss some of them below to provide context to this work.

Mukkavilli et al. approximated in [1] the channel quantization region corresponding to each code point based on the channel geometric property and derived a universal lower bound on the outage probability of quantized MISO beamforming systems with an arbitrary number of transmit antennas t over i.i.d. Rayleigh fading channels. Love and Heath [2, 3] related the problem to that of Grassmannian line packing [4]. Results on the density of Grassmannian line packings were derived and used to develop bounds on the codebook size given a capacity or SNR loss. Xia et al. [5, 6], Zhou et al. [7], and Roh and Rao [8] approximated the statistical distribution of the key random variable that characterizes the system performance. The distribution was used to analyze the performance of MISO systems with limited-rate feedback in the case of i.i.d. Rayleigh fading channels, and closed-form expressions of the capacity loss (or SNR loss) in terms of the feedback rate B, and the number of antennas t were obtained. Moreover, Roh and Rao extended in [10, 11] the results from MISO channels to the case of MIMO systems with quantized feedback. Narula et al. [12] related the quantization problem to rate distortion theory, and obtained an approximation to the expected loss of the received SNR due to finite-rate quantization of the beamforming vectors in an MISO system with a large number of antennas t. Furthermore, design and analysis of finite-rate feedback based multiple antenna systems have also been extended to multiuser areas in [17, 18], where efficient multiuser CSI feedback schemes were proposed and interesting observations of feedback requirement for MIMO broadcast channels were reported.

Despite all these recent results, the analysis of finite-rate feedback systems has proven to be difficult. All the aforementioned approaches are case specific, limited to i.i.d. channels, mainly MISO channels, and are hard to extend to more complicated schemes. Recently, in our work [19], a general framework for the analysis of quantized feedback multiple antenna systems was developed using a source coding perspective by leveraging the considerable work that exists in this area, particularly high resolution quantization theory. Specifically, the channel quantization was formulated as a general finite-rate vector quantization problem with attributes tailored to meet the general issues that arise in feedback based communication systems, including encoder side information, source vectors with constrained parameterizations, and general non-mean-squared distortion functions. By utilizing the proposed general framework, performance analysis of a finite-rate feedback MISO beamforming system transmitting over spatially correlated Rayleigh flat fading channels was provided in [20].

The general framework developed in [19] is versatile and has the potential for being adapted to deal with a variety of problems. This methodology, with suitable modifications, is used in this paper to enable the distortion analysis of a wide class of vector quantizers with transformed codebooks. *Transformed codebooks* are often used for simplicity and are obtained by a transformation of a given codebook to best match the statistical environment at hand. The procedure, though suboptimal, has recently been suggested for CSI feedback-based multiple antenna systems because of their simplicity and effectiveness. Love and Heath [13] and Xia and Giannakis [6] proposed a beamforming codebook design algorithm for correlated MIMO fading channels using a rotation-based transformation on the codebooks of the beamforming vectors originally designed for i.i.d. fading channels. The rotation is derived from the channel correlation matrix. However, to the authors' knowledge, limited analytical results are available characterizing the performance of transformed channel quantizers for multiple antenna systems with finite-rate feedback.

In this paper, we focus our attention on investigating the effects of codebook transformation on the performance of multiple antenna systems with finite-rate CSI feedback. The contributions of this paper are twofold. We first provide insight into the general problem of analyzing a vector quantizer with transformed codebook. Bounds on the average system distortion of this class of quantizers are provided. It exposes the effects of two kinds of suboptimality introduced by the transformed codebook on system performance. They are the loss caused by the suboptimal point density and the loss due to the mismatched Voronoi shape. We then focus our attention on the application of the proposed general framework to providing capacity analysis of a feedback-based MISO system with spatially correlated fading channels using channel quantizers with transformed codebooks. In particular, using system capacity as the objective function, upper and lower bounds on the average distortion of MISO systems with transformed codebooks are provided and compared to that of the optimal channel quantizers. It is shown that the average distortion of CSI quantizers with transformed codebooks can be upper and lower bounded by a scaling of the distortion of optimal quantizers. Furthermore, based on numerical and simulation results, the scaling factors are shown to be close to one for fading channels whose channel covariance matrix has small to moderate condition numbers. Preliminary version of these results have appeared in [21]. This paper provides more detailed (and complete) derivations along with discussions that could not be included in [21] due to space limitation.

2. BACKGROUND INFORMATION ON THE GENERALIZED VECTOR QUANTIZER

Multiple antenna systems with finite-rate CSI feedback were formulated as a generalized fixed-rate vector quantization problem in [19] and analyzed by adapting tools from high resolution quantization theory. In order to facilitate the understanding, we briefly summarize in this section some important results of the distortion analysis of the generalized vector quantizer (for readers that are familiar with the general distortion analysis provided in [19], the current section can be skipped without loss of continuity of the article). Extension of the distortion analysis to quantizers with a transformed codebook and its application to CSI-quantized MISO systems are provided in Section 3 and Section 5, respectively.

2.1. General vector quantization framework

It is assumed that the source variable $\mathbf{x}$ is a two-vector tuple denoted as $(\mathbf{y}, \mathbf{z})$, where vector $\mathbf{y} \in \mathbb{Q}$ represents the actual variable to be quantized (*quantization objective*) of dimension k_q, and $\mathbf{z} \in \mathbb{Z}$ is the additional side information of dimension k_z. The *side information* $\mathbf{z}$ is available at the encoder (receiver) but not at the decoder (transmitter). Quantization objective $\mathbf{y}$ and side information $\mathbf{z}$ have joint probability density function given by $p(\mathbf{y}, \mathbf{z})$, and a fixed-rate ($B$ bits per channel update) quantizer with $N = 2^B$ quantization levels is considered. Based on a particular source realization $\mathbf{x}$, the encoder (or the quantizer) represents vector $\mathbf{y}$ by one of the N vectors $\hat{\mathbf{y}}_1, \hat{\mathbf{y}}_2, \ldots, \hat{\mathbf{y}}_N$, which form the codebook. The encoding or the quantization process is denoted as $\hat{\mathbf{y}} = \mathcal{Q}(\mathbf{y}, \mathbf{z})$. The distortion of a finite-rate quantizer is defined as $D = E_{\mathbf{x}}[D_Q(\mathbf{y}, \hat{\mathbf{y}}; \mathbf{z})]$, where $D_Q(\mathbf{y}, \hat{\mathbf{y}}; \mathbf{z})$ is a *general distortion function* between $\mathbf{y}$ and $\hat{\mathbf{y}}$ that is parameterized by $\mathbf{z}$, not necessarily the mean square error. It is further assumed that the distortion function D_Q has a continuous second-order derivative (or Hessian matrix with respect to $\mathbf{y}$) $\mathbf{W}_{\mathbf{z}}(\hat{\mathbf{y}})$ with the ith and jth elements given by

$$w_{i,j} = \frac{1}{2} \cdot \frac{\partial^2}{\partial y_i \partial y_j} D_Q(\mathbf{y}, \hat{\mathbf{y}}; \mathbf{z}) \Big|_{\mathbf{y}=\hat{\mathbf{y}}}. \tag{1}$$

2.2. Asymptotic distortion integral of the general vector quantizer

Under high-resolution assumptions (large N), the distortion of a finite-rate feedback system has been shown to have the following form:

$$D = E[D_Q(\mathbf{y}, Q(\mathbf{y},\mathbf{z});\mathbf{z})] = 2^{-2B/k_q}\int_{\mathbb{Z}}\int_{\mathbb{Q}} m(\mathbf{y};\mathbf{z};\mathbb{E}_{\mathbf{z}}(\mathbf{y}))\,p(\mathbf{y},\mathbf{z})\lambda(\mathbf{y})^{-2/k_q}d\mathbf{y}\,d\mathbf{z}, \tag{2}$$

where $\mathbb{E}_{\mathbf{z}}(\mathbf{y})$ denotes the asymptotic (as N approaches infinity) projected Voronoi cell that contains $\mathbf{y}$ with side information $\mathbf{z}$ and captures the shape attribute of the quantization cell. In (2), $\lambda(\mathbf{y})$ is the point density function representing the relative density of the codepoints such that $\lambda(\mathbf{y})d\mathbf{y}$ is approximately the fraction of quantization points in a small neighborhood of $\mathbf{y}$. The function $m(\mathbf{y};\mathbf{z};\mathbb{E})$ is the normalized inertial profile that represents the asymptotic normalized distortion, or the relative distortion, of the quantizer $\mathcal{Q}$ at position $\mathbf{y}$ conditioned on side information $\mathbf{z}$ with Voronoi shape $\mathbb{E}$. It is given by

$$m(\mathbf{y};\mathbf{z};\mathbb{E}) \triangleq \left(\int_{\mathbf{y}'\in\mathbb{E}} d\mathbf{y}'\right)^{-(2+k_q)/k_q} \cdot\left(\int_{\mathbf{y}'\in\mathbb{E}}(\mathbf{y}'-\mathbf{y})^{\mathrm{T}}\cdot\mathbf{W}_{\mathbf{z}}(\mathbf{y})\cdot(\mathbf{y}'-\mathbf{y})d\mathbf{y}'\right). \tag{3}$$

The point density function $\lambda(\mathbf{y})$ and the normalized inertial profile $m(\mathbf{y};\mathbf{z};\mathbb{E})$ are the key characteristics that can be used to describe the behavior of a specific quantizer. Alternately, given a vector quantizer, one has to find these two functions, as indicated in [19], and the average system distortion can then be obtained using (2).

2.3. Minimization of the distortion integral

The distortion integral given by (2) allows the minimization of the overall distortion by optimizing the choice of the Voronoi shape $\mathbb{E}_{\mathbf{z}}(\mathbf{y})$ and the point density function $\lambda(\mathbf{y})$. First, the normalized inertial profile of an optimal quantizer can be defined as the minimum inertia of all admissible Voronoi regions (or shapes) $\mathbb{E}_{\mathbf{z}}(\mathbf{y})$, that is,

$$m_{\text{opt}}(\mathbf{y};\mathbf{z}) \triangleq \min_{\mathbb{E}_{\mathbf{z}}(\mathbf{y})\in\mathcal{H}_Q} m(\mathbf{y};\mathbf{z};\mathbb{E}_{\mathbf{z}}(\mathbf{y})), \tag{4}$$

where $\mathcal{H}_Q$ represents the set of all admissible tessellating polytopes that can tile the space $\mathbb{Q}_{\mathbf{z}}$. It is known that finding the optimal Voronoi region, as well as characterizing the exact optimal inertial profile, is hard. However, the inertial profile of any Voronoi shape, including the optimal inertial profile, can be tightly lower bounded by that of an "M-shaped" hyperellipsoid with the closed form expression given by

$$m(\mathbf{y};\mathbf{z};\mathbb{E}) \ge m_{\text{opt}}(\mathbf{y};\mathbf{z}) \gtrapprox \widetilde{m}_{\text{opt}}(\mathbf{y};\mathbf{z}) = \frac{k_q}{k_q+2}\cdot\left(\frac{|\mathbf{W}_{\mathbf{z}}(\mathbf{y})|}{\kappa_{k_q}^2}\right)^{1/k_q}, \quad \kappa_n = \frac{\pi^{n/2}}{\Gamma(n/2+1)}. \tag{5}$$

Second, by substituting the inertial profile lower bound (5) into the system distortion integral, as well as utilizing Holder's inequality to select the optimal point density, the asymptotic distortion of the generalized finite-rate quantization system can be lower bounded by $\widetilde{D}_{\text{Low}}$, given by

$$\widetilde{D}_{\text{Low}} = 2^{-2B/k_q}\cdot\left(\int_{\mathbb{Q}}\left(\widetilde{m}^{w}_{\text{opt}}(\mathbf{y})\cdot p(\mathbf{y})\right)^{k_q/(2+k_q)}d\mathbf{y}\right)^{(2+k_q)/k_q}, \tag{6}$$

where $\widetilde{m}^{w}_{\text{opt}}(\mathbf{y})$ is the average optimal inertial profile defined as

$$\widetilde{m}^{w}_{\text{opt}}(\mathbf{y}) = \int_{\mathbb{Z}}\widetilde{m}_{\text{opt}}(\mathbf{y};\mathbf{z})\cdot p(\mathbf{z}\mid\mathbf{y})d\mathbf{z}. \tag{7}$$

The optimal point density that minimizes the asymptotic system distortion is given by

$$\lambda^*(\mathbf{y}) = \left(\widetilde{m}^{w}_{\text{opt}}(\mathbf{y})\cdot p(\mathbf{y})\right)^{k_q/(2+k_q)} \cdot\left(\int_{\mathbb{Q}}\left(\widetilde{m}^{w}_{\text{opt}}(\mathbf{y})\cdot p(\mathbf{y})\right)^{k_q/(2+k_q)}d\mathbf{y}\right)^{-1}. \tag{8}$$

2.4. Distortion analysis of constrained source

The analysis discussed above is for the case where the input source $\mathbf{y}$ is a free random vector of dimension k_q. In some situations, it is required to quantize the k_q-dimensional source vector $\mathbf{y}\in\mathbb{Q}$ subject to a multidimensional constraint function $\mathbf{g}(\mathbf{y}) = \mathbf{0}$ of size $k_c \times 1$, for example, the scalar function $g(\mathbf{y}) = (\|\mathbf{y}\|^2 - 1)$ represents the unit norm constraint. In this case, the distortion analysis discussed above has been shown to still be valid with the following modification. First, the degrees of freedom in $\mathbf{y}$ are reduced from k_q to $k_q' = k_q - k_c$. Second, the sensitivity matrix is replaced by its constrained version $\mathbf{W}_{c,\mathbf{z}}(\mathbf{y})$, given by

$$\mathbf{W}_{c,\mathbf{z}}(\mathbf{y}) = \mathbf{V}_2^{\mathrm{T}}\cdot\mathbf{W}_{\mathbf{z}}(\mathbf{y})\cdot\mathbf{V}_2, \tag{9}$$

where $\mathbf{V}_2 \in \mathbb{R}^{k_q\times k_q'}$ is an orthonormal matrix with its columns constituting an orthonormal basis for the null space $\mathcal{N}((\partial/\partial\mathbf{y})\mathbf{g}(\mathbf{y}))$. Lastly, the multidimensional integrations used in evaluating the average distortions are over the constrained space $\mathbf{g}(\mathbf{y}) = 0$.

3. ASYMPTOTIC DISTORTION ANALYSIS OF QUANTIZERS WITH TRANSFORMED CODEBOOK

In certain situations, the underlying source distribution $p(\mathbf{y},\mathbf{z})$ or the distortion function D_Q of the source variable varies during the quantization process. It is practically infeasible to design separate codebooks optimized for every different source distribution and distortion function, or the encoder and the decoder may not have the ability to store a large number of codebooks. In these situations, it is convenient to use a quantizer whose codebook is constructed by a transformation of a fixed codebook based on the current statistical distribution of the source variable. These types of quantizers are generally called transformed quantizers [22, 23], and have been used in the conventional source coding area with a linear orthogonal transformation followed by a product quantizer. We provide in this subsection an analysis of the generalized vector quantizer, which is described in Section 2, when a transformed codebook is used. Detailed applications to finite-rate feedback MISO systems with a transformed codebook over spatially correlated fading channels are provided in Section 5.

3.1. Problem formulation

It is first assumed that all the codebooks are generated from one fixed codebook $\mathcal{C}_0$, which is designed to match the source distribution $p_0(\mathbf{y},\mathbf{z})$, and distortion function $D_{0,Q}$ with sensitivity matrix $\mathbf{W}_{0,\mathbf{z}}(\mathbf{y})$. Codebook $\mathcal{C}_0$ has a point density given by $\lambda_0(\mathbf{y})$, and a normalized inertial profile $I_0(\mathbf{y};\mathbf{z};\mathbb{E}_{0,\mathbf{z}}(\mathbf{y}))$ that is optimized to match the distortion function $D_{0,Q}$, with $\mathbb{E}_{0,\mathbf{z}}(\mathbf{y})$ representing the asymptotic Voronoi cell that contains $\mathbf{y}$ with side information $\mathbf{z}$. Let the source distribution change from $p_0(\mathbf{y},\mathbf{z})$ to $p(\mathbf{y},\mathbf{z})$, and let the distortion function become D_Q instead of $D_{0,Q}$ with sensitivity matrix $\mathbf{W}_\mathbf{z}(\mathbf{y})$ instead of $\mathbf{W}_{0,\mathbf{z}}(\mathbf{y})$. The encoder and decoder are assumed to adapt a transformed codebook $\mathcal{C}$ obtained from $\mathcal{C}_0$ by using a general one-to-one mapping $\mathbf{F}(\cdot)$ with both its domain and codomain in space $\mathbb{Q}$, that is,

$$\mathcal{C} = \{\mathbf{F}(\hat{\mathbf{y}}) \mid \hat{\mathbf{y}} \in \mathcal{C}_0\}. \tag{10}$$

3.2. Suboptimal point density and suboptimal voronoi shape

Assuming the codebook transformation function $\mathbf{F}(\cdot)$ has a continuous first order derivative, two types of suboptimality arise when the transformed quantizer is used. One comes from the suboptimal point density $\lambda_{\text{tr}}(\mathbf{y})$, which can be derived from $\lambda_0(\mathbf{y})$ as

$$\begin{aligned}\lambda_{\text{tr}}(\mathbf{y}) &= \frac{\lambda_0(\mathbf{F}^{-1}(\mathbf{y}))}{|\mathbf{F}_d(\mathbf{F}^{-1}(\mathbf{y}))|},\\ \mathbf{F}_d(\mathbf{y}) &= \frac{\partial \mathbf{F}(\mathbf{y})}{\partial \mathbf{y}}.\end{aligned} \tag{11}$$

If the source variable is subject to k_c constraints given by vector equation $\mathbf{g}(\mathbf{y}) = \mathbf{0}$, the transformed point density is given by

$$\lambda_{c\text{-tr}}(\mathbf{y}) = \frac{\lambda_0(\mathbf{F}^{-1}(\mathbf{y}))}{\left|\mathbf{V}_2(\mathbf{y})^{\mathrm{T}}\cdot\mathbf{F}_d(\mathbf{F}^{-1}(\mathbf{y}))\cdot\mathbf{V}_2(\mathbf{F}^{-1}(\mathbf{y}))\right|}, \tag{12}$$

where $\mathbf{V}_2(\mathbf{y})$ is an orthonormal matrix whose columns constitute an orthonormal basis for the null space $\mathcal{N}((\partial/\partial\mathbf{y})\mathbf{g}(\mathbf{y}))$. Compared to the optimal point density $\lambda^*(\mathbf{y})$ given by (8), which corresponds to the optimally designed codebook, $\lambda_{\text{tr}}(\mathbf{y})$ given by (11) is always suboptimal and hence leads to performance degradation. The other suboptimality arises from the constraints on the code points in the transformed codebook $\mathcal{C}$ in the sense that the Voronoi shape of the transformed code is not matched to the distortion function D_Q, and hence is not optimized to minimize the inertial profile. Note that these two suboptimalities, named as point density loss and cell shape loss, were also discussed in [22] in the setting of the conventional product quantizers and further applied to study the distortion performance of conventional quantizers with transformed codebooks.

3.3. Characterizing the inertial profile of the transformed codebook

Unfortunately, the Voronoi region $\mathbb{E}_{\text{tr},\mathbf{z}}(\hat{\mathbf{y}}_i')$ of the transformed codebook, which is defined to be

$$\begin{aligned}&\mathbb{E}_{\text{tr},\mathbf{z}}(\hat{\mathbf{y}}_i')\\ &\triangleq \{\mathbf{y} \mid D_Q(\mathbf{y},\hat{\mathbf{y}}_i';\mathbf{z}) \le D_Q(\mathbf{y},\hat{\mathbf{y}}_j';\mathbf{z}),\ \forall \hat{\mathbf{y}}_j' \in \mathcal{C},\ \hat{\mathbf{y}}_j' \ne \hat{\mathbf{y}}_i'\}, \quad \hat{\mathbf{y}}_i' \in \mathcal{C},\end{aligned} \tag{13}$$

is hard to characterize and depends on both the transformation $\mathbf{F}$ as well as the distortion function D_Q. In order to characterize the effects of the transformed Voronoi shape on the system distortion, lower and upper bounds of the normalized inertial profile of the transformed code are provided. First, let us consider a suboptimal quantizer $\mathcal{Q}_{\text{sub}}(\cdot)$ with transformed codebook $\mathcal{C}$ that uses a suboptimal encoding process given by

$$\hat{\mathbf{y}} = \mathcal{Q}_{\text{sub}}(\mathbf{y},\mathbf{z}) = \mathbf{F}(\mathcal{Q}(\mathbf{F}^{-1}(\mathbf{y}),\mathbf{z})), \tag{14}$$

where $\mathcal{Q}(\cdot)$ is the optimal encoder that is matched to the distortion function $D_{0,Q}$. This suboptimal encoder can be viewed as an extension of the "companding" model introduced by Bennett [24] to the general vector quantization problem. It was originally used in conventional scalar quantizers, where the encoder is a combination of a monotonically increasing nonlinear mapping $E(x)$, the compressor, followed by a uniform quantizer; and the corresponding decoder is composed of a uniform decoder followed by an inverse mapping E^{-1}, the expander. In the case of the generalized vector quantizer discussed here, the Voronoi shape of the suboptimal transformed encoder $\mathcal{Q}_{\text{sub}}$ can be analytically characterized as

$$\mathbb{E}_{\text{sub},\mathbf{z}}(\mathbf{F}(\mathbf{y})) = \{\mathbf{F}(\mathbf{y}') \mid \mathbf{y}' \in \mathbb{E}_{0,\mathbf{z}}(\mathbf{y})\}, \tag{15}$$

where $\mathbb{E}_{0,\mathbf{z}}(\mathbf{y})$ is the optimal Voronoi shape of the original codebook $\mathcal{C}_0$ corresponding to distortion function $D_{0,Q}$. Due to the suboptimality of encoder $\mathcal{Q}_{\text{sub}}$, the normalized inertial profile of the transformed Voronoi shape $\mathbb{E}_{\text{tr},\mathbf{z}}(\mathbf{y})$ is upper bounded by the inertial profile of $\mathbb{E}_{\text{sub},\mathbf{z}}(\mathbf{y})$ given by (15), but lower bounded by the inertial profile of the optimal Voronoi shape $\mathbf{E}_\mathbf{z}(\mathbf{y})$ corresponding to the distortion function D_Q.

Proposition 1. *Under high resolution assumptions, the approximated inertial profile $\widetilde{m}_{\text{tr}}(\mathbf{F}(\hat{\mathbf{y}});\mathbf{z})$ of a quantizer with transformed codebook can be upper and lower bounded by the* *following form:*

$$
\begin{aligned}
&\frac{k_q}{k_q+2}\cdot\left(\frac{|\mathbf{W}_{\mathbf{z}}(\mathbf{F}(\mathbf{y}))|}{\kappa_{k_q}^2}\right)^{1/k_q}\\
&\quad=\widetilde{m}_{\text{opt}}(\mathbf{F}(\mathbf{y});\mathbf{z})\overset{a}{\leq}\widetilde{m}_{\text{tr}}(\mathbf{F}(\mathbf{y});\mathbf{z})\overset{b}{\leq}\widetilde{m}_{\text{sub}}(\mathbf{F}(\mathbf{y});\mathbf{z})\\
&\quad=\frac{|\mathbf{F}_d(\mathbf{y})|^{-2/k_q}}{k_q+2}\cdot\left(\frac{|\mathbf{W}_{0,\mathbf{z}}(\mathbf{y})|}{\kappa_{k_q}^2}\right)^{1/k_q}\\
&\qquad\cdot\mathbf{tr}\big(\mathbf{W}_{0,\mathbf{z}}(\mathbf{y})^{-1}\cdot\mathbf{F}_d(\mathbf{y})^{\mathrm{T}}\cdot\mathbf{W}_{\mathbf{z}}(\mathbf{F}(\mathbf{y}))\cdot\mathbf{F}_d(\mathbf{y})\big).
\end{aligned}\tag{16}
$$

Furthermore, if the source variable is subject to k_c constraints given by the vector equation $\mathbf{g}(\mathbf{y})=\mathbf{0}$, the constrained inertial profile $\widetilde{m}_{c\text{-tr}}(\mathbf{F}(\hat{\mathbf{y}});\mathbf{z})$ can be similarly bounded by

$$
\begin{aligned}
&\frac{k'_q}{k'_q+2}\cdot\left(\frac{|\mathbf{V}_2(\mathbf{F}(\mathbf{y}))^{\mathrm{T}}\cdot\mathbf{W}_{\mathbf{z}}(\mathbf{F}(\mathbf{y}))\cdot\mathbf{V}_2(\mathbf{F}(\mathbf{y}))|}{\kappa_{k_{q'}}^2}\right)^{1/k_{q'}}\\
&\quad=\widetilde{m}_{c\text{-opt}}(\mathbf{F}(\mathbf{y});\mathbf{z})\overset{a}{\leq}\widetilde{m}_{c\text{-tr}}(\mathbf{F}(\mathbf{y});\mathbf{z})\overset{b}{\leq}\widetilde{m}_{c\text{-sub}}(\mathbf{F}(\mathbf{y});\mathbf{z})\\
&\quad=\frac{|\mathbf{V}_2(\mathbf{F}(\mathbf{y}))^{\mathrm{T}}\cdot\mathbf{F}_d(\mathbf{y})\cdot\mathbf{V}_2(\mathbf{y})|^{-2/k'_q}}{k'_q+2}\\
&\qquad\cdot\left(\frac{|\mathbf{V}_2(\mathbf{y})^{\mathrm{T}}\cdot\mathbf{W}_{0,\mathbf{z}}(\mathbf{y})\cdot\mathbf{V}_2(\mathbf{y})|}{\kappa_{k'_q}^2}\right)^{1/k'_q}\\
&\qquad\cdot\mathbf{tr}\big((\mathbf{V}_2(\mathbf{y})^{\mathrm{T}}\cdot\mathbf{W}_{0,\mathbf{z}}(\mathbf{y})\cdot\mathbf{V}_2(\mathbf{y}))^{-1}\cdot\mathbf{V}_2(\mathbf{y})^{\mathrm{T}}\\
&\qquad\quad\cdot\mathbf{F}_d(\mathbf{y})^{\mathrm{T}}\cdot\mathbf{W}_{\mathbf{z}}(\mathbf{F}(\mathbf{y}))\cdot\mathbf{F}_d(\mathbf{y})\cdot\mathbf{V}_2(\mathbf{y})\big),
\end{aligned}\tag{17}
$$

where $\mathbf{V}_2(\mathbf{y})$ is an orthonormal matrix with its columns constituting an orthonormal basis for the null space $\mathcal{N}((\partial/\partial\mathbf{y})\mathbf{g}(\mathbf{y}))$.

Proof. Due to the constraints on the code points in the transformed codebook $\mathcal{C}$, which cannot be optimized to minimize the normalized inertial profile, it is evident that the transformed inertial profile $\widetilde{m}_{\text{tr}}$ is lower bounded by the optimal inertial profile $\widetilde{m}_{\text{opt}}$ given by (5). Hence, inequality (a) in (16) can be obtained after some manipulations. The same reasonings are valid for inequality (a) in (17) for the constrained source.

As for inequality (b) in (16), since function $\mathbf{F}(\cdot)$ is first order continuous, any points in the vicinity of the transformed code point $\mathbf{F}(\hat{\mathbf{y}})$ has a first-order Taylor series expansion given by

$$
\begin{aligned}
\mathbf{F}(\mathbf{y})&\approx\mathbf{F}(\hat{\mathbf{y}})+\mathbf{F}_d(\hat{\mathbf{y}})\cdot(\mathbf{y}-\hat{\mathbf{y}}),\\
\mathbf{F}_d(\hat{\mathbf{y}})&=\frac{\partial}{\partial\mathbf{y}}\bigg|_{\mathbf{y}=\hat{\mathbf{y}}}\mathbf{F}(\mathbf{y}).
\end{aligned}\tag{18}
$$

Moreover, due to the fact that $\mathbf{F}(\cdot)$ is a one-to-one mapping, for any point $\mathbf{y}'$ in the vicinity of $\mathbf{F}(\hat{\mathbf{y}})$, there exists a unique point $\mathbf{y}$ in the neighborhood of $\hat{\mathbf{y}}$ such that $\mathbf{y}'=\mathbf{F}(\mathbf{y})$. Therefore, under high resolutions, the distortion function D_Q can be expanded around point $\mathbf{F}(\hat{\mathbf{y}})$ as follows:

$$
\begin{aligned}
&D_Q(\mathbf{y}',\mathbf{F}(\hat{\mathbf{y}});\mathbf{z})\\
&\quad\approx(\mathbf{y}'-\mathbf{F}(\hat{\mathbf{y}}))^{\mathrm{T}}\mathbf{W}_{\mathbf{z}}(\mathbf{F}(\hat{\mathbf{y}}))(\mathbf{y}'-\mathbf{F}(\hat{\mathbf{y}}))\\
&\quad\approx(\mathbf{y}-\hat{\mathbf{y}})^{\mathrm{T}}\cdot\big(\mathbf{F}_d(\hat{\mathbf{y}})^{\mathrm{T}}\cdot\mathbf{W}_{\mathbf{z}}(\mathbf{F}(\hat{\mathbf{y}}))\cdot\mathbf{F}_d(\hat{\mathbf{y}})\big)\cdot(\mathbf{y}-\hat{\mathbf{y}}),
\end{aligned}\tag{19}
$$

which has quadratic form but with transformed sensitivity matrix. By substituting (19), as well as the Voronoi shape of the suboptimal encoder given by (15), into the definition of the inertial profile given by (3), we can obtain the following normalized inertial profile of the transformed code with suboptimal encoder:

$$
\begin{aligned}
&\widetilde{m}_{\text{tr}}(\mathbf{F}(\hat{\mathbf{y}});\mathbf{z})\\
&\quad\leq\widetilde{m}_{\text{sub}}(\mathbf{F}(\hat{\mathbf{y}});\mathbf{z})=\frac{|\mathbf{F}_d(\hat{\mathbf{y}})|^{-2/k_q}}{k_q+2}\cdot\left(\frac{|\mathbf{W}_{0,\mathbf{z}}(\hat{\mathbf{y}})|}{\kappa_{k_q}^2}\right)^{1/k_q}\\
&\qquad\cdot\mathbf{tr}\big(\mathbf{W}_{0,\mathbf{z}}(\hat{\mathbf{y}})^{-1}\cdot\mathbf{F}_d(\hat{\mathbf{y}})^{\mathrm{T}}\cdot\mathbf{W}_{\mathbf{z}}(\mathbf{F}(\hat{\mathbf{y}}))\cdot\mathbf{F}_d(\hat{\mathbf{y}})\big),
\end{aligned}\tag{20}
$$

which corresponds to inequality (b) in (16).

If the source variable (vector) $\mathbf{y}$ is further subject to k_c *constraints* given by the vector equation $\mathbf{g}(\mathbf{y})=\mathbf{0}$, the distortion function D_Q can be similarly expanded around point $\mathbf{F}(\hat{\mathbf{y}})$ as

$$
\begin{aligned}
&D_Q(\mathbf{y}',\mathbf{F}(\hat{\mathbf{y}});\mathbf{z})\\
&\quad\approx(\mathbf{y}-\hat{\mathbf{y}})^{\mathrm{T}}\cdot(\mathbf{F}_d(\hat{\mathbf{y}})^{\mathrm{T}}\cdot\mathbf{W}_{\mathbf{z}}(\mathbf{F}(\hat{\mathbf{y}}))\cdot\mathbf{F}_d(\hat{\mathbf{y}}))\cdot(\mathbf{y}-\hat{\mathbf{y}})\\
&\quad=\mathbf{e}^{\mathrm{T}}\cdot(\mathbf{V}_2(\hat{\mathbf{y}})^{\mathrm{T}}\mathbf{F}_d(\hat{\mathbf{y}})^{\mathrm{T}}\mathbf{W}_{\mathbf{z}}(\mathbf{F}(\hat{\mathbf{y}}))\mathbf{F}_d(\hat{\mathbf{y}})\mathbf{V}_2(\hat{\mathbf{y}}))\cdot\mathbf{e},
\end{aligned}\tag{21}
$$

where $\mathbf{e}$ is the projected error vector with respect to point $\hat{\mathbf{y}}$ given by

$$
\mathbf{e}=\mathbf{V}_2(\hat{\mathbf{y}})^{\mathrm{T}}\cdot(\mathbf{y}-\hat{\mathbf{y}}).\tag{22}
$$

By substituting (21) and the suboptimal Voronoi shape (15) into the inertial profile definition (3), we can obtain the suboptimal inertial profile of the transformed code with constrained source

$$
\begin{aligned}
&\widetilde{m}_{\text{tr-}c}(\mathbf{F}(\hat{\mathbf{y}});\mathbf{z})\\
&\quad\leq\widetilde{m}_{\text{sub-}c}(\mathbf{F}(\hat{\mathbf{y}});\mathbf{z})=\frac{|\mathbf{V}_2(\hat{\mathbf{y}})^{\mathrm{T}}\cdot\mathbf{F}_d(\hat{\mathbf{y}})\cdot\mathbf{V}_2(\hat{\mathbf{y}})|^{-2/k'_q}}{k'_q+2}\\
&\qquad\cdot\left(\frac{|\mathbf{V}_2(\hat{\mathbf{y}})^{\mathrm{T}}\cdot\mathbf{W}_{0,\mathbf{z}}(\hat{\mathbf{y}})\cdot\mathbf{V}_2(\hat{\mathbf{y}})|}{\kappa_{k_q}^2}\right)^{1/k_q}\\
&\qquad\cdot\mathbf{tr}\big((\mathbf{V}_2(\hat{\mathbf{y}})^{\mathrm{T}}\cdot\mathbf{W}_{0,\mathbf{z}}(\hat{\mathbf{y}})\cdot\mathbf{V}_2(\hat{\mathbf{y}}))^{-1}\cdot\mathbf{V}_2(\hat{\mathbf{y}})^{\mathrm{T}}\\
&\qquad\quad\cdot\mathbf{F}_d(\hat{\mathbf{y}})^{\mathrm{T}}\cdot\mathbf{W}_{\mathbf{z}}(\mathbf{F}(\hat{\mathbf{y}}))\cdot\mathbf{F}_d(\hat{\mathbf{y}})\cdot\mathbf{V}_2(\hat{\mathbf{y}})\big),
\end{aligned}\tag{23}
$$

which corresponds to inequality (b) in (17). □

3.4. Distortion integral of the transformed codebook

By substituting the transformed point density (11) and the bounds of the transformed inertial profile given by (16) into the distortion integration (2), we can upper and lower bound the asymptotic system distortion of a transformed quantizer by the following form:

$$\begin{aligned}
&\widetilde{D}_{\text{tr-Low}} \\
&\quad = 2^{-2B/k_q}\cdot\left(\int_{\mathbb{Z}}\int_{\mathbb{Q}}\widetilde{m}_{\text{opt}}(\mathbf{y};\mathbf{z})\cdot p(\mathbf{y},\mathbf{z})\cdot\lambda_{\text{tr}}(\mathbf{y})^{-2/k_q}d\mathbf{y}\,d\mathbf{z}\right) \\
&\quad \leq \widetilde{D}_{\text{tr}} \\
&\quad = 2^{-2B/k_q}\cdot\left(\int_{\mathbb{Z}}\int_{\mathbb{Q}}\widetilde{m}_{\text{tr}}(\mathbf{y};\mathbf{z})\cdot p(\mathbf{y},\mathbf{z})\cdot\lambda_{\text{tr}}(\mathbf{y})^{-2/k_q}d\mathbf{y}\,d\mathbf{z}\right) \\
&\quad \leq \widetilde{D}_{\text{tr-Upp}} \\
&\quad = 2^{-2B/k_q}\cdot\left(\int_{\mathbb{Z}}\int_{\mathbb{Q}}\widetilde{m}_{\text{sub}}(\mathbf{y};\mathbf{z})\cdot p(\mathbf{y},\mathbf{z})\cdot\lambda_{\text{tr}}(\mathbf{y})^{-2/k_q}d\mathbf{y}\,d\mathbf{z}\right).
\end{aligned} \tag{24}$$

Similarly, by substituting (12) and (17) into (2), the asymptotic distortion of a constrained quantizer with transformed codebook is bounded by

$$\begin{aligned}
&\widetilde{D}_{c\text{-tr-Low}} \\
&\quad = 2^{-2B/k'_q}\cdot\left(\int_{\mathbb{Z}}\int_{\mathbb{Q}}\widetilde{m}_{c\text{-opt}}(\mathbf{y};\mathbf{z})\cdot p(\mathbf{y},\mathbf{z})\cdot\lambda_{c\text{-tr}}(\mathbf{y})^{-2/k'_q}d\mathbf{y}\,d\mathbf{z}\right) \\
&\quad \leq \widetilde{D}_{c\text{-tr}} \\
&\quad = 2^{-2B/k'_q}\cdot\left(\int_{\mathbb{Z}}\int_{\mathbb{Q}}\widetilde{m}_{c\text{-tr}}(\mathbf{y};\mathbf{z})\cdot p(\mathbf{y},\mathbf{z})\cdot\lambda_{c\text{-tr}}(\mathbf{y})^{-2/k'_q}d\mathbf{y}\,d\mathbf{z}\right) \\
&\quad \leq \widetilde{D}_{c\text{-tr-Upp}} \\
&\quad = 2^{-2B/k'_q}\cdot\left(\int_{\mathbb{Z}}\int_{\mathbb{Q}}\widetilde{m}_{c\text{-sub}}(\mathbf{y};\mathbf{z})\cdot p(\mathbf{y},\mathbf{z})\cdot\lambda_{c\text{-tr}}(\mathbf{y})^{-2/k'_q}d\mathbf{y}\,d\mathbf{z}\right).
\end{aligned} \tag{25}$$

Similar to conventional product transformed quantizers [22], there exist trade-offs between the two suboptimalities: point density loss and Voronoi shape loss. To be specific, it is always possible to find a transformation $\mathbf{F}(\cdot)$ such that the transformed point density $\lambda_{\text{tr}}(\mathbf{y})$ matches exactly the optimal point density $\lambda^*(\mathbf{y})$. However, by doing so, the transformation might cause shape loss of the transformed Voronoi cells in some cases, which will lead to significant increase in the normalized inertial profile. Therefore, a transformation that optimally balances two types of losses should be employed. This tradeoff is directly reflected in the distortion bound $\widetilde{D}_{\text{tr,Upp}}$ where both $\widetilde{m}_{\text{sub}}(\mathbf{y};\mathbf{z})$ and $\lambda_{\text{tr}}(\mathbf{y})$ in (24) depend on the transformation $\mathbf{F}(\cdot)$. So is the distortion bound $\widetilde{D}_{c\text{-tr,Upp}}$ given by (25).

4. MISO SYSTEMS USING FINITE-RATE CSI QUANTIZERS WITH TRANSFORMED CODEBOOK

4.1. System model of MISO fading channels

We consider a MISO system, with t transmit antennas and one receive antenna, signaling through a frequency flat fading channel. The channel model can be represented as

$$y = \mathbf{h}^{\text{H}}\cdot\mathbf{x} + n, \tag{26}$$

where y is the received signal (scalar), n is the additive complex Gaussian noise with zero mean and unit variance, and $\mathbf{h}^{\text{H}} \in \mathbb{C}^{1\times t}$ is the correlated MISO channel response with distribution given by $\mathbf{h} \sim \mathcal{N}_c(\mathbf{0},\mathbf{\Sigma}_h)$. For the sake of fair comparisons, we normalize the channel covariance matrix such that the mean of the eigen values equals one (equal to the i.i.d. channel case $\mathbf{\Sigma}_h = I_t$). Moreover, the statistical information (i.e., channel covariance matrix $\mathbf{\Sigma}_h$) of the MISO channel response is assumed to be perfectly known at both the transmitter and the receiver. The transmitted signal vector $\mathbf{x}$ is normalized to have a power constraint given by $E[\|\mathbf{x}\|^2] = \rho$, with ρ representing the average signal-to-noise ratio at each receive antenna.

4.2. Beamforming with finite-rate CSI feedback

In this paper, the channel state information $\mathbf{h}$ is assumed to be perfectly known at the receiver but only partially available at the transmitter through a finite-rate feedback link of B bits per channel update between the transmitter and receiver. To be specific, a quantization codebook $\mathcal{C} = \{\hat{\mathbf{v}}_1,\dots,\hat{\mathbf{v}}_N\}$, which is composed of unit-norm transmit beamforming vectors, is assumed known to both the receiver and the transmitter. Based on the channel realization $\mathbf{h}$, the receiver selects the best code point $\hat{\mathbf{v}}$ from the codebook and sends the corresponding index back to the transmitter. At the transmitter, the unit-norm vector $\hat{\mathbf{v}}$ is employed as the beamforming vector, and the resulting received signal can be represented as

$$\begin{gathered}
y = \langle\mathbf{h},\hat{\mathbf{v}}\rangle\cdot s + n = \|\mathbf{h}\|\cdot\langle\mathbf{v},\hat{\mathbf{v}}\rangle\cdot s + n, \\
E\left[|s|^2\right] = \rho,
\end{gathered} \tag{27}$$

where $\mathbf{v}$ is the channel direction vector given by $\mathbf{v} = \mathbf{h}/\|\mathbf{h}\|$.

4.3. Problem of channel quantizers with transformed codebook

According to [8], it is clear that the statistical information of the fading channel is very important for the design of MISO transmit precoders. The resulting optimal beamforming codebook obtained by utilizing a vector quantization (VQ) approach depends on the channel covariance matrix. In practical situations, the spatial correlation conditions of the fading channel responses may change during the transmission process. However, for a real system, it is impossible to design different codebooks optimized for every instantiation of the channel covariance matrix and it might also be infeasible for the transmitter and receiver to store a large number of codebooks and use them adaptively. In these cases, it is convenient to use a channel quantizer whose codebook is generated from a fixed pre-designed codebook through a transformation parameterized by the channel covariance matrix. (Imperfect knowledge of the channel covariance matrix will also impact the system performance.

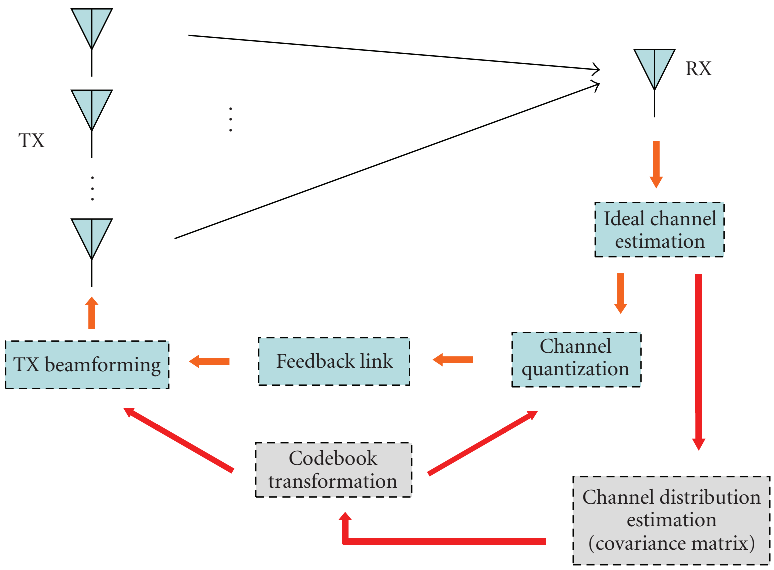

Figure 1: System diagram of a MISO beamforming system with limited CSI feedback.

Interested readers are referred to [20, Sections IV-C and V-B], where a detailed analysis of MISO beamforming systems employing channel quantizers designed with mismatched channel covariance matrix is provided.)

To be specific, suppose $\mathcal{C}_0$ is the optimal codebook designed for the i.i.d. MISO fading channels. When the elements of the fading channel response $\mathbf{h}$ are correlated, that is, $\mathbf{h} \sim \mathcal{N}_c(\mathbf{0}, \mathbf{\Sigma}_h)$, it is evident that codebook $\mathcal{C}_0$ is no longer optimal. In order to compensate for the mismatch between $\mathcal{C}_0$ and the current channel statistics, a transformed codebook $\mathcal{C}$ can be generated by the following manner:

$$\mathcal{C} = \{\mathbf{F}(\hat{\mathbf{v}}) \mid \hat{\mathbf{v}} \in \mathcal{C}_0\}, \tag{28}$$

where $\mathbf{F}(\cdot)$ is a general nonlinear transformation that depends on the channel statistics. Optimization of the transformation $\mathbf{F}(\cdot)$ turns out to be difficult, and hence a simple suboptimal transformation,

$$\mathbf{F}(\hat{\mathbf{v}}) = \frac{\mathbf{G}\hat{\mathbf{v}}}{\|\mathbf{G}\hat{\mathbf{v}}\|}, \tag{29}$$

was proposed in [6, 13] where $\mathbf{G} \in \mathbb{C}^{t\times t}$ is a fixed matrix which depends on the channel covariance matrix $\mathbf{\Sigma}_h$. Distortion analysis of CSI-quantizers with transformed codebooks is provided in next section.

In order to facilitate understanding, a top level diagram of a MISO beamforming system with finite rate CSI feedback is shown in Figure 1. The exchange of the CSI information between the transmitter and receiver is demonstrated. Major modules of the channel quantization process are also depicted.

4.4. *Capacity loss as system performance metric*

According to the received signal model given by (27), the corresponding ergodic capacity, or the maximum system mutual information rate, of the quantized MISO beamforming system is given by

$$C_Q = E\Big[\log_2\Big(1 + \rho \cdot \|\mathbf{h}\|^2 \cdot |\langle \mathbf{v}, \hat{\mathbf{v}}\rangle|^2\Big)\Big]. \tag{30}$$

On the other hand, with perfect channel state information available at the transmitter, which corresponds to the case of infinite rate feedback $B = \infty$, it is optimal to choose $\mathbf{v} = \mathbf{h}/\|\mathbf{h}\|$ as the transmit beamforming vector, and the corresponding system ergodic capacity is given by

$$C_P = E\big[\log_2\big(1 + \rho \cdot \|\mathbf{h}\|^2\big)\big]. \tag{31}$$

Therefore, the performance of a CSI-feedback-based MISO system can be characterized by the capacity loss C_{Loss} due to the finite-rate quantization of the transmit beamforming vectors, which is defined as the expectation of the instantaneous mutual information rate loss $C_L(\mathbf{h}, \hat{\mathbf{v}})$, that is,

$$C_{\text{Loss}} = C_P - C_Q = E\big[C_L(\mathbf{h}, \hat{\mathbf{v}})\big],$$
$$C_L(\mathbf{h}, \hat{\mathbf{v}}) = -\log_2\left(1 - \frac{\rho \cdot \|\mathbf{h}\|^2}{1 + \rho \cdot \|\mathbf{h}\|^2} \cdot \big(1 - |\langle \mathbf{v}, \hat{\mathbf{v}}\rangle|^2\big)\right). \tag{32}$$

This performance metric was also used in [11, 19]. From an information theoretical point of view, a CSI feedback scheme should be designed to minimize this performance metric.

5. CAPACITY ANALYSIS OF MISO CSI QUANTIZERS WITH TRANSFORMED CODEBOOK

By utilizing the distortion analysis of the transformed codebooks provided in Sections 2 and 3, this section provides an investigation of the capacity loss of a finite-rate CSI-quantized MISO beamforming system over spatially correlated fading channels, that uses transformed CSI quantizers.

5.1. Reformulation of the CSI-quantized MISO beamforming system

By employing the general framework described in Section 2, the finite-rate quantized MISO beamforming system can be formulated as a general fixed-rate vector quantization problem by adopting a direct mapping between CSI and source variables, given by $(\mathbf{v},\alpha)\rightarrow(\mathbf{y},\mathbf{z})$. Specifically, the source variable to be quantized is denoted as $\overline{\mathbf{v}} = [\mathbf{v}_R^{\mathrm{T}}, \mathbf{v}_I^{\mathrm{T}}]^{\mathrm{T}}$ of $2t$ real dimensions with $\mathbf{v}_R$ and $\mathbf{v}_I$ representing the real and imaginary parts of the complex channel directional vector $\mathbf{v}$. The encoder side information is denoted as $\alpha = \|\mathbf{h}\|^2$ of dimension $k_\alpha = 1$ representing the power of the vector channel. For vectors in the vicinity of $\hat{\mathbf{v}}$ (with $\hat{\mathbf{v}}_R$ and $\hat{\mathbf{v}}_I$ representing its real and imaginary parts), source variable $\overline{\mathbf{v}}$ is restricted under the constraint function given by

$$\mathbf{g}(\mathbf{v}) = \begin{bmatrix} \mathbf{v}_R^{\mathrm{T}}\mathbf{v}_R + \mathbf{v}_I^{\mathrm{T}}\mathbf{v}_I - 1 \\ \mathbf{v}_R^{\mathrm{T}}\hat{\mathbf{v}}_I - \mathbf{v}_I^{\mathrm{T}}\hat{\mathbf{v}}_R \end{bmatrix} = 0, \tag{33}$$

where the first element represents the norm constraint $\|\mathbf{v}\| = 1$, and the second element represents the phase constraint $\measuredangle\langle \mathbf{v}, \hat{\mathbf{v}}\rangle = 0$. The function $\mathbf{g}(\mathbf{v})$ has size $k_c = 2$, which leads to the actual degrees of freedom of the quantization variable $\mathbf{v}$ to be $k_q' = 2t - 2$. The instantaneous capacity loss due to effects of finite-rate CSI quantization is taken to be the system distortion function $D_Q(\mathbf{v}, \hat{\mathbf{v}}; \alpha)$, which has the following form according to (32)

$$D_Q(\mathbf{v}, \hat{\mathbf{v}}; \alpha) = C_L(\mathbf{h}, \hat{\mathbf{v}}) \triangleq -\log_2\left(1 - \frac{\rho\alpha}{1+\rho\alpha}\cdot\left(1 - |\langle \mathbf{v}, \hat{\mathbf{v}}\rangle|^2\right)\right), \tag{34}$$

where α is the instantaneous channel power given by $\alpha = \|\mathbf{h}\|^2$.

5.2. Distortion anslysis of optimal CSI quantizers

In order to understand CSI quantizers with transformed codebooks, it is worth investigating the optimal CSI quantization scheme first. For correlated MISO fading channels, by substituting the distortion function (34) into (5), the optimal normalized inertial profile of a MISO system is tightly lower bounded by the following form:

$$\widetilde{m}_{c,\mathrm{opt}}(\hat{\mathbf{v}}; \alpha) = \frac{(t-1)\cdot\gamma_t^{-1/(t-1)}\cdot\rho\alpha}{\ln 2\cdot t\cdot(1+\rho\alpha)}, \qquad \gamma_t = \frac{\pi^{t-1}}{(t-1)!}. \tag{35}$$

Moreover, by substituting the inertial profile lower bound $\widetilde{m}_{c,\mathrm{opt}}(\hat{\mathbf{v}}; \alpha)$ into the distortion integral (6), the average distortion (or capacity loss) of a CSI-quantized MISO system can be lower bounded by

$$\widetilde{D}_{c\text{-Low}}(\mathbf{\Sigma}_h) = \left(\frac{(t-1)\gamma_t^{-t/(t-1)}\cdot\rho\cdot\beta_1(\rho, t, \mathbf{\Sigma}_h)}{\ln 2\cdot|\mathbf{\Sigma}_h|}\right)\cdot 2^{-B/(t-1)}. \tag{36}$$

Note that $\beta_1(\rho, t, \mathbf{\Sigma}_h)$ is a constant coefficient that only depends on the number of antennas t, channel correlation matrix $\mathbf{\Sigma}_h$, and system SNR ρ, and is given by

$$\beta_1(\rho, t, \mathbf{\Sigma}_h) = \left(\int_{\mathbf{v}:\mathbf{g}(\mathbf{v})=0}\left(\left(\mathbf{v}^{\mathrm{H}}\mathbf{\Sigma}_h^{-1}\mathbf{v}\right)^{-(t+1)} \cdot {}_2F_0\left(t+1, 1;;-\frac{\rho}{\mathbf{v}^{\mathrm{H}}\mathbf{\Sigma}_h^{-1}\mathbf{v}}\right)\right)^{(t-1)/t} d\mathbf{v}\right)^{t/(t-1)}, \tag{37}$$

with ${}_2F_0(;;)$ representing the generalized hypergeometric function. The optimal point density $\lambda^*(\mathbf{v})$ that achieves the minimal distortion is given by

$$\lambda^*(\mathbf{v}) = \beta_1(\rho, t, \mathbf{\Sigma}_h)^{-(t-1)/t} \cdot \left(\left(\mathbf{v}^{\mathrm{H}}\mathbf{\Sigma}_h^{-1}\mathbf{v}\right)^{-(t+1)}\cdot {}_2F_0\left(t+1, 1;;-\frac{\rho}{\mathbf{v}^{\mathrm{H}}\mathbf{\Sigma}_h^{-1}\mathbf{v}}\right)\right)^{(t-1)/t}. \tag{38}$$

As a special case, when the fading channel responses are spatially uncorrelated, that is, $\mathbf{\Sigma}_h = I_t$, the average system distortion has the following form:

$$\widetilde{D}_{c\text{-Low}} = \left(\frac{t-1}{\ln 2}\cdot {}_2F_0(t+1, 1;;-\rho)\cdot\rho\right)\cdot 2^{-B/(t-1)}, \tag{39}$$

with the optimal point density $\lambda^*(\mathbf{v})$ being a uniform distribution given by

$$\lambda^*(\mathbf{v}) = \gamma_t^{-1}, \quad \mathbf{v} \in \{\mathbf{v} \mid \mathbf{g}(\mathbf{v}) = 0\}. \tag{40}$$

Due to space limitations and to avoid overlap with our previous work, the derivations in this subsection have been condensed by skipping some manipulations used in obtaining the final expressions. Please refer to [19, 25] for more details.

5.3. Distortion analysis of quantizers with transformed codebook

First, according to the codebook transformation given by (29) as well as the optimal point density function of i.i.d. channels given by (40), the transformed point density function $\lambda_{c\text{-tr}}(\mathbf{v})$ from (12) has the following form:

$$\lambda_{c\text{-tr}}(\mathbf{v}) = \gamma_t^{-1}\cdot|\mathbf{\Sigma}|^{-1}\cdot\left(\mathbf{v}^{\mathrm{H}}\mathbf{\Sigma}^{-1}\mathbf{v}\right)^{-t}, \quad \mathbf{\Sigma} = \mathbf{G}\cdot\mathbf{G}^{\mathrm{H}}, \tag{41}$$

which is equivalent to the PDF of a unit-norm complex vector $\mathbf{x}/\|\mathbf{x}\|$ with $\mathbf{x}$ having complex Gaussian distribution $\mathbf{x}\sim\mathcal{N}_c(\mathbf{0}, \mathbf{\Sigma})$. It is evident that the transformed point density given by (41) does not match the optimal point density function $\lambda^*(\mathbf{v})$ given by (38) in the general case. However, for MISO systems with a large number of antennas and in high-SNR and low-SNR regimes, it can be shown that the optimal point density $\lambda^*(\mathbf{v})$ reduces to be the source distribution $p_{\mathbf{v}}(\mathbf{x})$ given by the following form:

$$\lim_{t\to\infty}\lambda^*(\mathbf{x}) = p_{\mathbf{v}}(\mathbf{x}) = \gamma_t^{-1}\cdot|\mathbf{\Sigma}_h|^{-1}\cdot\left(\mathbf{x}^{\mathrm{H}}\mathbf{\Sigma}_h^{-1}\mathbf{x}\right)^{-t}. \tag{42}$$

In this case, by choosing matrix $\mathbf{G}$ as $\mathbf{G} = \mathbf{U}\mathbf{\Lambda}^{1/2}$ with matrices $\mathbf{U}$ and $\mathbf{\Lambda}$ obtained from the eigen-value decomposition of the channel covariance matrix, that is, $\mathbf{\Sigma}_h = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\mathrm{H}}$, one can generate a transformed codebook $\mathcal{C}$ whose point density $\lambda_{c\text{-tr}}(\mathbf{v})$ is equal to the optimal point density function $\lambda^*(\mathbf{v})$. By utilizing this codebook transformation, there is no distortion loss caused by the point density mismatch (when t is large). However, the system still suffers from the suboptimal Voronoi shape due to the transformation.

By substituting the transformation given by (29) into (17), the inertial profile of the transformed codebook with suboptimal encoder $\mathcal{Q}_{\text{sub}}$ (or encoding process) is given by

$$\begin{aligned}\widetilde{m}_{c\text{-sub}}(\mathbf{v};\alpha) &= \frac{\gamma_t^{-1/(t-1)}\cdot\rho\alpha\cdot\left(\mathbf{v}^{\mathrm{H}}\mathbf{\Sigma}^{-1}\mathbf{v}\right)}{t\cdot\ln 2\cdot(1+\rho\alpha)}\cdot\mathbf{tr}\left(\left(I-\mathbf{v}\mathbf{v}^{\mathrm{H}}\right)\cdot\mathbf{\Sigma}\right)\\ &\geq \widetilde{m}_{c\text{-opt}}(\mathbf{v};\alpha),\end{aligned} \tag{43}$$

where $\widetilde{m}_{c\text{-opt}}(\mathbf{v};\alpha)$ is the optimal inertia profile given by (35). It is evident from (43) that except for unitary rotations of the i.i.d. codebook, any nontrivial transformation of codebook $\mathcal{C}_0$ will lead to mismatched Voronoi shapes and hence causes inertial profile loss. Therefore, a codebook transformation that makes the best compromise between the point density loss and the inertial profile loss is favored.

Finding the optimal codebook transformation $\mathbf{F}$ that minimizes the system distortion turns out to be a difficult problem. In this paper, instead of optimizing the overall distortion with respect to matrix $\mathbf{G}$, we provide a distortion analysis of MISO systems with transformed CSI-quantizers using codebooks generated by the heuristic choice $\mathbf{\Sigma}_h = \mathbf{G}\cdot\mathbf{G}^{\mathrm{H}}$ (or $\mathbf{G} = \mathbf{U}\mathbf{\Lambda}^{1/2}$). (Note that the codebook transformation is not unique. Any right unitary rotation $\mathbf{G}\cdot\mathbf{P}$ on matrix $\mathbf{G}$, with $\mathbf{P}\cdot\mathbf{P}^{\mathrm{H}} = I$, can generate another codebook transformation (or codebook) with the same performance.) To be specific, by substituting the transformed point density (41) and the transformed inertia profile (43) into the distortion integral given by (25), the corresponding upper and lower bounds of the average system distortion of a MISO CSI-quantizer with transformed codebook has the following forms:

$$\begin{aligned}\widetilde{D}_{c\text{-tr-Low}} = {} & \frac{(t-1)\cdot\left|\mathbf{\Sigma}_h\right|^{1/(t-1)}}{\ln 2\cdot t}\\ &\cdot E\left[\frac{\rho\cdot\left(\mathbf{h}^{\mathrm{H}}\mathbf{\Sigma}_h^{-1}\mathbf{h}\right)^{t/(t-1)}}{\left(1+\rho\cdot\|\mathbf{h}\|^2\right)\cdot\|\mathbf{h}\|^{2/(t-1)}}\right]\\ &\cdot 2^{-B/(t-1)},\end{aligned} \tag{44}$$

$$\begin{aligned}\widetilde{D}_{c\text{-tr-Upp}} = {} & \frac{\left|\mathbf{\Sigma}_h\right|^{1/(t-1)}}{\ln 2\cdot t}\\ &\cdot E\left[\frac{\rho\cdot\left(\mathbf{h}^{\mathrm{H}}\mathbf{\Sigma}_h^{-1}\mathbf{h}\right)^{(2t-1)/(t-1)}\cdot\left(t\cdot\|\mathbf{h}\|^2-\mathbf{h}^{\mathrm{H}}\mathbf{\Sigma}_h\mathbf{h}\right)}{\left(1+\rho\cdot\|\mathbf{h}\|^2\right)\cdot\|\mathbf{h}\|^{(4t-2)/(t-1)}}\right]\\ &\cdot 2^{-B/(t-1)}.\end{aligned} \tag{45}$$

5.4. *Performance comparison of CSI-quantizers with optimal and transformed codebooks*

In order to assess the suboptimality caused by codebook transformation, one would like to compare the system performance in terms of the average distortion of quantizers using transformed codebooks with that of the optimally designed codebooks. Interestingly, in high-SNR and low-SNR regimes with a large number transmit antennas t, the average system distortion of CSI quantizers with transformed codebook can be upper and lower bounded by some multiplicative factors of the distortion of optimal quantizers.

Proposition 2. *For* MISO *systems with a large number of transmit antennas, that is, $t\to\infty$, the following inequalities are satisfied:*

$$\begin{aligned}\widetilde{D}^{H\text{-dim},H\text{-SNR}}_{c\text{-}Low} &\overset{a}{=} \widetilde{D}^{H\text{-dim},H\text{-SNR}}_{c\text{-tr-}Low} \leq \widetilde{D}^{H\text{-dim},H\text{-SNR}}_{c\text{-tr}}\\ &\leq \widetilde{D}^{H\text{-dim},H\text{-SNR}}_{c\text{-tr-Upp}} \overset{b}{\leq} c_1\cdot\widetilde{D}^{H\text{-dim},H\text{-SNR}}_{c\text{-}Low,1},\end{aligned} \tag{46}$$

$$\begin{aligned}\widetilde{D}^{H\text{-dim},L\text{-SNR}}_{c\text{-}Low} &\overset{a}{=} \widetilde{D}^{H\text{-dim},L\text{-SNR}}_{c\text{-tr-}Low} \leq \widetilde{D}^{H\text{-dim},L\text{-SNR}}_{c\text{-tr}}\\ &\leq \widetilde{D}^{H\text{-dim},L\text{-SNR}}_{c\text{-tr-Upp}} \overset{b}{\leq} c_2\cdot\widetilde{D}^{H\text{-dim},L\text{-SNR}}_{c\text{-}Low,1},\end{aligned} \tag{47}$$

where the superscript "H-dim" represents the high-dimensional distortion (t large), and "H-SNR" (or "L-SNR") represents the distortion in high-SNR (or low-SNR) regimes. In (46), the constant coefficients c_1 and c_2 are given by the following form:

$$c_1 = \frac{\left(\dfrac{\delta(t-2)}{\lambda_{h,1}\cdot\lambda_{h,2}}-(t-1)(t-2)\displaystyle\sum_{i=1}^{t}\frac{\left(\ln\lambda_{h,i}\right)/\lambda_{h,i}^2}{\prod_{k\neq i}\left(1-\lambda_{h,k}/\lambda_{h,i}\right)}\right)}{c_1^{t/(t-1)}}, \tag{48}$$

$$c_2 = (t-1)\sum_{i=1}^{t}\frac{\left(\ln\lambda_{h,i}\right)/\lambda_{h,i}}{\prod_{k\neq i}\left(1-\lambda_{h,k}/\lambda_{h,i}\right)}. \tag{49}$$

Proof. See Appendix A. □

Note from Proposition 2 that constants c_1 and c_2 can be viewed as the upper bounds of the penalty paid for using a transformed codebook instead of the optimal design. Numerical examples of the loss factors c_1 and c_2 as well as corresponding discussions are provided in Section 6.

5.5. *Discussion on quantization resolutions*

The proposed system distortion bounds, as well as the corresponding observations made in previous sections, are all derived based on the high-resolution assumption. However, the feedback rate of the channel state information is always constrained to be low (a few bits per channel update) due to various practical considerations, for example, reduced transmission overhead, latency, and uplink spectral efficiency loss. Fortunately, as a well-known result in the conventional source coding, the high-rate distortion bounds agree well with the real simulation results when the resolution is larger than 3 bits per dimensions ($B/k_q \geq 3$) [26]. In this paper, due to "log-like" nature of the distortion function (system capacity loss), the distortion bounds converge even faster (about 1.5 bits per dimension), which is verified by simulation results in the following section. Therefore, the proposed distortion lower bounds are tight, and hence are able to characterize the system performance well even for CSI quantizers with small to moderate quantization rates.

6. NUMERICAL AND SIMULATION RESULTS

Some numerical experiments were conducted to get a better feel for the utility of the bounds. Figure 2 shows the system capacity loss due to the finite-rate quantization of the CSI versus feedback rate B for a 3×1 MISO system over correlated Rayleigh fading channels under different system SNRs, $\rho = -10$ and 20 dB, respectively. The spatially correlated channel is simulated by the correlation model in [27]: a linear antenna array with antenna spacing of half wavelength, that is, $D/\lambda = 0.5$, uniform angular spread in $[-30^\circ, 30^\circ]$ and angle of arrival $\phi = 0^\circ$. Simulation results of both the optimal designed codebook using the minimal mean-squared weighted inner product (MSwIP) criterion proposed in [10], as well as the suboptimal transformed codebook, are plotted. For comparison purposes, the distortion lower bound $\widetilde{D}_{c\text{-tr-Low}}$ given by (44) and the distortion upper bound $\widetilde{D}_{c\text{-tr-Upp}}$ given by (45) are also included in the plot. Note that the capacity losses (y-axis) are demonstrated using unit of bits per channel update. To get a relative sense, the channel capacity assuming perfect CSIT for the same 3×1 MISO system is 7.78 bits per channel update for an SNR of 20 dB, and 0.36 bits per channel update for an SNR of -10 dB. It can be observed from Figure 2 that the distortion lower bound $\widetilde{D}_{c\text{-tr-Low}}$ is tight and the performance of the CSI quantizer with transformed codebook is close to that of the optimal codebooks.

In order to see the effects of channel correlation on CSI quantizations, we plot in Figure 3 the normalized capacity losses (or capacity loss ratios) versus the adjacent antenna spacing D/λ of a 3×1 MISO system using both optimal CSI quantizers and quantizers with transformed codebooks. In the plot, the normalized capacity loss is defined to be the distortion ratio of correlated fading channels over i.i.d. fading channels. The reasons of choosing the capacity loss ratio as a major performance metric are twofold. First, intuitively uncorrelated Gaussian distribution has the maximum amount of "uncertainty" among all possible channel distributions. It imposes greater challenges in terms of quantizing the CSI than spatially correlated fading channels. Therefore, normalizing the system capacity loss w.r.t that of i.i.d. fading channels would make this ratio a positive number between 0 and 1, which characterizes the relative quality of the channel quantizer. Second, according to (36), (39), (44), and (45), the system distortion (in terms of capacity loss) of both optimal and transformed codebooks can be expressed as a weighted exponential function given by $D = c\cdot 2^{-B/(t-1)}$, where c is a constant coefficient that is independent of the quantization resolution B. Therefore, the proposed capacity loss ratio does not depend on the feedback rate, and only reflects the impact of the channel statistical distributions as well as the type of channel quantizers used.

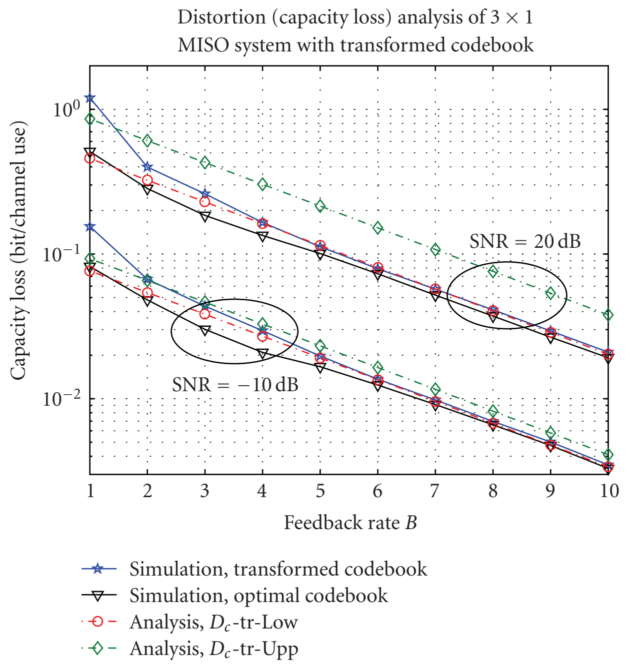

Figure 2: Capacity loss of a 3×1 correlated MISO system with normalized antenna spacing $d = D/\lambda = 0.5$ versus CSI feedback rate B using different channel quantization codebooks (optimal codebook versus transformed codebook).

In Figure 3, the capacity loss ratio is demonstrated with respect to the adjacent antenna spacing D/λ, which is directly related to the spatial correlation of the MISO channel response. When D/λ is sufficiently large, the channels can be viewed as i.i.d. Gaussian distributed, while $D/\lambda = 0$ means the channel is completely correlated (line of sight cases). In the plot, the average system signal to noise ratio is chosen in the low SNR regimes where $\rho = -10$ dB, and the quantization resolution is $B = 10$ bits per channel update. Simulation results in high SNR regimes, which are not shown here due to space limitations, show very similar results. Moreover, for comparison purpose, the ratio of the distortion bounds, that is, $\widetilde{D}_{c\text{-tr-Low}}(\mathbf{\Sigma}_h)/\widetilde{D}_{c\text{-tr-Low}}(I_t)$ and $\widetilde{D}_{c-\text{tr}-\text{Upp}}(\mathbf{\Sigma}_h)/\widetilde{D}_{c-\text{tr}-\text{Upp}}(I_t)$, is also included in the plot. One can first learn from Figure 3 that the system capacity loss increases as the adjacent antenna spacing increases (channel correlation decreases). It means higher feedback rate or finer resolution of the channel quantizer has to be used to maintain the same level of capacity losses, which is consistent with our earlier intuition. Moreover, it can be observed from the plot that the transformed codebook performs very close to the optimally designed codebook across all channel correlations. Finally, the plot also indicates that the analytical bounds agree well with the obtained simulation results. Therefore, we can analytically characterize the performance of beamforming systems using transformed channel quantizers without cumbersome numerical simulations.

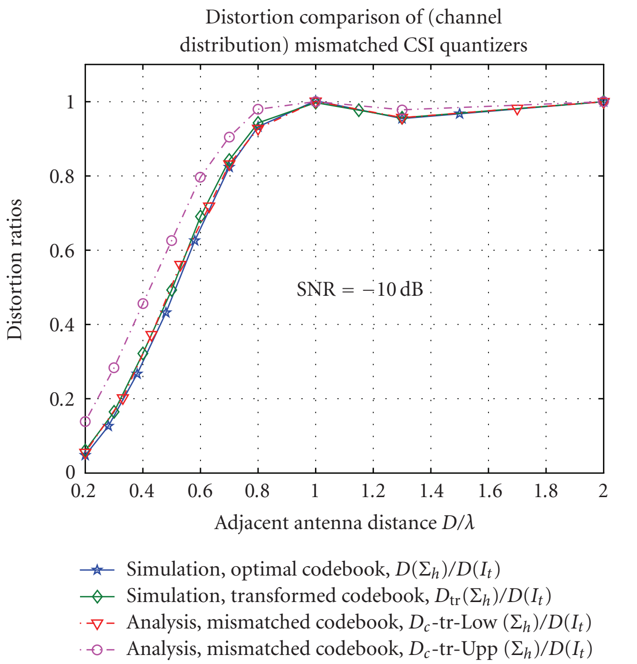

FIGURE 3: Normalized capacity loss (with respect to the capacity loss of uncorrelated fading channels) comparison of a 3×1 MISO transmit beamforming with optimal and transformed codebooks versus antenna spacing $d = D/\lambda$, in low-SNR regimes ($\rho = -10$ dB).

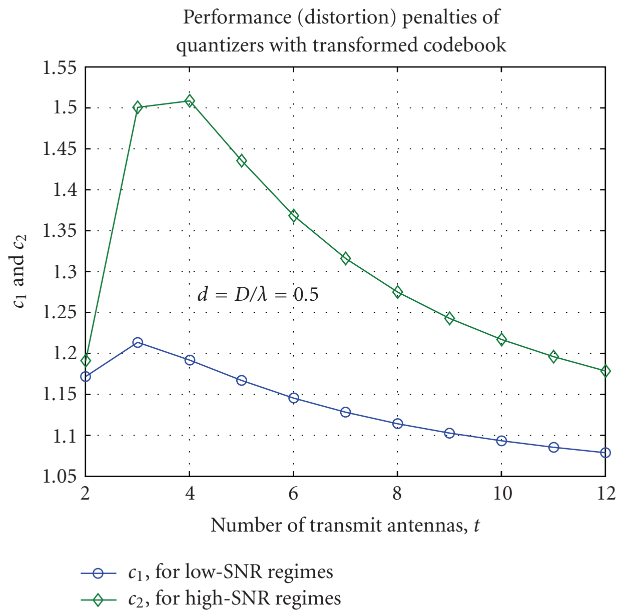

FIGURE 4: Demonstration of the distortion penalties of a MISO system using transformed codebooks over correlated fading channels with different number of transmit antennas of antenna spacing $d = D/\lambda = 0.5$.

In order to demonstrate the penalties of using transformed codebooks in high-SNR and low-SNR regimes, Figure 4 plots the constant coefficients c_1 and c_2 versus the number of transmit antennas t for correlated MISO channels with adjacent antenna spacing $D/\lambda = 0.5$. From the plot, it can be observed that (*the upper bound of*) the performance degradation caused by the transformed codebook is less than 10% in low-SNR regimes and 22% in high-SNR regimes for MISO systems with more than 10 transmit antennas. This means that the intuitive choice of $\mathbf{F}$ given in [6, 13] is a fairly good solution especially for cases when the channel covariance matrix has a relatively small condition number.

7. CONCLUSION

This paper extends the high-resolution quantization theory approach to study the effects of a finite-rate MISO CSI-quantizer employing a transformed codebook while transmitting over correlated fading channels. The contributions of this paper are twofold. First, analysis is provided for a generalized vector quantizer with a transformed codebook. Bounds on the average system distortion of this class of quantizers are provided. It exposes the effects of two kinds suboptimality, which include the suboptimal point density loss and the mismatched Voronoi shape. Second, we focused our attention on the application of the proposed general framework to provide the capacity analysis of a feedback-based MISO system over correlated fading channels using channel quantizers with transformed codebooks. In particular, upper and lower bounds on the channel capacity loss of MISO systems with transformed codebooks are provided and compared to that of the optimal quantizers. It was further proven that the average distortion of CSI quantizers with transformed codebooks can be upper and lower bounded by some multiplicative factors of the distortion of optimal quantizers. These factors were shown to be close to one for fading channels whose channel covariance matrix has small to moderate condition numbers. Numerical and simulation results were presented, which confirms the tightness of the theoretical distortion bounds.

APPENDIX

A. PROOF OF PROPOSITION 2

Proof. First, in the high-SNR regimes, distortion bounds $\widetilde{D}^{H\text{-dim},H\text{-SNR}}_{c\text{-tr-Low}}$ and $\widetilde{D}^{H\text{-dim},H\text{-SNR}}_{c\text{-tr-Upp}}$ can be represented as

$$\widetilde{D}^{H\text{-dim},H\text{-SNR}}_{c\text{-tr-Low}} = \left(\frac{(t-1)\cdot\left|\boldsymbol{\Sigma}_h\right|^{1/(t-1)}\cdot\beta_2}{\ln 2\cdot t}\right)\cdot 2^{-B/(t-1)} \approx \widetilde{D}^{H\text{-dim},H\text{-SNR}}_{c\text{-Low},1}, \tag{A.1}$$

$$\widetilde{D}^{H\text{-dim},L\text{-SNR}}_{c\text{-tr-Upp}} \leq \left(\frac{(t-1)\cdot\left|\boldsymbol{\Sigma}_h\right|^{1/(t-1)}\cdot\beta_3}{\ln 2\cdot t}\right)\cdot 2^{-B/(t-1)} = \left(\beta_3\cdot\beta_2^{-t/(t-1)}\right)\cdot\widetilde{D}^{H\text{-dim},H\text{-SNR}}_{c\text{-Low},1}, \tag{A.2}$$

where coefficients β_2 and β_3 can be expressed as the expected powers of the ratios of Gaussian quadratic variables, which are given by

$$\beta_2 = E\left[\frac{\mathbf{h}^{\mathrm{H}}\boldsymbol{\Sigma}_h^{-1}\mathbf{h}}{\mathbf{h}^{\mathrm{H}}\mathbf{h}}\right], \qquad \beta_3 = E\left[\left(\frac{\mathbf{h}^{\mathrm{H}}\boldsymbol{\Sigma}_h^{-1}\mathbf{h}}{\mathbf{h}^{\mathrm{H}}\mathbf{h}}\right)^2\right]. \tag{A.3}$$

The moments of ratios of random variables, including central quadratic forms in normal variables, were investigated in [28], and the results can be described by the following integrals:

$$E\left[\left(\frac{X}{Y}\right)^n\right] = \Gamma(n)^{-1}\int_0^\infty v^{n-1}M_{X,Y}^{(n)}(0,-v)dv, \tag{A.4}$$

where $M_{X,Y}(u,v)$ is the joint moment generating function (m.g.f.) of random variables X and Y, and $M_{X,Y}^{(n)}(0,-v)$ stands for $\partial^n M_{X,Y}(u,-v)/\partial v^n$ evaluated at $u=0$. Therefore, by setting $X = \mathbf{h}^{\mathrm{H}}\boldsymbol{\Sigma}_h^{-1}\mathbf{h}$ and $Y = \mathbf{h}^{\mathrm{H}}\mathbf{h}$, the joint m.g.f. of variables X and Y can be represented as

$$\begin{aligned} M_{X,Y}(u,v) &= \frac{1}{\mathbf{det}(I-(u\cdot I + v\cdot\boldsymbol{\Sigma}_h))} \\ &= \left(\prod_{k=1}^{t}(1-u-v\cdot\lambda_{h,k})\right)^{-1}. \end{aligned} \tag{A.5}$$

By substituting the joint m.g.f. given by (A.5) into the integral in (A.4) with $n=1$, the coefficient β_2, after some manipulations, has the following closed-form expression:

$$\beta_2 = (t-1)\sum_{i=1}^{t}\frac{(\ln\lambda_{h,i})/\lambda_{h,i}}{\prod_{k\neq i}(1-\lambda_{h,k}/\lambda_{h,i})}. \tag{A.6}$$

Finally, by substituting (A.6) into (A.1), equality (a) of (46) is proven. With similar reasoning, by substituting the joint m.g.f. (A.5) into (A.4) with $n=2$, coefficient β_3 is obtained. Correspondingly, a closed-form expression of the coefficient $c_1 = \beta_3\cdot\beta_2^{-t/(t-1)}$, given by (48), can also be obtained, and inequality (b) of (46) is proven.

Similarly, in low-SNR regimes, distortion bounds $\widetilde{D}_{c\text{-tr-Low}}^{H\text{-dim},L\text{-SNR}}$ and $\widetilde{D}_{c\text{-tr-Upp}}^{H\text{-dim},L\text{-SNR}}$ have the following forms:

$$\begin{aligned} \widetilde{D}_{c\text{-tr-Low}}^{H\text{-dim},L\text{-SNR}} &= \left(\frac{(t-1)\cdot|\boldsymbol{\Sigma}_h|^{1/(t-1)}\cdot\beta_4\cdot\rho}{\ln 2}\right)\cdot 2^{-B/(t-1)} \\ &= \beta_4\cdot\widetilde{D}_{c\text{-Low},1}^{H\text{-dim},L\text{-SNR}}, \\ \widetilde{D}_{c\text{-tr-Upp}}^{H\text{-dim},L\text{-SNR}} &\leq \left(\frac{(t-1)\cdot|\boldsymbol{\Sigma}_h|^{1/(t-1)}\cdot\beta_5\cdot\rho}{\ln 2}\right)\cdot 2^{-B/(t-1)} \\ &= \beta_5\cdot\widetilde{D}_{c\text{-Low},1}^{H\text{-dim},L\text{-SNR}}, \end{aligned} \tag{A.7}$$

where the coefficients β_4 and β_5 are given by

$$\beta_4 = E\left[\frac{\mathbf{h}^{\mathrm{H}}\boldsymbol{\Sigma}_h^{-1}\mathbf{h}}{t}\right], \qquad \beta_5 = E\left[\frac{(\mathbf{h}^{\mathrm{H}}\boldsymbol{\Sigma}_h^{-1}\mathbf{h})^2}{t\cdot\mathbf{h}^{\mathrm{H}}\mathbf{h}}\right]. \tag{A.8}$$

From (A.8), it is evident that $\beta_4 = 1$, and hence the equality (a) of (47), can be proven. Moreover, by extending the results of the moments of the quadratic forms provided in [28], the following expectation can be obtained after some manipulations:

$$E\left[\frac{X^2}{Y}\right] = \int_0^\infty \frac{\partial^2 M_{X,Y}(u,-v)}{\partial u^2}\bigg|_{u=0} dv. \tag{A.9}$$

Therefore, by setting $X = \mathbf{h}^{\mathrm{H}}\boldsymbol{\Sigma}_h^{-1}\mathbf{h}$ and $Y = \mathbf{h}^{\mathrm{H}}\mathbf{h}$, and substituting the joint m.g.f. given by (A.5) into the integral in (A.9), the coefficient β_5 can be obtained. It is equivalent to coefficient c_2 given by (49), and hence the inequality (b) of (47) can be proven. □

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