Effective fluid mixture of tensor-multi-scalar gravity

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Abstract We apply to tensor-multi-scalar gravity the effective fluid analysis based on the representation of the gravitational scalar field as a dissipative effective fluid. This generalization poses new challenges as the effective fluid is now a complicated mixture of individual fluids mutually coupled to each other, and many reference frames are possible for its description. They are all legitimate, although not all convenient for specific problems, and they give rise to different physical interpretations. Two of these frames are highlighted, and implications for cosmology are pointed out.

1 Introduction

There are many motivations to modify Einstein's theory of gravity, general relativity (GR). Quantizing GR, or even introducing the lowest-order quantum corrections introduces deviations from it, which assume the form of extra degrees of freedom, higher-order derivatives in the field equations, non-local terms, or higher powers of the curvature in the action. The low-energy limit of bosonic string theory, which is the simplest string theory, consists of a Brans–Dicke gravity [1] with coupling parameter $\omega = -1$ [2, 3]. Starobinski inflation, which was historically the first model of inflation in the early universe [4] and is the model favoured by cosmological observations [5, 6], has quadratic quantum corrections to the curvature as its essential ingredient and can be seen as an $f(R)$ theory (where R is the Ricci scalar), which is equivalent to a class of scalar–tensor theories [7–9].

The most compelling motivation to study modified gravity comes, no doubt, from the need to explain the present acceleration of the universe. The 1998 discovery that our universe expands in an accelerated way made with the study of the distance-redshift relation of type Ia supernovae, begs for an explanation. The standard cosmological model based on general relativity, known as the Λ cold dark matter (ΛCDM) model, postulates a problematic (astonishingly small) cosmological constant Λ or else a mysterious dark energy with very negative pressure. The latter was introduced in the theory in an *ad hoc* fashion [10] and is, no doubt, deeply unsatisfactory. An alternative to dark energy consists of modifying gravity on Hubble scales, which is consistent with current knowledge because GR is, thus far, tested only in certain regimes [11, 12]. Modifications of GR abound in the literature [13], with the class of scalar–tensor [1, 14–16] and $f(R)$ [17, 18] theories being apparently the most popular.

It is well known that the scalar–tensor gravity field equations can be written as effective Einstein equations with an effective *dissipative* fluid in their right-hand side, built out of the Brans–Dicke-like scalar field ϕ present in the theory and of its first and second covariant derivatives [19–25]. The formalism has been generalized to "viable" Horndeski gravity [26–28] and applied to Friedmann–Lemaître–Robertson–Walker (FLRW) cosmology [29], to theories containing non-propagating scalar degrees of freedom [30, 31], and to specific scalar-tensor solutions [32, 33]. But what is the analogue of a multi-component fluid? Naturally, the simplest multi-fluid equivalent of a theory of gravity is tensor-multi-scalar gravity. Here we extend the effective fluid formalism to this class of theories. The task is much less obvious than it would appear at first sight because all the gravitational scalar fields couple to gravity, which makes them all couple to each other. In general, there can also be direct mutual couplings through their kinetic and potential terms in the action. In the presence of multiple *real* fluids decoupled from each other, one can describe this mixture in the frame of an observer with timelike four-velocity u^μ. This four-velocity can be that of the comoving frame of one of the fluids, or it can be associated with any other observer. In general, it is difficult to define an average fluid [34]. This means that the total stress-energy tensor $T_{\mu\nu}$ of the effective fluid mixture, which is a tensor defined unambiguously, can be decomposed in many ways according to the four-velocity u^μ selected. Each of these descriptions is legitimate, but the description of the total mixture

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and its physical interpretation will depend on the observer u^μ selected to decompose $T_{\mu\nu}$. In particular, the density, pressure, heat flux density, and anisotropic stresses of each fluid as "seen" from a particular observer u^μ will differ from those measured in the comoving frame of that fluid. To appreciate the difference between the descriptions of a fluid in different frames, consider a perfect fluid with four-velocity u^*_μ that, in its comoving frame, is described by the stress-energy tensor[1]

$$T_{\mu\nu} = \rho^* u^*_\mu u^*_\nu + P^* h^*_{\mu\nu}. \tag{1}$$

In the frame of a different observer with timelike four-velocity u^μ related to u^μ_* by

$$u^*_\mu = \gamma\left(u_\mu + v_\mu\right), \tag{2}$$

$$\gamma = \frac{1}{\sqrt{1-v^2}} = -u^*_\mu u^\mu, \tag{3}$$

$$v^2 \equiv v^\alpha v_\alpha > 0, \qquad v^\alpha u_\alpha = 0, \qquad 0 \le v^2 < 1, \tag{4}$$

this perfect fluid (now "tilted") will appear dissipative, with the different stress-energy tensor decomposition [36–38]

$$T_{\mu\nu} = \rho\, u_\mu u_\nu + P h_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu + \pi_{\mu\nu}, \tag{5}$$

where

$$h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu, \tag{6}$$

$$\rho = \rho^* + \gamma^2 v^2\left(\rho^* + P^*\right) = \gamma^2\left(\rho^* + v^2 P^*\right) \tag{7}$$

is the energy density,

$$P = P^* + \frac{\gamma^2 v^2}{3}\left(\rho^* + P^*\right) \tag{8}$$

is the pressure,

$$q^\mu = \left(1 + \gamma^2 v^2\right)\left(\rho^* + P^*\right)v^\mu = \gamma^2\left(\rho^* + P^*\right)v^\mu \tag{9}$$

is the energy flux density, and

$$\pi^{\mu\nu} = \gamma^2\left(\rho^* + P^*\right)\left(v^\mu v^\nu - \frac{v^2}{3} h^{\mu\nu}\right) \tag{10}$$

is the anisotropic stress tensor. It is clear that the (spatial) vector q^μ arises solely due to the relative motion between the two frames, *i.e.*, to the (spatial) vector v^μ. In this context it is problematic to interpret this purely convective current as a heat flux according to Eckart's generalization of Fourier's law [39]

$$q_\mu = -K\left(h_{\mu\nu}\nabla^\nu T + T\,\dot{u}_\mu\right), \tag{11}$$

where T is the temperature and K is the thermal conductivity. This law expresses the fact that heat conduction is caused not only by spatial temperature gradients but also by an "inertial" contribution due to the fluid acceleration [39–42].

The situation becomes more complicated when multiple fluids are coupled to each other and even more when they are *effective* fluids, and they all couple explicitly with the curvature[2] (more precisely, with the Ricci scalar R) and to each other, which is the situation in tensor-multi-scalar gravity. In this work, we discuss two possibilities, but other frames may be more convenient for specific problems.

Rather surprisingly, in tensor-multi-scalar gravity formulated in the Jordan conformal frame, one can obtain a particular frame as a sort of fictitious "average" frame, which is generally not possible with real fluids [34]. It is obtained by identifying the coupling function of the scalars to R (which depends on *all* the scalar fields in the theory) with a new field ψ and amounts to a redefinition of the scalar fields. This procedure is routine in tensor-single-scalar gravity, in which the only Brans–Dicke-like field is redefined for convenience, without much consequence or interpretation. In tensor-multi-scalar gravity, instead, this redefinition takes a new meaning. It identifies a four-velocity and a sort of "average" frame because there is only one Ricci scalar R and all the scalar fields in the theory couple to it. This ingredient is missing for real fluids, which do not couple to the curvature and have no "average" frame [34].

In the following, we analyse tensor-multi-scalar gravity in its Jordan (conformal) frame formulation. It is possible to discuss it from the point of view of the "average" observer, or from the comoving frame of each fluid, or from that of any other timelike observer u^μ. It is important to remember that these descriptions will be different and will provide different physical interpretations

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[1] We follow the notation and conventions of Ref. [35]: the metric signature is $-+++$, $\kappa \equiv 8\pi G$, G is Newton's constant, and units are used in which the speed of light c is unity.

[2] We do not consider derivative couplings in this work.

of the mechanical and thermal aspects of the fluid mixture and that these are all legitimate (hence one should not strive to identify the "correct" one). The point is that some of these formulations (originating different decompositions of the total $T_{\mu\nu}$ based on different u^α) will be more convenient, and some others will be less convenient, for specific physical problems. One should adopt the formulation that is most convenient for the particular problem at hand without prejudice. For example, analyses of the quark-gluon plasma created in heavy ion collisions universally employ the Landau (or energy) frame [43–47] in which there is no heat flux[3] while in FLRW cosmology, where comoving coordinates are the standard, relativistic fluids are routinely described in their comoving (or Eckart) frame [34, 35].

Here we are interested in the fluid-mechanical equivalent and in the thermal description of tensor-multi-scalar gravity, where the fluids in the mixture are *effective* fluids, and they all couple explicitly with R and with each other. This is a very specific situation, and our choices, although convenient in this problem, are not meant to be recipes with universal convenience (although aspects of our discussion may apply to other situations as well). After this discussion, we present an alternative view of the first-order thermodynamics of tensor-multi-scalar gravity in the Einstein conformal frame, while the last section summarizes our conclusions.

2 Tensor-multi-scalar gravity in the Jordan conformal frame

Let us begin with a convenient Jordan frame formulation of tensor-multi-scalar gravity (without derivative couplings). We adopt most of the notations specific to tensor-multi-scalar gravity used in Ref. [49]. There are N scalar fields of gravitational nature $\{\phi^A\}$, with $A = 1, 2, \ldots, N$, all coupled non-minimally with the Ricci scalar R and between themselves, as described by the action

$$S_{\text{TMS}} = \frac{1}{2\kappa} \int d^4 x \sqrt{-g} \Big[F(\phi^J) R - Z_{AB}(\phi^J) g^{\mu\nu} \nabla_\mu \phi^A \nabla_\nu \phi^B - V(\phi^J) \Big] + S^{(\text{m})}, \tag{12}$$

where capital indices A, B, J, $\ldots$ label the scalar fields in the multiplet $\{\phi^1, \ldots, \phi^N\}$, g is the determinant of the spacetime metric $g_{\mu\nu}$, ∇_μ is the associated covariant derivative, and V is a scalar field potential. We include the matter action $S^{(\text{m})}$ because, in many realistic situations, matter is present. For example, in FLRW cosmology, even if the dynamics of the present universe is dominated by the mysterious dark energy which could be described by modifying Einstein gravity, the present universe still contains baryons, radiation, neutrinos, and dark matter, some of which have been dominant in the past (for example, in the radiation era). The Einstein summation convention is used also on the multiplet indices J. The coupling function $F(\phi^1, \ldots, \phi^N)$ depends on all the ϕ^A, *i.e.*, $\partial F / \partial \phi^I \neq 0 \,\forall I \in \{1, \ldots, N\}$, or else some of the scalar fields would not be coupled directly to R and would lose their status of gravitational scalar fields.[4] F is assumed to be positive to keep the effective gravitational coupling $G_{\text{eff}} \simeq 1/F$ positive.

The matrix $Z_{AB}(\phi^1, \ldots, \phi^N)$ acts as a Riemannian metric on the scalar field space of coordinates $\{\phi^1, \ldots, \phi^N\}$. Z_{AB} can be taken to be symmetric without loss of generality because it multiplies the combination of kinetic terms $\nabla^\alpha \phi^A \nabla_\alpha \phi^B$ symmetric in A and B. The elements of Z_{AB} are all positive to avoid introducing unstable phantom fields. In general, also the potential V $(\phi^1, \ldots, \phi^N)$ depends on multiple fields (although it is not important that it depends on *all* these fields, which is instead crucial for the coupling function F).

Since the matrix Z_{AB} is real and symmetric, it can be diagonalized at each spacetime point x^μ and has positive eigenvalues, turning the sum of kinetic terms appearing in the action (12) into

$$\begin{aligned} Z_{AB}(\phi^J) g^{\mu\nu} \nabla_\mu \phi^A \nabla_\nu \phi^B &= \bar{Z}_{AB}(\bar{\phi}^J) g^{\mu\nu} \nabla_\mu \bar{\phi}^A \nabla_\nu \bar{\phi}^A \\ &= \sum_{A=1}^{N} \bar{Z}_A(\bar{\phi}^J) g^{\mu\nu} \nabla_\mu \bar{\phi}^A \nabla_\nu \bar{\phi}^A, \end{aligned} \tag{13}$$

where a bar denotes fields in the system of principal axes of the matrix Z_{AB} in field space, and

$$\bar{Z}_{AB} = \text{diag}(\bar{Z}_1, \ldots, \bar{Z}_N) \tag{14}$$

is the diagonal form of Z_{AB}. This diagonalization, however, is not crucial, and we will not use it explicitly, retaining the non-diagonal form of Z_{AB} in our formulae.

3 Multi-fluid decomposition

The total stress-energy tensor is obtained by varying the action (12) with respect to $g^{\mu\nu}$. Using $\partial_A \equiv \partial/\partial \phi^A$ and $D_{AB} \equiv Z_{AB} + \partial_{AB} F$, the associated equation of motion reads

$$G_{\mu\nu} = \kappa\, T_{\mu\nu} + \frac{\kappa\, T_{\mu\nu}^{(\text{m})}}{F}, \tag{15}$$

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[3] This frame is found to be non-unique in Ref. [48].

[4] The nature of these scalar fields (gravitational or not) depends on the conformal frame [50]. Here we refer to the Jordan conformal frame.

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor, $T_{\mu\nu}^{(m)} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S^{(m)}}{\delta g^{\mu\nu}}$ is the matter stress-energy tensor and

$$
\kappa \, T_{\mu\nu} = \frac{1}{F} \, \partial_A F \left[\nabla_\mu \nabla_\nu \phi^A - g_{\mu\nu} \Box \phi^A \right] + \frac{D_{AB}}{F} \, \nabla_\mu \phi^A \nabla_\nu \phi^B
$$
$$
- \frac{1}{2F} (Z_{AB} + 2\partial_{AB} F) g_{\mu\nu} \nabla_\rho \phi^A \nabla^\rho \phi^B - \frac{V}{2F} \, g_{\mu\nu} \, . \tag{16}
$$

The equation of motion obtained by variation of the action with respect to ϕ^A reads

$$
2 Z_{AB} \Box \phi^A = + \partial_B F \, R - \partial_B V - \partial_B Z_{AC} \nabla_\alpha \phi^A \nabla^\alpha \phi^C
$$
$$
+ 2 \partial_A Z_{CB} \nabla_\alpha \phi^C \nabla^\alpha \phi^A \, . \tag{17}
$$

We can obtain the expression of the Ricci scalar from (15),

$$
R = \frac{2V}{F} + \frac{3\partial_A F}{F} \Box \phi^A - \frac{\kappa \, T^{(m)}}{F} + \frac{(3\partial_{AB} F + Z_{AB})}{F} \, \nabla_\alpha \phi^B \nabla^\alpha \phi^A \, ,
$$

where $T^{(m)} \equiv g^{\mu\nu} T_{\mu\nu}^{(m)}$ is the trace of the matter stress-energy tensor. With this expression, Eq. (17) turns into

$$
0 = \frac{(3\partial_A F \partial_B F + 2F Z_{AB})}{F} \Box \phi^A + \frac{2\partial_B F \, V}{F} - \partial_B V - \frac{\partial_B F}{F} \kappa \, T^{(m)}
$$
$$
+ \frac{\partial_B F}{F} (3\partial_{AC} F + Z_{AC}) \nabla_\alpha \phi^A \nabla^\alpha \phi^C
$$
$$
+ (2\partial_A Z_{BC} - \partial_B Z_{AC}) \nabla_\alpha \phi^A \nabla^\alpha \phi^C \, .
$$

Assuming $\det(3\partial_A F \partial_B F + 2F Z_{AB}) \neq 0$, we use the matrix $M^{AB} \equiv (3\partial_A F \partial_B F + 2F Z_{AB})^{-1}$ to isolate $\Box \phi^A$, obtaining

$$
\Box \phi^A = M^{AB} \left[F \partial_B V - 2V \partial_B F + \kappa T^{(m)} \partial_B F - \partial_B F (3\partial_{AC} F + Z_{AC}) \nabla_\alpha \phi^A \nabla^\alpha \phi^C - F(2\partial_A Z_{BC} - \partial_B Z_{AC}) \nabla_\alpha \phi^A \nabla^\alpha \phi^C \right]. \tag{18}
$$

The goal of the decomposition given here is to separate $T^{\mu\nu}$ so that each part can be decomposed in the frame of a given fluid. Each fluid then receives an individual stress-energy tensor contribution. The number of purely convective terms is minimized by such a decomposition to allow for a clearer description of the intrinsic dissipative properties of each fluid.

Assuming the gradient of each scalar field to be timelike,

$$
X^A \equiv -\frac{1}{2} \, \nabla^\mu \phi^A \nabla_\mu \phi^A > 0 \, , \tag{19}
$$

we define the ϕ^A-fluid four-velocity

$$
u_\mu^A \equiv \frac{\nabla_\mu \phi^A}{\sqrt{2X^A}} \, . \tag{20}
$$

At this point, in order to avoid ambiguities, all the multiplet summations in this section will be written with an explicit summation symbol. The above identification between a scalar field and an associated effective fluid allows us to rewrite the scalar field derivatives in terms of kinematic quantities [27]. The second derivative $\nabla_\mu \nabla_\nu \phi^A$ in Eq. (16) can be expanded as

$$
\nabla_\mu \nabla_\nu \phi^A = \sqrt{2X^A} \left(\sigma^A{}_{\mu\nu} + \frac{1}{3} \, \Theta^A h_{\mu\nu}^A - 2\dot{u}^A{}_{(\mu} u^A{}_{\nu)} \right)
$$
$$
- \left(\Box \phi^A - \sqrt{2X^A} \, \Theta^A \right) u^A{}_\mu u^A{}_\nu \, , \tag{21}
$$

where $h^A{}_{\mu\nu} \equiv g_{\mu\nu} + u^A{}_\mu u^A{}_\nu$ is the three-metric of the hypersurface orthogonal to the four-vector $u^A{}_\mu$, $\Theta^A \equiv \nabla_\mu u^{A\mu}$ is the expansion tensor associated with the A-fluid, and $\sigma^A{}_{\mu\nu} \equiv \frac{1}{2} \left(h^A{}_{ac} \nabla^c u^A{}_b + h^A{}_{bc} \nabla^c u^A{}_a \right) - \frac{1}{3} \, \Theta^A h^A{}_{ab}$.

With this result, Eq. (16) becomes

$$
\kappa \, T_{\mu\nu} = \sum_{A,B} \left\{ \frac{1}{F} \left(2\sqrt{X^A X^B} D_{AB} + \sqrt{2X^A} \partial_A F \, \Theta^A \delta_{AB} \right) u^A{}_\mu u^B{}_\nu \right.
$$
$$
+ \frac{\partial_B F}{F} \delta_{AB} \left(-\Box \phi^A + \frac{\sqrt{2X^A}}{3} \Theta^A \right) h^A{}_{\mu\nu}
$$
$$
- \frac{1}{2F} \left[2\sqrt{X^A X^B} (Z_{AB} + 2\partial_{AB} F) \, u^A{}_\rho u^{B\rho} \right] g_{\mu\nu}
$$
$$
\left. + \frac{\sqrt{2X^A} \partial_B F}{F} \delta_{AB} \left(\sigma^A{}_{\mu\nu} - 2\dot{u}^A{}_{(\mu} u^A{}_{\nu)} \right) \right\} - \frac{V}{2F} g_{\mu\nu} \, , \tag{22}
$$

where

$$\Box\phi^A = \sum_{A,B}\left\{ M^{AB}\left[F\,\partial_B V - 2V\,\partial_B F + \kappa\,T^{(m)}\,\partial_B F \right.\right.$$
$$- 2\sum_C \left(\sqrt{X^A X^C}\partial_B F(3\partial_{AC}F + Z_{AC})u^A{}_\alpha u^{C\alpha} \right.$$
$$\left.\left.\left. + \sqrt{X^A X^C}\,F(2\partial_A Z_{BC} - \partial_B Z_{AC})u^A{}_\alpha u^{C\alpha}\right)\right]\right\}. \tag{23}$$

Since this equation does not depend on four-velocity gradients, we can interpret it as a purely inviscid contribution to the stress-energy tensor mixture.

If we rewrite the metric as

$$g_{\mu\nu} = h^A{}_{\mu\nu} - u^A{}_\mu u^A{}_\nu = \frac{1}{N}\sum_J\left(h^J{}_{\mu\nu} - u^J{}_\mu u^J{}_\nu \right)$$
$$= \frac{1}{N}\sum_{A,B}\delta_{AB}\left(h^B{}_{\mu\nu} - u^A{}_\mu u^B{}_\nu \right) \tag{24}$$

and we define

$$\mathcal{T} \equiv -\frac{1}{N}\sum_{A,B}\left\{ \sqrt{X^A X^B}(Z_{AB} + 2\partial_{AB}F)\,u^A{}_\rho u^{B\rho} \right\} - \frac{V}{2N} \tag{25}$$

then, writing explicitly the summations, the stress-energy tensor assumes the form

$$\kappa\,T_{\mu\nu} = \sum_{A,B}\left\{ \frac{1}{F}\left(2\sqrt{X^A X^B}\,D_{AB} + \sqrt{2X^A}\,\partial_A F\,\Theta^A\delta_{AB} - \mathcal{T}\delta_{AB} \right)u^A{}_\mu u^B{}_\nu \right.$$
$$+ \frac{\partial_B F}{F}\delta_{AB}\left(\mathcal{T} - \Box\phi^B + \frac{\sqrt{2X^B}}{3}\Theta^B \right)h^A{}_{\mu\nu} + \frac{\sqrt{2X^B}\partial_B F}{F}\delta_{AB}\left(\sigma^B{}_{\mu\nu} - 2\dot{u}^A{}_{(\mu}u^B{}_{\nu)} \right)\right\}$$
$$= \sum_A\left\{ \frac{1}{F}\left(\partial_A F\,\Theta^A - \mathcal{T} + \mathcal{U}^A \right)u^A{}_\mu u^A{}_\nu + \frac{\partial_A F}{F}\left(\mathcal{T} - \Box\phi^A + \frac{\sqrt{2X^A}}{3}\Theta^A \right)h^A{}_{\mu\nu} \right.$$
$$\left. + \frac{\sqrt{2X^A}\partial_A F}{F}\sigma^A{}_{\mu\nu} + \frac{1}{F}\left(\mathcal{V}^A{}_{(\mu} - 2\sqrt{2X^A}\partial_A F\dot{u}^A{}_{(\mu} \right)u^A{}_{\nu)} \right\}. \tag{26}$$

The last equality of Eq. (26) is obtained by decomposing $u^B{}_\nu = h^{A\rho}{}_\nu u^B{}_\rho - u^A{}_\nu(u^{A\rho}u^B{}_\rho)$, where there is no summation on the repeated indices, and by defining

$$\sum_{AB}2\sqrt{X^A X^B}\,D_{AB}u^A{}_\mu u^B{}_\nu$$
$$= \sum_A \mathcal{U}^A u^A{}_\mu u^A{}_\nu + \mathcal{V}^A{}_{(\mu}u^A{}_{\nu)}$$

where

$$\sum_B 2\sqrt{X^A X^B}u^{A\rho}u^B{}_\rho D_{AB} = -\mathcal{U}^A, \tag{27}$$

and

$$\sum_B 2\sqrt{X^A X^B}D_{AB}h^{A\rho}{}_\nu u^B{}_\rho = \mathcal{V}^A{}_\nu. \tag{28}$$

In the form given by Eq. (26), the stress-energy tensor is the sum of dissipative stress-energy tensors decomposed along the four velocity of each fluid. It can be interpreted as a mixture of interacting (effective) imperfect fluids.

4 "Average" or "ψ-" description

Let us discuss another possible procedure. In the following, we redefine the fields ϕ^A but, before proceeding, it is essential to note (and remember through the rest of this work) that *all* these fields couple directly with the Ricci scalar R through F, and they all play

a role in determining the properties of the effective fluid equivalent to the tensor-multi-scalar theory and the effective gravitational coupling $G_{\text{eff}} \equiv F^{-1}$. (Their role may be different as, in general, $F(\phi^1, \ldots, \phi^N)$ is not symmetric in all its arguments.) In particular, the effective temperature of this multi-component fluid is determined by *all* the fields ϕ^A and the upcoming redefinition of these fields does not change this fact.

We proceed to redefine the scalar field multiplet as in Ref. [49], which is standard practice in single-scalar-tensor gravity. We can rename the coupling function by electing it to be a Brans-Dicke-like scalar,

$$\psi \equiv F\left(\phi^1, \ldots, \phi^N\right), \tag{29}$$

and we then have the N scalar fields $\left\{\psi, \phi^1, \ldots, \phi^{N-1}\right\}$. This mathematically convenient procedure effectively makes only the field ψ couple explicitly to R, but the reader should not be fooled into believing that the remaining fields ϕ^A do not couple to gravity. In fact, all the fields ϕ^A are coupled to ψ (and also to each other), which makes them couple also to gravity. Indeed, they were explicitly coupled to gravity before the field redefinition $\psi \equiv F$ and the physics does not change. The action (12) is recast as [49]

$$S_{\text{TMS}} = \frac{1}{2\kappa} \int d^4x \sqrt{-g}\left[\psi R - \frac{\omega}{\psi} \nabla^\alpha \psi \nabla_\alpha \psi - Z_{AB} \nabla^\mu \phi^A \nabla_\mu \phi^B - V\right] + S^{(\text{m})}, \tag{30}$$

where

$$\omega = \omega\left(\psi, \phi^A\right), \qquad 2\omega + 3 > 0, \tag{31}$$

$$Z_{AB} = Z_{AB}\left(\psi, \phi^J\right) > 0, \tag{32}$$

$$V = V\left(\psi, \phi^A\right). \tag{33}$$

The field equations for $g_{\mu\nu}, \psi, \phi^A$ obtained by varying the action (30) are [49]

$$G_{\mu\nu} = \frac{1}{\psi}\left(\nabla_\mu \nabla_\nu \psi - g_{\mu\nu} \Box \psi\right) + \frac{\omega}{\psi^2}\left(\nabla_\mu \psi \nabla_\nu \psi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \psi \nabla^\alpha \psi\right)$$
$$+ \frac{Z_{AB}}{\psi}\left(\nabla_\mu \phi^A \nabla_\nu \phi^B - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \phi^A \nabla^\alpha \phi^B\right) - \frac{g_{\mu\nu} V}{2\psi} + \frac{\kappa}{\psi} T_{\mu\nu}^{(\text{m})}, \tag{34}$$

$$Z_{AB} \Box \phi^B = \left(\frac{1}{2} \partial_A Z_{BC} - \partial_B Z_{AC}\right) \nabla_\alpha \phi^C \nabla^\alpha \phi^B + \frac{1}{2\psi} \partial_A \omega \nabla_\alpha \psi \nabla^\alpha \psi$$
$$- \partial_\psi Z_{AB} \nabla_\alpha \psi \nabla^\alpha \phi^B + \frac{1}{2} \partial_A V, \tag{35}$$

$$\Box \psi = \frac{\psi}{2\omega}\left(\partial_\psi V + \partial_\psi Z_{AB} \nabla_\alpha \phi^B \nabla^\alpha \phi^A - R\right)$$

$$- \frac{\partial_A \omega}{\omega} \nabla_\alpha \psi \nabla^\alpha \phi^A + \frac{(\omega - \psi \partial_\psi \omega)}{2\psi\omega} \nabla_\alpha \psi \nabla^\alpha \psi, \tag{36}$$

where we have used the notation $\partial_A \equiv \partial/\partial\phi^A$ and $\partial_\psi \equiv \partial/\partial\psi$.

Using the metric field equations, we can express the Ricci scalar in terms of the matter and effective stress-energy tensors,

$$R = \frac{3}{\psi} \Box \psi + \frac{\omega}{\psi^2} \nabla_\alpha \psi \nabla^\alpha \psi$$
$$+ \frac{Z_{AB}}{\psi} \nabla_\alpha \phi^A \nabla^\alpha \phi^B + \frac{2V}{\psi} - \frac{\kappa}{\psi} T^{(\text{m})} \tag{37}$$

where $T^{(\text{m})} \equiv g^{\mu\nu} T_{\mu\nu}^{(\text{m})}$. Then, the equation of motion for ψ turns into

$$\Box \psi = \frac{1}{3 + 2\omega}\Big[\left(\psi \partial_\psi Z_{AB} - Z_{AB}\right) \nabla_\alpha \phi^A \nabla^\alpha \phi^B$$
$$- 2\partial_A \omega \nabla_\alpha \psi \nabla^\alpha \phi^A - \partial_\psi \omega \nabla_\alpha \psi \nabla^\alpha \psi$$
$$+ \psi \partial_\psi V - 2V + \kappa T^{(\text{m})}\Big]. \tag{38}$$

Finally, we define the effective stress-energy tensor as

$$\kappa T_{\mu\nu} \equiv \frac{1}{\psi}\left(\nabla_\mu \nabla_\nu \psi - g_{\mu\nu} \Box \psi\right) + \frac{\omega}{\psi^2}\left(\nabla_\mu \psi \nabla_\nu \psi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \psi \nabla^\alpha \psi\right)$$
$$+ \frac{Z_{AB}}{\psi}\left(\nabla_\mu \phi^A \nabla_\nu \phi^B - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \phi^A \nabla^\alpha \phi^B\right) - \frac{g_{\mu\nu} V}{2\psi}. \tag{39}$$

We can now move to the effective fluid picture.

5 Comoving (Eckart) frame of ψ-fluid

Assume that the gradient of ψ is timelike; using

$$X \equiv -\frac{1}{2} \nabla^\mu \psi \nabla_\mu \psi > 0 \tag{40}$$

we define the effective fluid four-velocity

$$u^\mu = \frac{\nabla^\mu \psi}{\sqrt{2X}} \tag{41}$$

which is normalized, $u^\mu u_\mu = -1$. In general, the ϕ^A-fluids are tilted with respect to the ψ-fluid, *i.e.*, $u^{A\mu}$ and u^μ have different directions. Using the u^μ of the ψ-fluid we perform the usual $3 + 1$ splitting of spacetime into the time direction and the 3-space "seen" by the observer with four-velocity u^μ. This 3-space has Riemannian metric

$$h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu . \tag{42}$$

The kinematic quantities (expansion tensor $\Theta_{\mu\nu}$, expansion scalar $\Theta = \nabla_\mu u^\mu$, shear tensor $\sigma_{\mu\nu}$, shear scalar, and acceleration $\dot{u}^\mu$) associated with u^μ are the same as those calculated for single-scalar-tensor gravity in [21]. In fact, their definitions are purely kinematic and theory-independent since they do not use the field equations but only the definition (41) of u^μ. These kinematic quantities are straightforward, although lengthy to compute. Since they are used here, we report them in "Appendix A".

The field Eq. (34) has the form of effective Einstein equations with an effective stress-energy tensor in their right-hand side, which can be seen as the stress-energy tensor of a *dissipative* multi-component fluid of the form

$$T_{\mu\nu} = (P + \rho)u_\mu u_\nu + P g_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu + \pi_{\mu\nu} \tag{43}$$

where

$$\rho = T_{\mu\nu} u^\mu u^\nu \tag{44}$$

is the effective energy density,

$$q_\mu = -T_{\alpha\beta} u^\alpha h_\mu{}^\beta \tag{45}$$

is the effective heat current density describing heat conduction,

$$\Pi_{\alpha\beta} = P h_{\alpha\beta} + \pi_{\alpha\beta} = T_{\mu\nu} h_\alpha{}^\mu h_\beta{}^\nu \tag{46}$$

is the effective stress tensor,

$$P = \frac{1}{3} g^{\alpha\beta} \Pi_{\alpha\beta} = \frac{1}{3} h^{\alpha\beta} T_{\alpha\beta} \tag{47}$$

is the effective isotropic pressure, and the trace-free part of the stress tensor

$$\pi_{\alpha\beta} = \Pi_{\alpha\beta} - P h_{\alpha\beta} \tag{48}$$

is the effective anisotropic stress tensor. q^μ, $\Pi_{\alpha\beta}$, and $\pi_{\alpha\beta}$ are purely spatial with respect to u^μ. The fluid description is obtained by expressing the derivatives of ψ in terms of the relative effective fluid four-velocity (41) and kinematic quantities,

$$\nabla_\mu \psi = \sqrt{2X} \, u_\mu , \tag{49}$$

$$\nabla_\mu X = -\dot{X} u_\mu + h_{\mu\nu} \nabla^\nu X = -\dot{X} u_\mu - 2X \dot{u}_\mu , \tag{50}$$

$$\nabla_\mu \nabla_\nu \psi = \nabla_\mu \left(\sqrt{2X} \, u_\nu \right)$$

$$= \sqrt{2X} \, \nabla_\mu u_\nu - \frac{\dot{X}}{\sqrt{2X}} u_\mu u_\nu - \sqrt{2X} \dot{u}_\mu u_\nu$$

$$= \sqrt{2X} \left(\sigma_{\mu\nu} + \frac{1}{3}\Theta h_{\mu\nu} - 2\dot{u}_{(\mu} u_{\nu)} \right) - \frac{\dot{X}}{\sqrt{2X}} u_\mu u_\nu . \tag{51}$$

Furthermore, we have

$$\Box\psi = \sqrt{2X} \, \Theta + \frac{\dot{X}}{\sqrt{2X}} , \tag{52}$$

and therefore, the ψ-equation of motion reads

$$
\begin{aligned}
\frac{\dot{X}}{\sqrt{2X}} = & -\sqrt{2X}\,\Theta \\
& + \frac{1}{3+2\omega}\Big[\big(\psi\,\partial_\psi Z_{AB} - Z_{AB}\big)\nabla_\alpha\phi^A\nabla^\alpha\phi^B \\
& - 2\sqrt{2X}\,\partial_A\omega\,u_\alpha\nabla^\alpha\phi^A + 2\partial_\psi\omega X \\
& + \psi\,\partial_\psi V - 2V + \kappa\,T^{(m)}\Big].
\end{aligned}
\tag{53}
$$

We need these equations to eliminate the dependence of $T_{\mu\nu}$ on $\dot{X}$ and on $\Box\psi$. Indeed, prior to using the equation of motion for ψ, one obtains

$$
\begin{aligned}
\kappa T_{\mu\nu} = & \left(\frac{V}{2\psi} + \frac{X\omega}{\psi^2} + \frac{\sqrt{2X}}{\psi}\Theta\right)u_\mu u_\nu \\
& + \left(-\frac{V}{2\psi} + \frac{X\omega}{\psi^2} - \frac{\dot{X}}{\sqrt{2X}\psi} - \frac{2\sqrt{2X}}{3\psi}\Theta\right)h_{\mu\nu} \\
& - 2\frac{\sqrt{2X}}{\psi}\dot{u}_{(\mu}u_{\nu)} + \frac{\sqrt{2X}}{\psi}\sigma_{\mu\nu} \\
& + \frac{Z_{AB}\nabla_\mu\phi^A\nabla_\nu\phi^B}{\psi} - \frac{g_{\mu\nu}Z_{AB}\nabla_\alpha\phi^A\nabla^\alpha\phi^B}{2\psi}.
\end{aligned}
\tag{54}
$$

Using the decomposition $\nabla_\mu = h_\mu{}^\nu\nabla_\nu - u_\mu u^\nu\nabla_\nu$, defining $\dot{\phi}^A \equiv u^\alpha\nabla_\alpha\phi^A$, and taking into account the symmetry $Z_{AB} = Z_{BA}$, the interacting terms contribute to the density, pressure, heat flux and anisotropic stress,

$$
Z_{AB}\nabla_\mu\phi^A\nabla_\nu\phi^B = Z_{AB}\Big(\dot{\phi}^A\dot{\phi}^B u_\mu u_\nu + h_\mu{}^\rho h_\nu{}^\sigma \nabla_\rho\phi^A\nabla_\sigma\phi^B - 2h_{(\mu}{}^\alpha u_{\nu)}\nabla_\alpha\phi^A\dot{\phi}^B\Big)
\tag{55}
$$

and the stress-energy tensor reads

$$
\begin{aligned}
\kappa\,T_{\mu\nu} = & \left(\frac{V}{2\psi} + \frac{X\omega}{\psi^2} + \frac{\sqrt{2X}}{\psi}\Theta + \frac{Z_{AB}}{\psi}\dot{\phi}^A\dot{\phi}^B + \frac{Z_{AB}}{2\psi}\nabla_\alpha\phi^A\nabla^\alpha\phi^B\right)u_\mu u_\nu \\
& + \left(-\frac{V}{2\psi} + \frac{X\omega}{\psi^2} - \frac{\dot{X}}{\sqrt{2X}\psi} - \frac{2\sqrt{2X}}{3\psi}\Theta - \frac{Z_{AB}}{2\psi}\nabla_\alpha\phi^A\nabla^\alpha\phi^B\right)h_{\mu\nu} \\
& - \frac{2\sqrt{2X}}{\psi}\dot{u}_{(\mu}u_{\nu)} - 2\frac{Z_{AB}}{\psi}h_{(\mu}{}^\alpha u_{\nu)}\nabla_\alpha\phi^A\dot{\phi}^B \\
& + \frac{\sqrt{2X}}{\psi}\sigma_{\mu\nu} + \frac{Z_{AB}}{\psi}h_\mu{}^\rho h_\nu{}^\sigma\nabla_\rho\phi^A\nabla_\sigma\phi^B.
\end{aligned}
\tag{56}
$$

Then, it is straightforward to obtain the effective fluid quantities

$$
\begin{aligned}
\kappa\,\rho = & \frac{1}{2\psi^2}\Big[\psi\,V + 2X\omega + 2\psi\sqrt{2X}\,\Theta \\
& + \psi\,Z_{AB}\big(\nabla_\alpha\phi^A\nabla^\alpha\phi^B + 2\dot{\phi}^A\dot{\phi}^B\big)\Big],
\end{aligned}
\tag{57}
$$

$$
\begin{aligned}
\kappa\,q^\alpha = & -h^\alpha{}_\mu u_\nu T^{\mu\nu} \\
\\
= & -\frac{\sqrt{2X}}{\psi}\dot{u}^\alpha - \frac{Z_{AB}}{\psi}\dot{\phi}^A h^\alpha{}_\mu\nabla^\mu\phi^B,
\end{aligned}
\tag{58}
$$

$$
\begin{aligned}
\kappa\,P = & -\frac{V}{2\psi} + \frac{X\omega}{\psi^2} - \frac{\dot{X}}{\sqrt{2X}\psi} - \frac{2\sqrt{2X}}{3\psi}\Theta \\
& - \frac{Z_{AB}}{2\psi}\nabla_\alpha\phi^A\nabla^\alpha\phi^B + \frac{Z_{AB}}{3\psi}h_{\mu\nu}\nabla^\mu\phi^A\nabla^\nu\phi^b \\
= & \frac{\sqrt{2X}}{3\psi}\Theta - \frac{\kappa\,T^{(m)}}{\psi(3+2\omega)} - \frac{(2\omega-1)}{2\psi(3+2\omega)}V + \frac{X}{\psi^2}\omega \\
& - \frac{(\partial_\psi V\,\psi + 2X\,\partial_\psi\omega)}{\psi(3+2\omega)}
\end{aligned}
$$

$$+ \frac{Z_{AB}(3 - 2\omega) - 6\partial_\psi Z_{AB}\psi}{6\psi(3 + 2\omega)} \nabla_\alpha \phi^B \nabla^\alpha \phi^A$$

$$+ \frac{2\sqrt{2X}}{\psi(3 + 2\omega)} \partial_A \omega \dot{\phi}^A + \frac{Z_{AB}}{3\psi} \dot{\phi}^A \dot{\phi}^B,$$

$$\kappa \pi^{\rho\sigma} = \frac{\sqrt{2X}}{\psi} \sigma^{\rho\sigma} + \left(h^{\mu\rho}h^{\nu\sigma} - \frac{1}{3}h^{\rho\sigma}h^{\mu\nu} \right) \frac{Z_{AB}}{\psi} \nabla_\mu \phi^A \nabla_\nu \phi^B$$

$$= \frac{\sqrt{2X}}{\psi} \sigma^{\rho\sigma} + \frac{Z_{AB}\nabla^\rho\phi^A\nabla^\sigma\phi^B}{\psi} - \frac{h^{\rho\sigma}Z_{AB}\nabla_\alpha\phi^A\nabla^\alpha\phi^B}{3\psi}$$

$$+ \frac{2Z_{AB}\dot{\phi}^B u^{(\rho}\nabla^{\sigma)}\phi^A}{\psi} - \frac{(g^{\rho\sigma} - 2u^\rho u^\sigma)Z_{AB}\dot{\phi}^A\dot{\phi}^B}{3\psi}$$

$$= \frac{\sqrt{2X}}{\psi} \sigma^{\rho\sigma} + \frac{Z_{AB}\nabla^\rho\phi^A\nabla^\sigma\phi^B}{\psi} - \frac{h^{\rho\sigma}Z_{AB}\nabla_\alpha\phi^A\nabla^\alpha\phi^B}{3\psi}$$

$$+ \frac{2Z_{AB}\dot{\phi}^B u^{(\rho}h^{\sigma)\alpha}\nabla_\alpha\phi^A}{\psi} - \frac{(g^{\rho\sigma} + 4u^\rho u^\sigma)Z_{AB}\dot{\phi}^A\dot{\phi}^B}{3\psi}, \tag{59}$$

where an overdot denotes differentiation along the lines of the ψ-fluid, i.e., $\dot{\phi}^A \equiv u^\alpha \nabla_\alpha \phi^A$.

At this point, we can identify the various contributions to the effective energy tensor as

$$P = P_{\mathrm{inv}} + P_{\mathrm{vis}} + P_\phi \tag{60}$$

$$= P_{\mathrm{inv}} - \zeta\Theta + P_\phi, \tag{61}$$

$$\rho = \rho_{\mathrm{inv}} + \rho_{\mathrm{vis}} + \rho_\phi \tag{62}$$

$$= \rho_{\mathrm{inv}} - 3\zeta\Theta + \rho_\phi, \tag{63}$$

$$q^\mu = -\frac{\sqrt{2X}}{\psi}\dot{u}^\mu + q_\phi^\mu, \tag{64}$$

$$\pi^{\mu\nu} = -2\eta\sigma^{\mu\nu} + \pi_\phi^{\mu\nu}, \tag{65}$$

where

$$\kappa P_{\mathrm{inv}} = \frac{X}{\psi^2}\omega - \frac{(2\omega - 1)}{2\psi(3 + 2\omega)}V - \frac{\kappa T^{(\mathrm{m})}}{\psi(3 + 2\omega)}$$

$$- \frac{(\partial_\psi V \psi + 2X\partial_\psi \omega)}{\psi(3 + 2\omega)},$$

$$\kappa P_\phi = \frac{Z_{AB}(3 - 2\omega) - 6\partial_\psi Z_{AB}\psi}{6\psi(3 + 2\omega)} \nabla_\alpha \phi^B \nabla^\alpha \phi^A$$

$$+ \frac{2\sqrt{2X}}{\psi(3 + 2\omega)} \partial_A \omega \dot{\phi}^A + \frac{Z_{AB}}{3\psi} \dot{\phi}^A \dot{\phi}^B,$$

$$\kappa \rho_{\mathrm{inv}} = \frac{1}{2\psi^2}\left(\psi V + 2X\omega + 2\psi\sqrt{2X} \right), \tag{66}$$

$$\kappa \rho_\phi = \frac{Z_{AB}}{2\psi}\left(\nabla_\alpha\phi^A\nabla^\alpha\phi^B + 2\dot{\phi}^A\dot{\phi}^B \right), \tag{67}$$

$$\kappa q_\phi^\mu = -\frac{Z_{AB}}{\psi}\dot{\phi}^A h^\mu_{\ \alpha}\nabla^\alpha\phi^B, \tag{68}$$

$$\kappa \pi_\phi^{\mu\nu} = \left(h^{\mu\rho}h^{\nu\sigma} - \frac{1}{3}h^{\rho\sigma}h^{\mu\nu} \right) \frac{Z_{AB}}{\psi} \nabla_\rho\phi^A\nabla_\sigma\phi^B, \tag{69}$$

while

$$\zeta = -\frac{\sqrt{2X}}{3\kappa\psi}, \qquad \eta = \frac{\sqrt{2X}}{2\kappa\psi} \tag{70}$$

are the bulk and shear viscosity coefficients, respectively.

In the particular case in which the Lagrangian is linear in X, the ψ-equation of motion reveals that $\Box\psi$ does not contain derivatives of the ψ-fluid four-velocity; therefore, it only contributes to the inviscid pressure. However, it contains ϕ-terms related to the interactions.

Finally, the ϕ-terms contribute only to the inviscid part of the effective stress-energy tensor (because P_ϕ and ρ_ϕ depend only on first derivatives of the fields), to the heat flux, and to the shear viscosity. In the general case of the previous section, all the ϕ fields contribute to both viscous and inviscid parts.

6 FLRW cosmology

Thus far, we have discussed the general tensor-multi-scalar theory of gravity, which conceivably has many applications. Given that the most compelling motivation (although certainly not the only one) for modifying gravity is explaining the present acceleration of the universe without the *ad hoc* dark energy, it is natural to discuss FLRW cosmology in the context of tensor-multi-scalar gravity.

Assume a spatially flat FLRW universe with line element

$$ds^2 = -dt^2 + a^2(t)\big(dx^2 + dy^2 + dz^2\big) \tag{71}$$

in comoving coordinates (t, x, y, z), where the scale factor $a(t)$ embodies the expansion history of the cosmos. Due to spatial homogeneity and isotropy, the gravitational scalar fields of tensor-multi-scalar gravity depend only on time t and the anisotropic stress tensor and heat current density vanish, $\pi^{\mu\nu} = 0$, $q^\mu = 0$. It is natural to assume that the ψ-frame is the frame of the observers who see the cosmic microwave background homogeneous and isotropic around them (apart from tiny temperature fluctuations $\delta T/T \simeq 10^{-5}$). Then, $\nabla^\mu \psi$ is timelike, and we restrict to situations in which it is also future-oriented. We have $\psi = \psi(t)$, $\phi^A = \phi^A(t)$, $X = -\dot{\psi}^2/2$, and $\Theta = 3H$, where $H \equiv \dot{a}/a$ is the Hubble function and an overdot denotes differentiation with respect to the comoving time t. Since the anisotropic stresses and the heat flux must vanish, the only dissipative fluid feature respecting the FLRW spatial symmetries is bulk viscosity. The fluid quantities reduce to

$$\kappa \rho = \frac{1}{2\psi^2}\Big[\psi V - \omega \dot{\psi}^2 + 6H\psi|\dot{\psi}| + Z_{AB}\,\dot{\phi}^A\dot{\phi}^B\psi\Big]$$
$$= \rho_{\text{inv}} + \rho_{\text{vis}} + \rho_\phi, \tag{72}$$

where

$$\kappa \rho_{\text{inv}} = \frac{1}{2\psi^2}\big(\psi V - \omega \dot{\psi}^2\big), \tag{73}$$

$$\kappa \rho_{\text{vis}} = 6\psi|\dot{\psi}|, \tag{74}$$

$$\kappa \rho_\phi = \frac{Z_{AB}}{2\psi}\,\dot{\phi}^A\dot{\phi}^B. \tag{75}$$

The pressure is

$$\kappa P = -\frac{\kappa T^{(\text{m})}}{(2\omega + 3)\psi} + \frac{H|\dot{\psi}|}{\psi} + \frac{(1 - 2\omega)}{2\psi(2\omega + 3)} - \frac{\omega \dot{\psi}^2}{2\psi^2}$$
$$+ \frac{\dot{\psi}^2\partial_\psi \omega - \psi \partial_\psi V}{\psi(2\omega + 3)} + \left[\frac{2Z_{AB} - \psi \partial_\psi Z_{AB}}{\psi(2\omega + 3)}\right]\dot{\phi}^A\dot{\phi}^B, \tag{76}$$

where $P = P_{\text{inv}} + P_{\text{vis}} + P_\phi$ with

$$\kappa P_{\text{inv}} = -\frac{\dot{\psi}^2\omega}{2\psi^2} + \frac{(1 - 2\omega)V}{2\psi(2\omega + 3)} - \frac{\kappa T^{(\text{m})}}{\psi(2\omega + 3)} + \frac{\dot{\psi}^2\partial_\psi \omega - \psi \partial_\psi V}{\psi(2\omega + 3)}, \tag{77}$$

$$\kappa P_{\text{vis}} = \frac{H|\dot{\psi}|}{\psi}, \tag{78}$$

$$\kappa P_\phi = \frac{\big[6\psi \partial_\psi Z_{AB} - (3 - 2\omega)Z_{AB}\big]}{6\psi(2\omega + 3)}\,\dot{\phi}^A\dot{\phi}^B + \frac{2|\dot{\psi}|}{\psi(2\omega + 3)}\,\dot{\phi}^A\partial_A\omega + \frac{Z_{AB}}{2\psi}\,\dot{\phi}^A\dot{\phi}^B. \tag{79}$$

The bulk viscosity coefficient is

$$\zeta = -\frac{|\dot{\psi}|}{3\kappa\psi}\,. \tag{80}$$

It is well known that bulk viscosity can generate a negative pressure that could, in principle, accelerate the universe (cf., *e.g.*, the recent review [51]). The specific microphysical implementation of bulk viscosity during the present era of the universe is not simple. For example, particle production in curved spacetime, which leads to bulk viscosity, is insufficient to explain away dark energy. In tensor-multi-scalar gravity, bulk viscosity comes as an essential feature of the nonminimal coupling between the scalar fields and the Ricci scalar. Although scalar fields and tensor-single-scalar gravity (perhaps in its $f(R)$ incarnation [7–9]) have been widely used to model the present cosmic acceleration and are the subjects of vast literatures (see, e.g., [10, 34, 52, 53]), tensor-multi-scalar gravity

offers new possibilities. Modifying gravity at galactic scales has been proposed as a solution to the dark matter problem, even in the Newtonian regime with the introduction of MOND gravity and in tensor-single-scalar theories, in which the extra gravitational field produces Yukawa instead of Coulombic potentials (e.g., [53]). In tensor-multi-scalar gravity, the multiplet ϕ^A can in principle explain simultaneously dark energy (through the "average" field ψ) and dark matter (through the multiplet fields ϕ^A)—Occam's razor would dictate the presence of a single field ϕ, but this is not a given.

The possibility of explaining simultaneously dark energy and dark matter by replacing GR with a tensor-multi-scalar theory is intriguing and will be explored in future research. An appealing feature is that, in such a scenario, dark energy would unavoidably couple to dark matter. Coupled dark energy and dark matter have been studied extensively and have recently been revived in attempts to alleviate or solve the notorious Hubble tension plaguing the ΛCDM model of cosmology. A common problem of these scenarios is that the coupling has to be introduced by hand and, usually, the formulation is not covariant, being proposed exclusively in FLRW spaces [54]. Tensor-multi-scalar gravity, instead, is formulated as a general theory valid in arbitrary geometries and is completely covariant. Furthermore, the coupling between the fields ψ and ϕ^A is not *ad hoc* but is a necessary feature of the theory. Specific scenarios of coupled dark energy and dark matter in tensor-multi-scalar gravity necessarily require the specification of N, Z_{AB} $\left(\psi, \phi^A\right)$, $V\left(\psi, \phi^A\right)$, and $\omega\left(\psi, \phi^A\right)$: a detailed study is beyond the purpose of this paper.

7 Conclusions

The picture of the effective fluid equivalent of tensor-multi-scalar gravity that emerges from the previous sections is the following. Because all the N original gravitational scalar fields couple explicitly to the Ricci scalar, they are automatically coupled to each other. In addition, they may have explicit couplings to each other through the functions Z_{AB} and V, but this is not necessary for them to be mutually coupled. In the multi-fluid interpretation, this property could correspond to these fields being thermalized, but this interpretation is not corroborated in any obvious way by the field equations and remains rather arbitrary.

Data availability No Data associated in the manuscript.

Declarations

Conflict of interest The authors declare no conflict of interest.

Appendix A: Kinematic quantities of the ψ-fluid

The (double) projection of the velocity gradient onto the 3-space orthogonal to u^c

$$V_{\alpha\beta} \equiv h_\alpha{}^\mu \, h_\beta{}^\nu \, \nabla_\nu u_\mu \,, \tag{81}$$

is decomposed into its symmetric and antisymmetric parts. The latter is identically zero because the ψ-fluid is derived from a scalar field. The symmetric part is further decomposed into its trace-free and pure trace parts. This results in

$$V_{\alpha\beta} = \Theta_{\alpha\beta} + \omega_{\alpha\beta} = \sigma_{\alpha\beta} + \frac{\Theta}{3} h_{\alpha\beta} + \omega_{\alpha\beta} \,, \tag{82}$$

where the expansion tensor $\Theta_{\alpha\beta} = V_{(\alpha\beta)}$ is the symmetric part of $V_{\alpha\beta}$, $\Theta \equiv \Theta^\rho{}_\rho = \nabla_\rho u^\rho$ is its trace, the vorticity tensor $\omega_{\alpha\beta} = V_{[\alpha\beta]} = 0$, and the symmetric, trace-free shear tensor is

$$\sigma_{\alpha\beta} \equiv \Theta_{\alpha\beta} - \frac{\Theta}{3} h_{\alpha\beta} \,. \tag{83}$$

Expansion, vorticity, and shear are purely spatial,

$$\Theta_{\alpha\beta}u^{\alpha} = \Theta_{\alpha\beta}u^{\beta} = \omega_{\alpha\beta}\,u^{\alpha} = \omega_{\alpha\beta}\,u^{\beta} = \sigma_{\alpha\beta}u^{\alpha}$$
$$= \sigma_{\alpha\beta}u^{\beta} = 0. \tag{84}$$

For general fluids, it is [34, 55]

$$\nabla_{\beta}u_{\alpha} = \sigma_{\alpha\beta} + \frac{\Theta}{3}\,h_{\alpha\beta} + \omega_{\alpha\beta} - \dot{u}_{\alpha}u_{\beta} = V_{\alpha\beta} - \dot{u}_{\alpha}u_{\beta}. \tag{85}$$

The kinematic quantities of the ψ-fluid relevant for the present discussion are computed in [21] and are as follows:

$$\nabla_{\beta}u_{\alpha} = \frac{1}{\sqrt{-\nabla^{\rho}\psi\nabla_{\rho}\psi}}\left(\nabla_{\alpha}\nabla_{\beta}\psi - \frac{\nabla_{\alpha}\psi\nabla^{\rho}\psi\nabla_{\beta}\nabla_{\rho}\psi}{\nabla^{\sigma}\psi\nabla_{\sigma}\psi}\right), \tag{86}$$

the acceleration is

$$\dot{u}_{\alpha} \equiv u^{\beta}\nabla_{\beta}u_{\alpha} = \left(-\nabla^{\rho}\psi\nabla_{\rho}\psi\right)^{-2}\nabla^{\beta}\psi$$
$$\times\left[(-\nabla^{\rho}\psi\nabla_{\rho}\psi)\nabla_{\alpha}\nabla_{\beta}\psi + \nabla^{\rho}\psi\nabla_{\beta}\nabla_{\rho}\psi\nabla_{\alpha}\psi\right], \tag{87}$$

$$V_{\alpha\beta} = \frac{\nabla_{\alpha}\nabla_{\beta}\psi}{\left(-\nabla^{\rho}\psi\nabla_{\rho}\psi\right)^{1/2}}$$
$$+ \frac{\left(\nabla_{\alpha}\psi\nabla_{\beta}\nabla_{\sigma}\psi + \nabla_{\beta}\psi\nabla_{\alpha}\nabla_{\sigma}\psi\right)\nabla^{\sigma}\psi}{\left(-\nabla^{\rho}\psi\nabla_{\rho}\psi\right)^{3/2}}$$
$$+ \frac{\nabla_{\delta}\nabla_{\sigma}\psi\nabla^{\sigma}\psi\nabla^{\delta}\psi}{\left(-\nabla^{\rho}\psi\nabla_{\rho}\psi\right)^{5/2}}\,\nabla_{\alpha}\psi\nabla_{\beta}\psi. \tag{88}$$

The expansion scalar reads

$$\Theta = \nabla_{\rho}u^{\rho} = \frac{\Box\psi}{\left(-\nabla^{\rho}\psi\nabla_{\rho}\psi\right)^{1/2}}$$
$$+ \frac{\nabla_{\alpha}\nabla_{\beta}\psi\nabla^{\alpha}\psi\nabla^{\beta}\psi}{\left(-\nabla^{\rho}\psi\nabla_{\rho}\psi\right)^{3/2}}, \tag{89}$$

while the shear tensor is

$$\sigma_{\alpha\beta} = \left(-\nabla^{\rho}\psi\nabla_{\rho}\psi\right)^{-3/2}\Big[-\left(\nabla^{\rho}\psi\nabla_{\rho}\psi\right)\nabla_{\alpha}\nabla_{\beta}\psi$$
$$- \frac{1}{3}\left(\nabla_{\alpha}\psi\nabla_{\beta}\psi - g_{\alpha\beta}\,\nabla^{\sigma}\psi\nabla_{\sigma}\psi\right)\Box\psi$$
$$- \frac{1}{3}\left(g_{\alpha\beta} + \frac{2\nabla_{\alpha}\psi\nabla_{\beta}\psi}{\nabla^{\rho}\psi\nabla_{\rho}\psi}\right)\nabla_{\sigma}\nabla_{\tau}\psi\nabla^{\tau}\psi\nabla^{\sigma}\psi$$
$$+ \left(\nabla_{\alpha}\psi\nabla_{\sigma}\nabla_{\beta}\psi + \nabla_{\beta}\psi\nabla_{\sigma}\nabla_{\alpha}\psi\right)\nabla^{\sigma}\psi\Big]. \tag{90}$$

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