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Contact interactions and strong resolvent convergence, a partly variational approach

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Abstract In Dell'Antonio (Eur. Phys. J. Plus 136:392, 2021) we considered several types of contact (zero range) interactions. Their Hamiltonians are limit, in strong resolvent topology, of a sequence of potentials with decreasing support. Here we review and improve these results and provide a new analysis Bose–Einstein condensation and of the Fermi sea.

1 Summary

In [1] we considered several aspects of contact (zero range) interactions. Here we give a review and improve on these results. There are two types (of Hamiltonians) of contact interactions, weak and strong. Both are self-adjoint extension of the free Hamiltonian restricted to functions that vanish in a neighbourhood of the *contact point* O.

Hamiltonian of strong contact or weak contact are limit *in strong resolvent sense* as $\epsilon \to 0$ of a sequence of Hamiltonians with potentials $V_\epsilon(|x_i - x_k|) = -|c|\frac{1}{\epsilon^k}V(\frac{|x_i-x_k|}{\epsilon})$ $\;V \in L^1(R^3) \cap L^2(R^3)$

where $k = 3$ for strong contact and $k = 2$ for weak contact.

We stress that both limits are in the strong resolvent sense. Divergences occur if one looks for quadratic form convergence.

With strong contact we describe the Efimov effect in low energy Physics and the high density Bose–Einstein condensate.

With weak contact we describe a three-particle system which satisfies the Gross–Pitaevskii equation and the low density Bose–Einstein condensate.

Remark Our approach to Bose–Einstein condensation is entirely different from the traditional one [2] that regards Bose–Einstein condensation as due to the interaction of a very large number N of particles through a two-body potential with range that becomes shorter as N increases.

In our description, the low density phase (Bogolyubov theory) is a collection of pairs of particles in which each particle of a pair is in weak contact with both particles of the other pair.

The high density phase is also a collection of four-particle systems in which there is a weak contact between two pairs of particles and each pair is made of particles in strong contact.

A collection of N such systems becomes a $4N$ body system due to Bose–Einstein statistics.

This collection of particles becomes a condensate because it is bound by a smooth external potential

2 Contact interactions, weak and strong

We consider three-dimensional systems. In two dimensions there is only one type of contact interaction.

In a two-body system strong contact is not defined, weak contact gives a zero energy resonance.

The Hamiltonians of weak or strong contact are self-adjoint extensions of the symmetric operator defined by the free Hamiltonian H_0 (the free Hamiltonian) restricted to functions that vanish in a neighbourhood of the contact point.

Contact interaction for systems of five or more particles can be reduced to separate contacts of subsystems.

We prove that the Hamiltonians of contact interaction are limits, *in strong resolvent sense* as $\epsilon \to 0$ of Hamiltonians that are the sum of a free Hamiltonian (a second-order elliptic operator) and a (negative) potential that scales as $V_\epsilon(|x_1 - x_2|) = -|c|\frac{1}{\epsilon^k}V(\frac{|x_1-x_2|}{\epsilon})$ $\;V \in L^1(R^3) \cap L^2(R^3)$ where $k = 2$ for weak contact and $k = 3$ for strong contact.

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3 Stable systems of contact interactions

It is convenient to assign weight $+2$ to each of the particles (the free Hamiltonian is a second-order elliptic operator), -2 to weak contact and -3 to strong contact (these are the powers of ϵ in the denominator of the approximate potentials).

With this convention contact interactions describe physical systems of *total weight zero* and such that the barycentre moves freely.

We call these systems *stable*.

Contact interactions cannot be described by standard methods of perturbation theory.

For a two-body system in R^6 strong contact is not defined; weak contact implies a zero energy resonance.

For a three-particle system, there are two stable irreducible configurations: strong separate contact of two particles with a third one and joint weak contact of three particles.

The first will lead to the Efimov effect in low energy Physics, the second will lead to a system which satisfies the Gross–Pitaevskii equation.

Also for a four-particle system there are two stable irreducible configurations: a system of two pairs of particles in which in which each has a weak contact with both members of the other pair (this will lead to the Bogolyubov system [3] i.e. a low density Bose–Einstein condensate).

The other irreducible configuration defines the high density B-E condensate, a system in which there is weak contact between two system each made of two particles in strong contact.

In both of these last two cases the Gross–Pitaevskii equation is the *variational equation* for the *Energy Functional*.

The Energy Functional has *the same expression* in both cases.

The difference is that in the case of weak contact the functional is non-negative and has at the bottom an infinite number critical points (ghosts), while in the other case the functional has infinitely many negative critical points (bound states) at energies that scale as $n^{-\frac{1}{2}}$.

In both cases the contact point O moves freely.

All other stable systems are compositions of these "fundamental ones" .

Of course if the particles are identical, one must take into account the appropriate statistics and contact interactions *may appear* to be an N-body problem for N arbitrary large.

Remark 1 Here we consider the case of bosons. The same results are obtained if one considers fermions and takes as free Hamiltonian the Pauli operator for spinor-valued functions.

Remark 2 The "stability" of contact interactions depends on the dimension of space.

In two dimensions the Hamiltonian is still a second-order operator and there is only one type of contact interaction.

In two dimension this contact *does not* produce resonances and any open chain of particles in contact is a stable system [4].

Remark 3 In case of strong joint contact one may seek to obtain by quadratic forms convergence. It is known that this procedure leads do divergences and one seeks ti obtain a finite results by *renormalisation*.

Indeed the outcome is a quadratic forms *and not a self-adjoint operator*. We shall comment on this. later.

4 The "Krein" maps

As in [1] we consider stable systems and introduce, as intermediate step, maps *to a space of more singular functions*.

There are two types of maps, one for strong contact and one for weak contact.

Both are *mixing* (they are not diagonal in the particle structure) and *fractioning* (the new space is made of more singular functions) and *the interactions are weaker*.

Both maps are *non-invertible*.

The mixing and fractioning property strongly suggest the use of Gamma convergence [5] a variational method introduced by E. de Giorgi and of common use in the theory of Composite Media.,

We call these maps *Krein maps* $\mathcal{K}$ and for historical reasons [6] we call $\mathcal{M}$ (Minlos space) the resulting spaces.

One can consider the Krein maps as *re-arrangement maps*.

Remark The introduction, as intermediate step, of a space of more singular functions goes back at least to Friederichs [7] in his analysis of self-adjoint extension the Laplacian on $(0, +\infty)$

5 The Krein maps: case I–IV

(i) The Krein map for separate strong contact of one particle with two particle Strong contact is formally represented by a delta "function".

The delta function is a quadratic forms *and not a self-adjoint operator*.

In this case the Krein map is chosen to act on the two delta functions as $\delta(x_i - x_k) \to H_0^{-\frac{1}{2}}\delta(x_i - x_k)H_0^{-\frac{1}{2}}$ where H_0 is the three-particle free Hamiltonian. This map is positivity preserving.,

Under the Krein map the free three-particle Hamiltonian H_0 is mapped into the operator $H_0^{-\frac{1}{4}}H_0H_0^{-\frac{1}{4}} = H_0^{\frac{1}{2}}$.

Notice that in this case the "smoothing action" is chosen to be stronger on the potential part that on the kinetic part. This is due to the fact that a delta function is too singular as perturbation of a second-order elliptic operator.

The map is *fractioning* (the target space is a space of more singular functions) and *mixing* (it does not preserve the particle structure).

The resulting two operators (kinetic energy and potential term) have in $\mathcal{M}$ a (polar) singularity at contact that has the same weight for both but opposite sign.

In $\mathcal{M}$ the system is represented therefore [8, 9] by an ordered one parameter family of self-adjoint operators.

Each of them has an infinite number of bound states with energies that scale as $c\frac{1}{n}$.

Remark It can be verified that the corresponding semiclassical system is Yukawa interaction.

(ii) The weak Krein map for joint weak contact of three particles.

We have seen that weak contact of two particles leads to a zero energy resonance. Joint weak contact of three particles requires more attention.

The weak Krein map $\mathcal{K}$ is also here a map from $L^2(R^9)$ to a space $\mathcal{M}$ of more singular functions; also this map is positivity preserving.

The map is different from that used in the case of strong contact. In the case of weak contact the map *acts in the same way* on the kinetic energy and on the potential.

Under the Krein map the quadratic potential W is mapped in $H_0^{-\frac{1}{4}}WH_0^{-\frac{1}{4}}$ and the operator H_0 into the operator $H_0^{-\frac{1}{4}}H_0H_0^{-\frac{1}{4}} = H_0^{\frac{1}{2}}$.

The map is also here *fractioning* (the target space is a space of more singular functions) and *mixing* (the map does not preserve the particle structure).

In $\mathcal{M}$ the free part is represented by three first-order differential operators and the interaction by a chain of three two-body potentials that have at a singularity of order 1 *in the difference of the coordinates*.

This system s homogeneous.

The semiclassical limit of this system is the three-body Coulomb system.

(iii) The weak Krein map for a system of two pair of particles in which each particle of a pair has a weak contact interaction with both particles of the other pair . [1].

The weak Krein map $\mathcal{K}$ is here a map from $L^2(R^{12})$ to a space $\mathcal{M}$ of more singular functions; the map is positivity preserving.

Also in the case the map *acts in the same way* on the kinetic energy and on the potential.

The Krein maps the quadratic potential W' into $H_0^{-\frac{1}{4}}W'H_0^{-\frac{1}{4}}$ and the operator H_0 into the operator $H_0^{-\frac{1}{4}}H_0H_0^{-\frac{1}{4}} = H_0^{\frac{1}{2}}$.

The map is also here *fractioning* (the target space is a space of more singular functions) and *mixing* (the map does not preserve the particle structure).

In $\mathcal{M}$ the free part is represented by a chain of four consecutive first-order differential operators and the interaction by a chain of four two-body potentials that have at the origin a singularity of order one.

Remark The semiclassical limit of this system is the Boltzmann equation.

(iv) The strong Krein map for a system in which there is a weak contact two pairs of particles in strong contact.

Also in this case the Krein map is chosen to act on the two delta functions as $W \to H_0^{-\frac{1}{2}}\delta H_0^{-\frac{1}{2}}$ but on the weak interaction potential W as $H_0^{-\frac{1}{4}}WH_0^{-\frac{1}{4}}$

Notice that in $L^2(R^{12})$ the Hamiltonian of the system has an additive form but it is a weak contact since the map is mixing this is no longer true in $\mathcal{M}$.

In $\mathcal{M}$ the system is still represented as in case I by a one-parameter family of self-adjoint operators with an infinite number of bound states but now there is a resonance at zero energy.

6 Gamma convergence

The fact that the Krein map is mixing and fractioning to a space of more singular functions and the presence in $\mathcal{M}$ of an ordered sequence of self-adjoint operators strongly suggests the use of Gamma Convergence [5], a variational method introduced by de Giorgi and of common usage in the theory of composite materials.

Since the Krein map changes the metric topology, in $L^2(R^N)$ the operators in $\mathcal{M}$ are turned into quadratic forms.

Gamma convergence selects the infimum of this ordered sequence of quadratic forms.

The Gamma limit F(y) of a sequence of quadratic form $F_n(y)$ is the quadratic forms is defined by the relations

$$\forall y \in Y, y_n \to y;\ F(y) = \lim \inf F(y_n) \quad \forall x \in Y_n \;\; \forall \{x_n\} : F(x) \leq \text{limsup}_n F_n(x_n) \tag{1}$$

The first condition implies that $F(y)$ is a common lower bound for the function F_n, the second implies that this lower bound is optimal.

The condition for the existence of the Gamma limit is that the sequence be contained in a compact set for the (Sobolev) topology of Y (so that a Palais-Smale convergent sequence exists).

In our case the topology of Y is the Frechet topology defined by Sobolev semi-norms.

In all systems we considered the condition for Gamma convergence are met (there are no zero energy resonances).. By a theorem of Kato [10] (strict convexity) the limit form admits strong closure and defines therefore a self-adjoint operator.

Gamma convergence of the forms implies strong resolvent convergence [5]. There is *no converge in the operator topology*

7 Gamma convergence for the Krein maps

We study now the outcome of Gamma convergence of system I to IV

In systems II and III there is in $\mathcal{M}$ a unique self-adjoint operator (different for the two cases).

This corresponds in $L^2(R^N)$ to a weakly closed quadratic form.

Since there is no resonance this form can be closed strongly; Gamma convergence "lifts" this quadratic form to be a self-adjoint operator in $L^2(R^{3N})$, $N = 3, 4$

This operator has in case II an infinite number of bound states that scale as $\frac{c}{\log n}$ and in case
III an infinite number of zero energy bound states.

In system I and IV on the contrary there is in $\mathcal{M}$ a one parameter ordered family of self-adjoint operators each with an infinite number of eigenvalues that scale as $\frac{c}{n}$.

In case *IV* there is also a zero energy resonance but since this is not the lowest energy it does not play a role in the proof of Gamma convergence i.e. the existence of a Palais-Smale converging subsequence.

Due to the difference in metric topology between $\mathcal{M}$ and $L^2(R^N)$, $N = 9 or N = 12$ in case I this operator has eigenvalues that scale with rate $\frac{c}{\sqrt{n}}$

8 Resolvent convergence of Hamiltonian for potentials with support converging to a point

For the study of strong contact interaction perturbation theory fails: it leads to divergences that must be *renormalized*.

In our approach through Gamma convergence we introduced, as intermediate step, a map to a space $\mathcal{M}$ of more singular functions.

In $\mathcal{M}$ the system is described by self-adjoint operators.

In $L^2(R^N)$ due to the difference in metric topology these self-adjoint operators are turned to weakly closed quadratic (in cases I and IV a sequence of well-ordered quadratic forms).

Gamma convergence selects *the infimum of the quadratic forms*.

This form can be closed strongly and provides a self-adjoint operator, the Hamiltonian of the system.

By Gamma convergence *the resolvents* of these Hamiltonians converge to *the resolvent* of the Hamiltonian associated to strong or weak contact.

One has also resolvent convergence of a sequence of approximate Hamiltonians H_ϵ.

In all cases we consider, when $\epsilon > 0$, however small, the system is represented in $L^2(R^3)$ by self-adjoint operator H_ϵ with two-body potentials $V_\epsilon(|x_1 - x_2|) = -|c|\frac{1}{\epsilon^3}V(\frac{|x_1-x_2|}{\epsilon})$ $\;V \in L^1(R^3) \cap L^2(R^3)$ where $k = 2$ for weak contact and $k = 3$ for strong contact.

There is in general no convergence in the strong operator topology.

The limit resolvent satisfies resolvent identities *and is therefore the resolvent of a limit operator*
$\mathcal{H}$ which is different in the four cases that we have discussed.

It would very interesting to compare the self-adjoint operator $\mathcal{H}$ with the "operator" obtained through *renormalization* i.e. to prove that "renormalization" is equivalent to cancellation of the difference between the (infinite) quadratic form provided by perturbation theory and the self-adjoint operator $\mathcal{H}$ obtained by Gamma convergence..

We have used the double commas since the "operator" which is obtained in perturbation theory *after renormalisation* is in general only a quadratic form and it is difficult to see whether it satisfies the continuity properties needed to be a self-adjoint operator.

9 Connection with physics

In the case of separate strong contact of two particles with a third one case the resulting Hamiltonian has a family of bound states with energies that scale with rate $cn^{-\frac{1}{3}}$. This is the Efimov effect in low energy Physics.

In the case of mutual weak contact of three particles the family scales as $\frac{1}{\log n}$. We shall see that is also the distribution of energy levels of conduction electron is an infinite crystal.

A system of three particles in mutual weak contact is described by the Gross–Pitaevskii equation, a cubic focusing equation.

$$i\partial_t \phi(x,t) = -\Delta\phi(x,t) - c\phi^3(x,t) \quad x \in R^3 \quad c > o \tag{2}$$

describes the motion of a system of three non-relativistic bosons in mutual weak contact. The *Gross–Pitaevskii Energy (Functional)*

$$E(\phi) = \int \left[\nabla\phi(x)\right|^2 - c\rho^2(x); \quad \rho(x) = |\phi(x)|^2 \tag{3}$$

is *the variational functional* for the Gross–Pitaevskii equation,

The same functional represents *the Energy* for two *different* systems.

The first one consist is of two pairs of particles in which there is a weak contact of each particle of a pair with both members of other pair.

The second one is also the energy of a four particles system but now a particle has separate strong contact with two particle *and weak contact with a fourth particle*.

The collection of system of the first type is also called Bogolyubov system (recall that the particles are identical, either Bosons or Fermions)

This first system has positive energy and an infinite number of zero energy bound states (called often *ghosts*).

Since the contact is weak (and can be easily broken) this Hamiltonian can be described using creation and annihilation operators.

The collection of the infinitely many states can be used as Fock space.

Also the second type there is a particle in weak contact but there is a *strong separate contact of a particle with a pair of particles*.

This system has an infinite number of bound states that scale as $\frac{c}{\sqrt{n}}$.

In this system the infinitely many zero energy states are missing and therefore for this system *there is no Bogolyubov system and no Fock description*.

This difference is relevant if one considers N-particle systems.

Remark 1 Strong and weak contact are limits *in strong resolvent sense* when $\epsilon \to 0$ of interactions through potentials $V^\epsilon(|x_i - x_j|) = \frac{1}{\epsilon^2} V(\frac{|x_i - x_j|}{\epsilon}) \; x_i \in R^3$ where $V \in L^1(R^3) \cup L^2(R^3)$.

Recall that resolvent convergence implies convergence of the spectrum and of the Wave Operators *but does not imply operator convergence*.

Remark 2 We recall that the traditional way (see e.g. [9]) of describing Bose–Einstein condensation is to consider a many-body system, with a two-body potential that scales as $V_{N,\epsilon}(x_i - x_j) = N^{-2} V(\frac{x_I - x_j}{\epsilon})$ where N is the number of particles in the system (*so that the interaction between two particles depends on the total number of particles*).

One considers the limit $\epsilon \to 0, \quad N \to \infty$. Bose–Einstein condensation is discussed within this approach which is entirely different from ours.

10 Bogolyubov theory

The condensate considered in Bogolyubov's theory is a low density collection of *quasi-particles*

Correctly Bogolyubov indicates the "particles" with the name "quasiparticles" because the theory considers the case of closed chains of four "particles" in a *in weak contact, case III above*.

The Gross–Pitaevskii Functional describes the energy of the system.

We consider collections of clusters of *two pairs of identical particles, either Bosons or Fermions*; in the case of Fermions they have opposite spin.

Since the particles are very loosely bound one can easily "break the bond" and "free the particles".

This suggest a Fock space formulations in which one introduces *creation* and *annihilation* operators; one can write in this formalism the interaction Hamiltonian.

If one considers a many particle system, in order to confine the system one can make use of an external potential that can be chosen of class $L^1(R^3) \cap L^2(R^3)$.

This confining potential does not alter the structure of the condensate since it can be chosen to vanish on the support of strong contact (a set of measure zero).

By means of these forces the gas of identical pair of wave functions in weak contact is confined i.e. it becomes a *condensate*.

The confining potential can be a function that grows sufficiently fast at infinity but also to a sufficiently intense electromagnetic field if the particle are charged.

Since Gamma convergence provides an infimum of a set of quadratic forms the same analysis we have made for a scalar potential can be repeated for the magnetic Hamiltonian H_M *defined by minimal coupling*.

Recall that both weak contact interactions and strong contact interactions are limit *in strong resolvent sense* of interactions through a sequence of potentials that have support of radius decreasing to zero.

The approach we take is variational; we have resolvent convergence but *there is no operator convergence and no estimate of the rate of convergence.*

Remark 1 Bogoliubov was led to the formulation of his theory from his work on superfluidity. In the Ginsburg-Landau theory of super-fluidity the system is made of (Cooper) pairs of particles which are in a very weak *repulsive contact* Since the repulsive force has zero range, *the energy functional is the same as the Gross–Pitaevskii energy functional*. Superfluidity is due to the breaking of this very weak attraction.. This process is often called *B.C.S. to B.E.C. transition*.

Remark 2 We have described Bogolyubov's theory *for weak contact interactions*.

The condensate depends of the parameter that indicates the strength of the interaction.

In the case of approximating regular potentials V_ϵ one should consider that the convergence holds *in strong resolvent sense* (and not in the operator topology).

One must therefore use (or develop) perturbation theory *for the resolvents*.

Remark 3 A Fermi-Dirac condensate.

The low density Bose–Einstein condensate is a gas of two pairs of identical wave functions in which particle of each pair is in weak contact with both particles of the other pair.

If one identifies each pair with a density matrix one has a pair of density matrices in weak contact.

Fermions with opposite spin orientations can have a weak contact and the resulting system has the properties of a density matrix; therefore it is possible to have a Fermi-Dirac condensate.

11 High density Bose–Einstein condensate

Consider now the case of weak contact between two pairs of particles which are in strong contact.

Also in this case the interaction is *too strong to allow the use of perturbation theory*. The traditional use of perturbation theory leads to divergences that must be renormalised.

This system can be studied using by Gamma convergence.

In the limit of contact interaction the system is still described by clusters of four particles but since there are now two strong contacts Bogolyubov's analysis is not correct as it stands.

One has still a Bose–Einstein condensate but it is now a *high density condensate*

It is still the limit of a system which can be described through regular potentials; also here the limit is taken in the strong resolvent sense.

Again a strong regular external potential can be used to constrain the system.

Remark 1 One can still introduce creation and annihilation operators but now the creation is of a pair of particles which are strongly bound (a quark, see Appendix 2).

We have noticed that two particles in strong contact cannot exist as an isolated system but two such pairs in weak contact are an element of a condensate.

If the condensate is made of spin $\frac{1}{2}$ fermions a pair of them in weak contact can represent a *current*.

For finite values of the parameter ϵ (the range of the potential') the "marking spots" have a finite width and one can determine (e.g. with a de Finetti analysis) the probability that the interaction between quarks takes place and therefore the strength of the interaction.

The energy of the system is still described by the Gross–Piraiewski energy functional

$$E(\phi) = \int (|\nabla\phi(x)|^2 - c\rho^2(x))d^3x, \qquad \rho(x) = |\phi(x)|^2 \tag{4}$$

but now this functional has a different interpretation, i.e. it is the energy functional for the weak contact interaction of two quarks (two particles in strong contact).

This energy functional differs from the functional for the Bogolyubov system because it has infinitely many critical point (bound states).

Remark 2 The Krein map can be extended to the relativistic case in a way that is compatible with the Fock space formalism. Therefore it may provide strong resolvent convergence for contact interaction in Quantum Field Theory.

Here there strong contacts leads to creation and destruction of a particle. Notice e.g. in the Φ_4 theory there are two strong and one weak contact as in the high density Bose–Einstein gas.

12 Contact interactions in solid state physics: energy spectrum of conduction electrons in a crystal

With this formalism we can also study the case [1] of weak joint contact interaction for three (conduction) electrons at the vertices of a Bloch lattice of a crystal.

In this case the weak contact of three electrons *has a different origin*. *It is not due* to interaction between electrons (electrons repel each other) but rather to the joint action at the vertex of a

Bloch cell on each of the three electrons by the nearby three nuclei in the lattice.

Electron are Fermi particles with spin $\frac{1}{2}$ that satisfy the Pauli equation.

Conduction electrons are the “outer-shell” electrons of an atom.

In a three dimensional crystal the nuclei are placed in the centre of *Bloch cells*.

At the corners of a Bloch cell each electron the attraction of “his” atom and also of the two “nearby” atoms.

This is a joint three-body weak contact interaction.

We considered joint weak contact for particles that are bosons and satisfy the Schrödinger equation. The same results are obtained in [1] for the Pauli equation and particles of spin $\frac{1}{2}$ (the non-relativistic Schrödinger operator is the “square” of the Pauli operator).

We can introduce also here the weak Krein map to a space $\mathcal{M}$ of more singular spin-valued functions, Also here this map is fractioning and mixing.

As in [1] the free Hamiltonian and the potential have in $\mathcal{M}$ the same degree of singularity as the kinetic term.

Therefore their sum represents in $\mathcal{M}$ a one parameter family a self-adjoint operator with an infinite number of bound states.

In $L^2(R^9) \times S$ (S is the spin space) this corresponds to quadratic forms bonded form below.

Gamma convergence selects the infimum; this form can be closed strongly and provide the Hamiltonian H of our system of three conduction electrons.

Its eigenvalues scale as $c\frac{1}{\log n}, \quad c < 0$.

This is therefore the spectrum of conduction electrons in a crystal.

The collection of energy levels that bound electrons can occupy is *the Fermi sea*

It is *a discrete set* but the eigenvalues near the upper part of the spectrum are so closely spaced that they *seem* to form a continuum (the Fermi sea).

Since electrons satisfies Fermi statistics and have spin $\frac{1}{2}$ no more than two of them can occupy the same state; the conduction electrons in an infinite crystal occupy therefore the entire (discrete) spectrum which has (multiplicity two and) negative eigenvalues that scale as $\frac{c}{\log n}$

We point out that, together with the Gibbs factor, this implies quantisation of resistivity at very low temperatures.

If the material is “doped” (it has many impurities) conductivity decreases because the number of bound states available increases and the highest occupied level is far from zero.

Remark Due to the scaling properties of the weak contact, the interaction of the conduction electrons *with the entire infinite lattice of nuclei* scales as Coulomb interaction.

13 Reduction to the graph

Since the contact interactions occur only at the vertices of a graph, the analysis of the energy spectrum of the Fermi sea in a crystal can be reduced, as done in [1], to an analysis of a self-adjoint operator on the graph defined by the boundary of the Bloch cells.

The interaction takes now place at the vertices of the graph and results is a change of direction (*but not of absolute value of the velocity*) at the vertices, with equal probability for the three possible directions.

This provides a stationary process with generator the Hamiltonian on the graph.

The solution for this first-order differential operator on spin space can be continued by unique continuation to the interior of the cell and give the Hamiltonian for the electrons in the crystal.

Remark 1 One can modify the boundary conditions at the vertices to obtain other diffusions on the graph and other spectra.

One can also consider random three-body contact at the vertices and study the corresponding stationary processes.

The continuation is a process that is continuous (as spin -valued function); it may have a spin flip at the centre of the cells. The $\frac{c}{\log n}$, $c < 0$ structure of the energy spectrum of the eigenstates makes this structure time independent (*the phases of the eigenfunction "move at the same speed"*.

As remarked in [1] the eigenfunctions of the deepest bound states are supported near the graph. This is confirmed by measurements performed with the aid of an electron microscope (the electron microscope "registers" only waves of sufficiently small wave length; it measures therefore only electron in "deep" bound states.

The measurement indicates that the density of those conduction electrons that are "more bound" is negligible outside a small neighbourhood of the graph and has a maximum at the vertex.

On the contrary eigenfunction near the top of the spectrum are almost plane waves (Bloch waves); the spectrum is very dense and may appear as continuous (but it is still a point spectrum).

The Fermi sea is the collection of the electrons in the bound states.

It is worth recalling that in Quantum Mechanics electrons are "identical particles" that satisfy the Fermi-Dirac statistics. No two of them can occupy the same state. The "Fermi sea" is the collection of electrons that are in a bound state.

In view of the periodic structure of the nuclei there is a collection of "Fermi seas" and they may overlap.

In highly "doped" crystals there are many more different bound state of the conduction electrons and the least bound electron may have still a relevant binding energy.

Therefore conductivity decreases.

14 Quantization of linear response

The experiments to measure superconductivity and superfluidity are performed at equilibrium.

The Hamiltonian has eigenvalues that scale as $-\frac{1}{\log n}$.

At finite temperature the equilibrium state of an ensemble is given by the Gibbs distribution $e^{-\frac{H}{T}}$.

The logarithmic distribution of eigenvalues, together with the exponential pf the Gibbs distribution, gives *at very small temperatures* an integer spacing (quantization) for the liner response for every physical property that depends on the spectrum of the conduction electrons (e.g. resistivity).

At finite temperature the Gibbs factor is no longer prominent and quantisation no longer holds.

Remark 1 The presence of an infinite number of bound states is of course an artefact of contact interaction, a singular interaction.

We proved that the weak contact interaction is limit, *in strong resolvent sense*, as $\epsilon \to 0$ of interactions due to potentials that have support of radius that decreases to zero

If $\epsilon > 0$ the potential has finite Rollnik norm and therefore the number of bound states is finite.

It is reasonable to expect that if the potential has very short (but finite) range the energy of the bound states has still an approximate logarithmic behaviour.

15 Diffusion on graphs

We have seen that weak *attractive* contact interaction gives the spectrum of conduction electrons in a crystal.

With the same formalism one can also study *repulsive* interactions.

In the case of repulsion joint weak contact repulsion corresponds to a process in which at the vertex there is an equal probability to exit in the three possible directions *without modifying the absolute values of the velocity*.

Therefore since the "free part" of the generator is a positive second-order differential operator, this process represents *diffusion on a tripartite graph*.

One can also use for the interaction at the vertex a random process to have a *random diffusion process* on the graph.

As we have seen in the analysis the problem of conduction electrons, this process can be "lifted" by unique continuation to become diffusion in R^3.

We plan to come back in the future to this problem.

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Appendix

In this Appendix we collect some processes that can be described by contact interactions.

1. The Mott problem [11]

Consider strong simultaneous repulsive contact of a cosmic particle with a nucleus of and atom and its conduction electron (an electron in the outer shell of the atom)

This strong repulsive contact leads to the simultaneous emission of the ion and of the electron.

Since the momentum of the cosmic particle is very high, in the reference frame of the laboratory the two particles are emitted in the same direction of the incoming particle and are strongly localised.

Therefore one can make use for them of a semiclassical description.

In a cloud chamber or a suitably prepared photographic plate these ejected charged particles produce ionisation tracks which are visible.

In view of their semiclassical nature the particles follow essentially classical trajectories.

In presence of a stationary magnetic field the trajectories of the ion and of the electron turn in opposite directions because they have is opposite charges.

The trajectories are short because the ion and the electron emitted loose momentum in the interaction with the atoms of the environment.

This interaction is described in the Physics literature as "the Mott problem".

2. Strong contact, the Higgs particle and the Quarks[12, 13]

A further example of contact interaction are the Higgs particles and the Quarks.

It is commonly believed that proton-antiproton annihilation in a high energy experiment is due to *simultaneous* collision of the proton and the antiproton *and a massive particle* (the "Higgs particle") [12].

The corresponding process is simultaneous strong contact interaction of the particles and the anti-particles with the Higgs particle.

Without the presence of this third particle the annihilation would contradict energy and momentum conservation and in any case would not be the result of strong contact.

Due to the smallness of their Bohr radius the collision of a particle and an antiparticle is *very rare event* unless the particle and antiparticle are forced to be (almost) in the same position, as in particle-antiparticle collision experiment.

Energy and momentum conservation imply that the Higgs particle are very massive and therefore their Bohr radius is very small; therefore the Higgs particles must be *very numerous* and one refers to their collection as *Higgs field*.

Under this condition there is small but finite probability that a *joint* strong contact interaction tales place between the particle-antiparticle pair and the Higgs particle.

Notice that joint strong is a very strong interaction which is not described by Gamma convergence. It occurs at a smaller scale.

Due to energy and momentum conservation and conservation of total electric and nuclear charge this process leads to the introduction of *Quarks* [13].

A Quark can be described as a product of a strong joint contact interaction between two particles. Therefore it is not a *"physical"* particle i.e. a particle that is produced in a "natural process".

In the theory described in [13] charge conservation and conservation of "baryonic" charge calls for the existence of three types of quarks and the corresponding antiquarks.

In order to satisfy electric and barynic charge conservation in the strong separate contact interaction between the particle - antiparticle pair and Higgs particle the three *Quarks* produced in the interaction have *fractional* electric and iso-charge (nuclear charge).

Quarks satisfy Jordan statistics.

A further strong joint contact interaction between these three Quarks gives a *different* system of three quarks of opposite nuclear and electric charge (antiquarks). The quark-antiquark system contains six elements [13].

A third joint strong contact interaction (*the three-fold way* of [13]) leads to a physical system made of three "physical" particles with integer electric and nuclear charges.

These particles interact with the environment and their interaction produces further *physical* particles.

Remark 4 In three dimensions strong contact of three particles is not allowed as a physical process.

Therefore Quarks with fractional charge *do not exist as physical objects*.

They are a useful book-keeping devise.

3. The Galilei invariant Lee model

The system of two particles in simultaneous strong contact with a third particle is also known in Physics under the name of Galileian invariant Lee Model [14].

The three particle sector was analysed in detail by Schrader [15] using separately the renormalisation group approach *and strong resolvent convergence*.

The analysis in [15] is very accurate and even describes scattering theory.

It does not however prove that the resolvent obtained by strong resolvent convergence is indeed the resolvent of the operator obtained using the renormalisation group.

Indeed it does not prove that the "operator" obtained through the renormalisation group is indeed a self-adjoint operator (as compared to a weakly closed quadratic form).

In the present case, our procedure through Gamma convergence gives a self-adjoint operator with infinitely many bound states, while in [15] only one bound state is predicted.

This problem deserves further attention.

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