



# Complex behavior of COVID-19's mathematical model

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**Abstract** It is almost more than a year that earth has faced a severe worldwide problem called COVID-19. In December 2019, the origin of the epidemic was found in China. After that, this contagious virus was reported almost all over the world with different variants. Besides all the healthcare system attempts, quarantine, and vaccination, it is needed to study the dynamical behavior of this disease specifically. One of the practical tools that may help scientists analyze the dynamical behavior of epidemic disease is mathematical models. Accordingly, here, a novel mathematical system is introduced. Also, the complex behavior of this model is investigated considering different dynamical analyses. The results represent that some range of parameters may lead the model to chaotic behavior. Moreover, comparing the two same bifurcation diagrams with different initial conditions reveals that the model has multi-stability.

## 1 Introduction

It is almost more than a year that a widespread disease caused by COVID-19 has turned into an international concern. COVID-19 is one of the newest Coronaviruses' members, which can cause illness in both mammals and birds [1]. This virus, covered by lipids, is turned into a protein on the host cell with its messenger RNA (mRNA). Therefore, it can be categorized as a positive-sense single-stranded RNA virus [2]. In December 2019, the first suspected patient with an intense respiratory sign has been reported in Wuhan, China. Following that, it has been observed almost all over the world with different variants, which has resulted in a worldwide pandemic announcement. A wide range of symptoms, from moderate symptoms to severe illness, has been reported by infected people with COVID-19 [3]. After the incubation period, the most general symptoms are fever, tiredness, and dry cough [4]. However, other patients may suffer from other severe symptoms [5]. Although vaccination is started, the world still experiences a high level of daily confirmed cases. There are still vast unknown aspects of this contagious virus that reveals the high importance of theoretical studies, such as mathematical modeling [6, 7]. Mathematical models

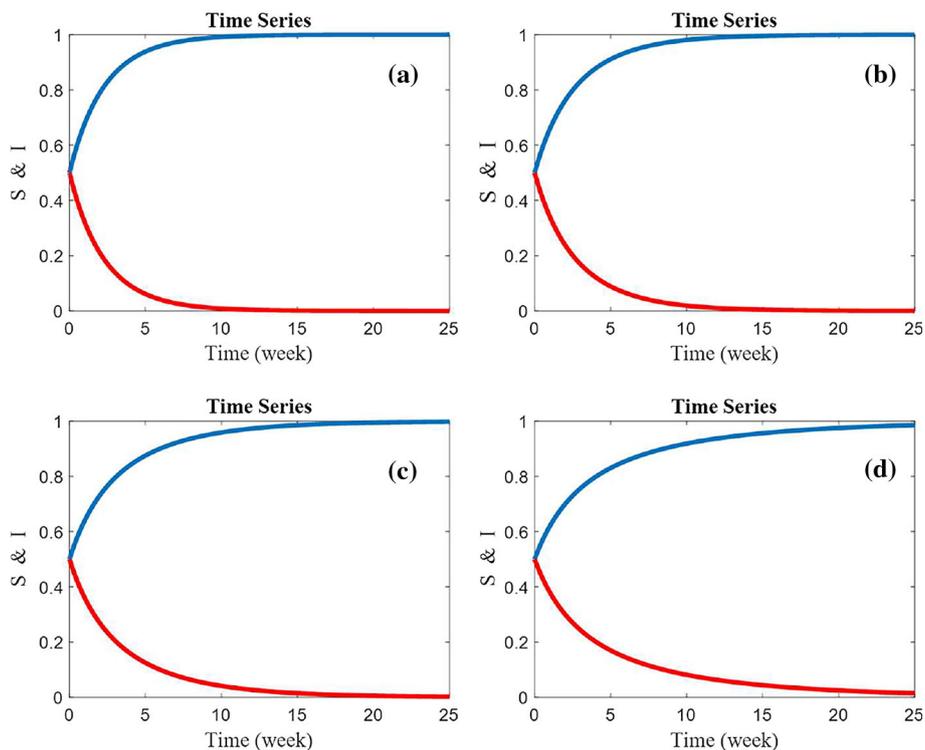
not only need to be accurate but also should be fast with low costs of simulation. A wide range of mathematical approaches can be utilized for computational modeling [8, 9]. A series of recent studies show that several computational models have been investigated the different dynamical aspects of the COVID-19 pandemic [10]. For instance, in February 2020, Wang et al. have used eight-dimensional differential equations to estimate the virus's future developments [11]. After that, Yang et al. have derived helpful information for forecasting the future growth of COVID-19 from 2003 SARS data using the artificial intelligence (AI) approach. Then, to find an effective cure, they have utilized both a 4-dimensional SEIR model and some updated epidemic COVID-19 data [12]. In early March 2020, Chen and Yu have used both a mathematical model and data of the first two months spread of COVID-19 to characterize China's coronavirus epidemic. They have claimed that, according to the results, coronavirus's dynamical behavior may be chaotic from the nonlinear point of view [13]. Corona Tracker Community Research Group has also used the real-time data query of this disease to predict coronavirus outbreak in the world [14]. In late March, Cano et al. have used simple stochastic simulations to emphasize the influence of prevention, such as social distancing. They have modeled the COVID-19 epidemic with a simple Markov chain model [15]. Control theory can also play an important role in analyzing and controlling the widespread disease [16]. Late in August, Rowthorn and Maciejowski used a SIR model and control theory as a new approach to control the

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**Fig. 1** The response of the Eq. 3 with a different value of the contact parameter when the other parameters are set to  $a = 0, c = 0.5, m = 1, q = 0.5$

and  $(S_0, I_0) = (0.5, 0.5)$  **a**  $\beta = 0.1$ , **b**  $\beta = 0.2$ , **c**  $\beta = 0.3$ , and **d**  $\beta = 0.4$ . The steady-state of the system experience a shorter transit time with a lower contact rate ( $\beta$ )

disease by quantifying the benefits of early intervention [17].

In this research, a two-dimensional modified version of the SEIR model is proposed, which may help scientists take one step closer to understanding the coronavirus’s complexity. A comprehensive dynamical analysis is done to investigate the different dynamical aspects of this model from the viewpoint of nonlinear dynamics. Furthermore, in line with previous studies, this model reveals interesting properties like chaos and multi-stability [18]. Multi-stability is considered one of the most important dynamical properties of the chaotic system, including many sub-branches such as bi-stability, extreme multi-stability, and mega-stability. These systems turn into a potential choice in chaos-based engineering applications.

In the rest of the paper, a brief description of the model is given in Sect. 2. Then, the existence of complex behavior is investigated in Sect. 3, which contains bifurcation analysis. Finally, Sect. 4 is the conclusion.

### 2 Model description

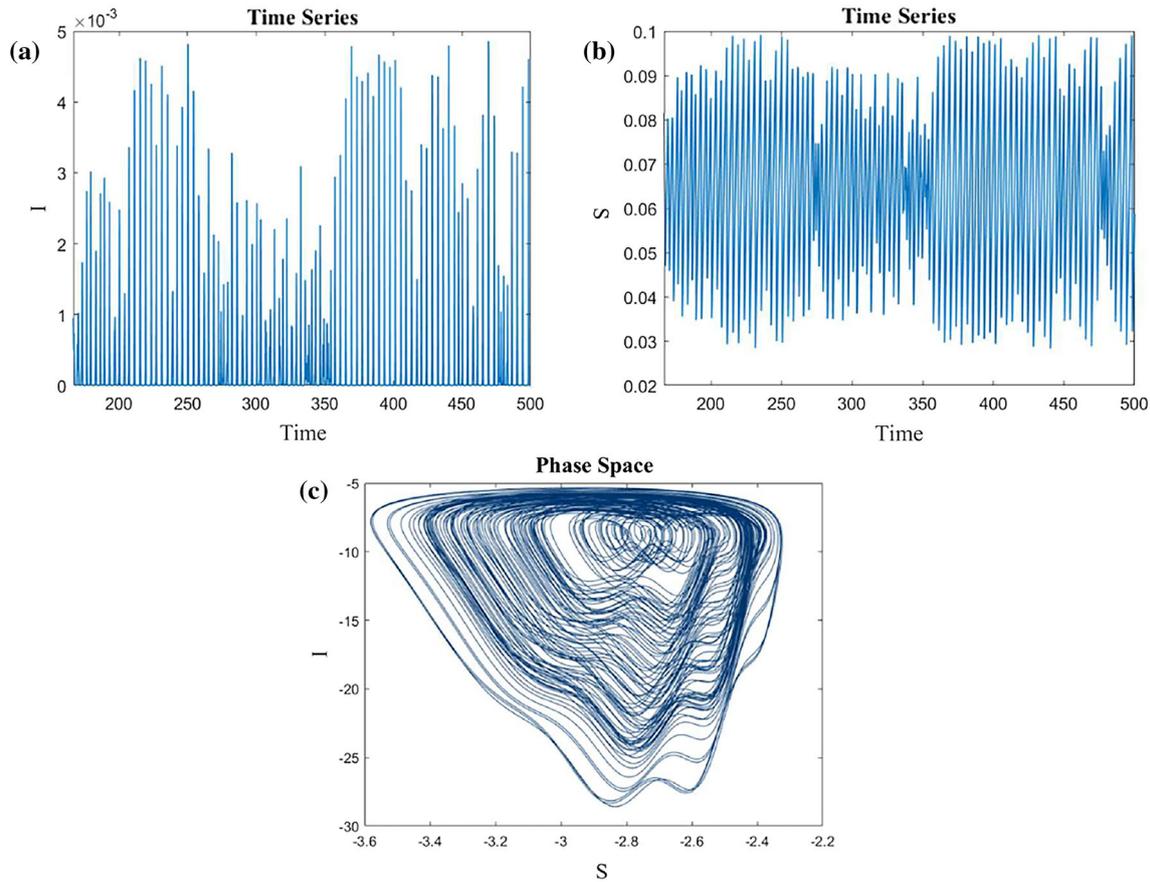
The SEIR model is one of the members of the computational compartmental models in epidemiology [19]. In 1927, Kermack and McKendrick proposed the first compartmental infection model, which divided the population into two susceptible and infected classes [20]. This model is a simple two-dimensional ordinary differential equation which is as follows:

$$\begin{aligned} \dot{S} &= -\beta SI \\ \dot{I} &= (\beta S - \alpha)I \end{aligned} \tag{1}$$

where  $\beta$  and  $\alpha$  show the infection and death rate. During the time, scientists developed some higher dimensional versions of the SI model to cover the new aspects of the epidemic disease [21]. The SEIR model is known as one of the modified SI models, which considers the effect of the disease’s incubation period. In this model, it has been assumed that the population is large enough [22]. Therefore, each people in the community can be a member of four dynamical classes, which are the Susceptible ( $S(t)$ ), Exposed ( $E(t)$ ), Infected ( $I(t)$ ), and Recovered ( $R(t)$ ).

$$\begin{aligned} \dot{S} &= -\beta SI + a - aS \\ \dot{E} &= \beta SI - (\alpha + a)E \\ \dot{I} &= \alpha E - (\gamma + a)I \\ \dot{R} &= \gamma I - aR \end{aligned} \tag{2}$$

where the  $\beta$ , and  $a$  are the contact and birth rate, respectively. The  $\alpha^{-1}$  and  $\gamma^{-1}$  show the mean latent and infection period. Also, the constant  $S + E + I + R = 1$ , which reveals the normalized number of the total population, reduces the equation’s dimension to three. Furthermore, theoretical and numerical results showed that a general relationship could be considered between the infected and exposed populations [23]. Therefore, the



**Fig. 2** **a** Time series corresponding to the  $S$  variable, **b** Time series corresponding to the  $I$  variable, and **c** Strange attractor of the system (1) with  $a = 0.02, b_0 = 1575, \delta = 0.2, e = 0.1, m = 0.3584, q = 35.8623$  and  $(S_0, I_0) = (0.5, 0.5)$

SEIR model’s simplified version is reduced to two-dimensional ordinary differential equations [24].

By considering the cure and death rate of the patients, the new two-dimensional SEIR system is reported here as follows:

$$\begin{aligned} \dot{S} &= -\beta SI + a - aS + cI \\ \dot{I} &= m\beta SI - qI \\ m &= \frac{\alpha}{a + \gamma}, q = a + \alpha + d - c \end{aligned} \tag{3}$$

where  $c$  and  $d$  are the cure and death rates. This model is a simple two-dimensional ordinary differential equation with two equilibria, which are  $\left(\frac{s^*}{I^*}\right) = \left(\frac{1}{0}\right)$  and  $\left(\frac{s^*}{I^*}\right) = \left(\frac{\frac{q}{m\beta}}{\frac{am\beta - da}{\beta(q - mc)}}\right)$ . Figure 1 shows the impact of the contact rate on the response time of the model.

To be more accurate, the seasonal fluctuation needs to be included in the model [24]. Hence, the time-varying contact rate is considered for the rest of the paper. Moreover, the scale time of the system is changed to per year. The final modified model is described as follows:

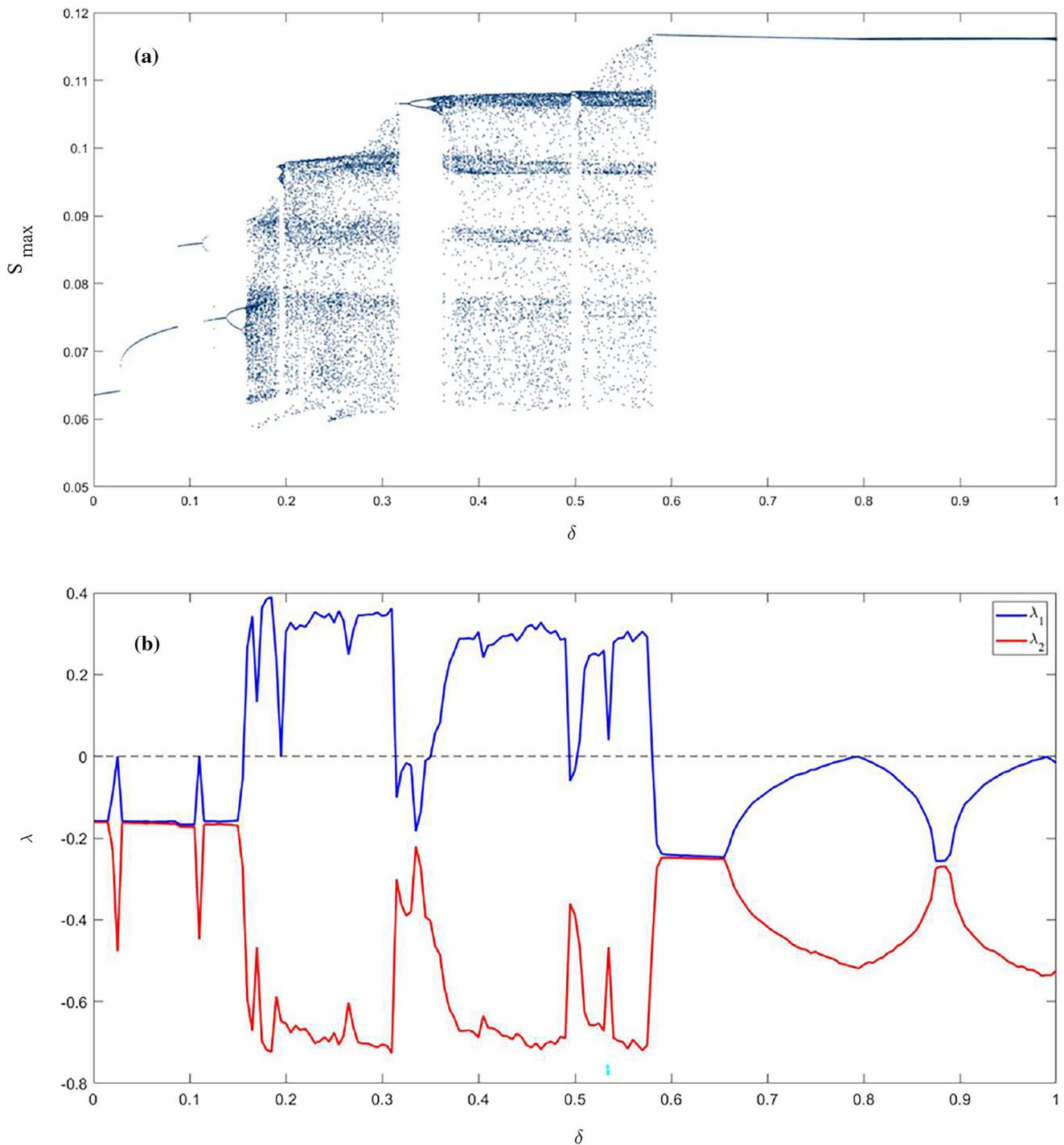
$$\dot{S} = -\beta(t)SI + a - aS + cI$$

$$\begin{aligned} \dot{I} &= m\beta(t)SI - qI \\ \beta(t) &= b_0(1 + \delta \sin(2\pi t)) \end{aligned} \tag{4}$$

where  $b_0$  is the base contact rate, and  $0 \leq \delta \leq 1$  indicates the weight of the seasonality. The time series and strange attractor corresponding to the model’s chaotic behavior after removing the transient part are plotted in Fig. 2.

### 3 Existence of chaos

The existence of the strange attractor in the previous part claims that the system (4) potentially can be chaotic. In order to investigate the nonlinear properties of the system (4) accurately, bifurcation analysis has been used. Some well-known patterns on the bifurcation diagram, such as period-doubling root to chaos, can be signs of chaos. Also, one positive Lyapunov exponents are the sufficient condition of chaos. Figure 3a represents the bifurcation diagram of the system (4) when the contact rate ( $\delta$ ) is smoothly changed as the bifurcation parameter. The corresponding Lyapunov exponents’ diagram also has been plotted in Fig. 3b.

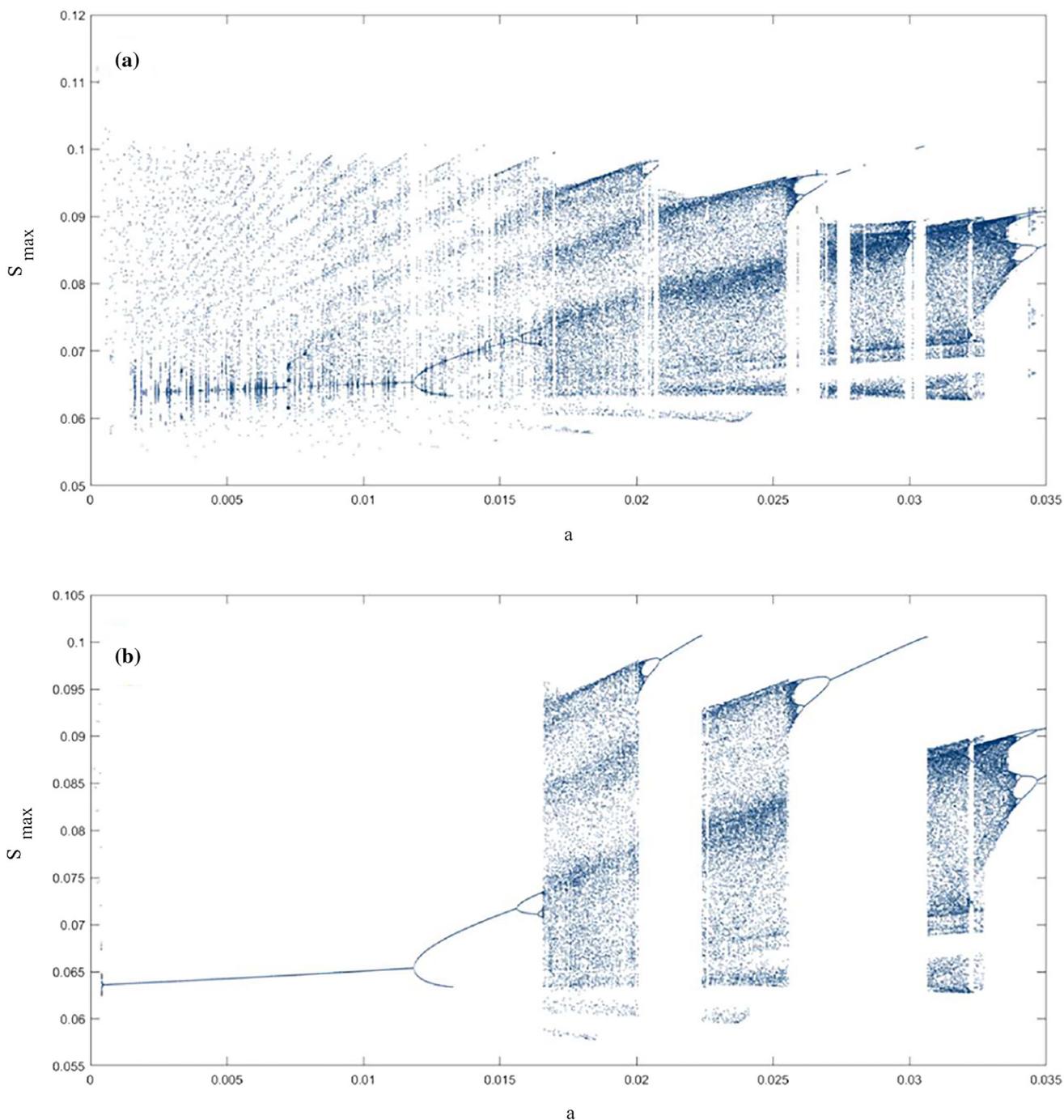


**Fig. 3** **a** Bifurcation diagram and **b** corresponding Lyapunov exponents of the system (4) according to smooth changes of contact rate ( $\delta$ ) when the other parameters are set to  $a = 0.02$ ,  $b_0 = 1575$ ,  $e = 0.1$ ,  $m = 0.3584$ ,  $q = 35.8623$  and  $(S_0, I_0) = (0.5, 0.5)$

As shown in Fig. 3, the system (4) can be chaotic in some range of the contact rate parameter, which confirms the system's rich dynamical potential (4).

Another basic parameter affecting the disease's spread differently in a different region is the birth rate. The bifurcation diagram of the system (4) has been explored concerning changing the birth rate as the bifurcation

parameter in Fig. 4 in two different approaches. Figure 4a represent the bifurcation diagram of the system (4) concerning the birth rate (a) with the fixed initial condition  $(S_0, I_0) = (0.5, 0.5)$  while Fig. 5b has been plotted with the help of the (upward/downward) continuation strategy. The different patterns between the



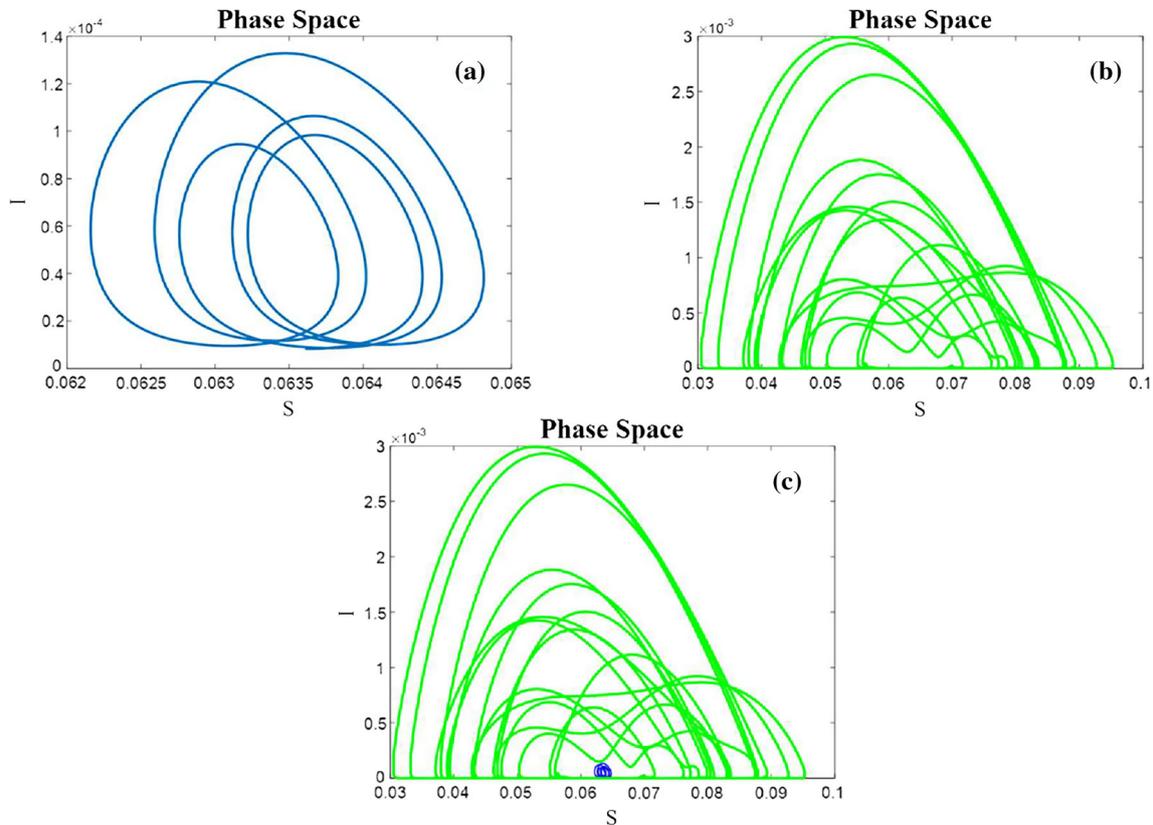
**Fig. 4** Bifurcation diagram of the system (4) concerning changing the birth rate **a** with fixed initial condition  $(S_0, I_0) = (0.5, 0.5)$  **b** with (upward/downward) continuation strategy

two bifurcations show that the system (4) has different attractors' coexistence.

As illustrated in Fig. 4, the system (4) is multistable and has coexisting different dynamical attractors. This point increases the importance of the initial condition in analyzing the dynamical behavior of the epidemic models. Figure 5 shows two different coexisting attractors in one frame to shed more light on the multi-stability and coexisting attractors.

### 4 Conclusion

COVID-19 is one of the biggest natural disasters in world history, which has affected the lives of millions of people all over the world during the last months. It is time for all the scientists with different majors, help each other to increase the public knowledge about this new virus. Computational modeling is a useful tool that can play an artificial safe lab to help scientists



**Fig. 5** Example of multistability and coexisting different attractors in same parameters ( $a = 0.02, b_0 = 1575, e = 0.1, m = 0.3584, q = 35.8623$ ) with different initial condi-

tions **a**  $(S_0, I_0) = (0.5, 0.5)$ , **b** random initial conditions, **c** plotting both **a** and **b** simultaneously to emphasize the difference between the attractors

investigate the different aspects of the epidemic. In this regard, a new mathematical model has been proposed in this paper by considering the effect of seasonality. This new model can be categorized as the simplified form of SEIR compartment models with the ability to produce complex behavior. Bifurcation and Lyapunov exponents analyses revealed that this model could be chaotic in the parameter's proper range. Also, multistability was another important feature of the proposed model.

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