



Constant-roll $f(R)$ inflation compared with cosmic microwave background anisotropies and swampland criteria

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Abstract Inflationary models derived from $f(R)$ gravity, where the scalaron rolls down with a constant rate from the top to the minimum of the effective potential, are considered. Specifically, we take into account three $f(R)$ models, i.e. Starobinsky R^2 , R^{2p} and the logarithmic corrected models. We compare the inflationary parameters derived from the models with the observational data of CMB anisotropies, i.e., the Planck and Keck/array datasets in order to find observational constraints on the parameters space. We find that although our $f(R)$ constant-roll models for $\gamma = 0$ show observationally acceptable values of r , they do not predict favored values of the spectral index. In particular, we have $n_s > 1$ for the Starobinsky R^2 and R^{2p} models and $0.996 < n_s < 0.999$ for logarithmic model. Finally, we study the models from the point of view of Weak Gravity Conjecture adopting the swampland criteria.

1 Introduction

Cosmological inflation is the most straightforward paradigm to describe the early universe phenomena. In particular, the inflationary scalar and tensor perturbations are unanimously considered responsible for structure formation and primordial gravitational waves, respectively [1–11]. The simplest inflationary model is based on a single scalar field, the so-called inflation, rolling down slowly from the top to the minimum point of potential under the slow-roll approximation. Then, the inflationary epoch terminates when inflation decays to standard particles at the final step of inflation due to the reheating process [12, 13]. The single field models have been studied broadly in inflationary literature and also have been compared with the observational datasets. Consequently, some of them are restricted or even ruled out [14] and some models are still viable with the recent high precision observations [15–17]. Moreover, this group of models does not show any non-Gaussianity in their primordial spectrum because of uncorrelated modes of the spectrum [18]. In such a case, if the future observations predict non-Gaussianity in the perturbations spectrum, then the single field models will be situated in an unstable status. Recently, a new approach to inflation has been proposed in which inflation is rolling down with a constant rate [19–21]. Hence, $\ddot{\phi}$ as a non-negligible term can be expressed as

$$\ddot{\phi} = \beta H \dot{\phi} \quad (1)$$

where $\beta = -(3 + \alpha)$ and α is a non-zero parameter. For $\alpha = -3$, the model is reduced to the standard slow-roll. Going beyond of the slow-roll approximation, we can consider a ultra slow-roll regime where the term of $\ddot{\phi}$ is non-negligible in the Klein–Gordon equation as $\ddot{\phi} = 3H\dot{\phi}$. The ultra slow-roll models show a limited amount for the non-decaying mode of curvature perturbations [22] and also predict a large η but unable to solve the η problem introduced in supergravity for the hybrid inflationary models [23]. Moreover, the ultra solutions are located in the non-attractor phase of inflation but reveal a scale-invariance of the scalar perturbation spectrum. Besides, the main problem of ultra models is that the non-Gaussianity consistency relation of single field models is violated in the presence of ultra conditions through super-Hubble evolution of the scalar perturbation [24]. The fast-roll models are another class of inflationary models introduced to go beyond the slow-roll approximation in which a fast-rolling stage is considered at the start of inflation and can be connected to the standard slow-roll only after a few e-folds [25, 26].

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Constant-roll approach has opened a new window to analyze cosmic inflation due to a constant rate of rolling for inflation. One can find a wide range of inflationary models investigated in the context of constant-roll idea [27–43], in particular, models coming from $f(R)$ gravity, where new scalar fields are not required to drive inflation and one only takes into account geometric extensions of the Hilbert–Einstein action. In [44], the authors considered two different approaches. First, they studied the constant-roll evolution of a scalar-tensor theory in the presence of $f(R)$ gravity and, as second approach, they applied the constant-roll condition directly into $f(R)$ gravity. Generalizations of the constant-roll condition in scalar-tensor gravity [45], in Gauss–Bonnet gravity [46] and in connection to reheating in $f(R)$ gravity [47] have been also developed. On the other hand, logarithmic-corrected R^2 gravity in presence of Kalb–Ramond fields has been considered in [48]. This model, i.e., logarithmic $f(R)$ + Kalb–Ramond, seems to be consistent with the observable values of inflationary parameters. Specifically, the authors consider both $\gamma = 0$ and non-zero γ . Finally, the constant slow-roll condition in Palatini formalism has been taken into account in [49]. In [50], the authors introduced the constant-roll condition in the Jordan frame, and by calculating the potential of scalaron in the Einstein frame, they obtained the form of $f(R)$ in the Jordan frame.

The main aim of the present manuscript is to find the observational constraints from Cosmic Microwave Background (CMB) anisotropies on the parameters space of some $f(R)$ models introduced in the context of the constant-roll inflation. In practice, we use as CMB data, the Planck and the BICEP2/Keck array data releases [51,52]. Also, we compare the predictions of the models with observations in order to find the effects of the new approach. As a secondary purpose, we investigate the $f(R)$ constant-roll inflationary models in view of swampland conditions since $f(R)$ models contain some stringy-like corrections to GR and also $f(R)$ gravity can be realized as a natural class of theories emerging from the string landscape because of the existence of Noether symmetries in this class of extended theories of gravity [53,54]. First, we study the Starobinsky R^2 model as the most well-known inflationary model in $f(R)$ gravity containing a quadratic term coming from higher-order curvature added to Ricci scalar in the Hilbert–Einstein action as $f(R) = R + R^2/6M^2$ [2]. The Starobinsky model is also well-known as one of the most successful inflationary models being in good agreement with current observations. Hence, it is now considered as a “target” model for several future CMB experiments as, for example, the Simons Observatory [55], CMB-S4 [56], and the LiteBIRD satellite experiment [57]. Also, R^2 model has been proposed as one of the possible alternatives to the cosmological constant of the concordance Λ CDM model [58–60,60,61,61–67]. Despite the mentioned achievements, the Starobinsky model predicts a very tiny tensor-to-scalar ratio $r \simeq 0.003$ for 60 e-folds which is out of the current and probably future observations. The goal of these future experiments is therefore to improve the experimental sensitivity to measure such a signal with enough statistical significance with $\delta s < 0.001$. Since the predicted value of r is the first approximation, we might consider some theoretical corrections to the Starobinsky model in order to improve the obtained value. Therefore, we focus on a generalized form of the Starobinsky inflation, the so-called R^{2p} model (with $p \approx 1$). These inflationary models were first proposed by [68,69] in the context of higher-derivative theories and subsequently were applied to inflation providing a simple and elegant generalization of the R^2 inflation [70–74]. Moreover, we consider the logarithmic corrected Starobinsky model $f(R) = R + \lambda R^2 + \nu R^2 \ln R$ where logarithmic corrections come from quantum gravity. Logarithmic $f(R)$ gravity can be considered a prototype model with quantum corrections, able to describe primordial and current accelerated expansions of the universe. Inflationary examples of logarithmic $f(R)$ models can be found in Refs. [2,71,75–90]. For other cosmological situations, see [79,91–100].

The above discussion motivates us to arrange the paper as follows. In Sect. 2, we introduce $f(R)$ gravity and its main properties in both Jordan and Einstein frames with the connecting relations of parameters in the two frames. Section 3 is devoted to the study of the constant-roll inflation in $f(R)$ gravity. In Sect. 4, we focus on the $f(R)$ models and obtain the corresponding potentials and also the inflationary parameters. In Sect. 5, we analyze the obtained results by comparing with the observational datasets coming from the Planck and the BICEP2/Keck array satellites. Also, we investigate the models from the viewpoint of the Weak Gravity Conjecture using the swampland criteria. In Sect. 6, we conclude the analysis of the models and draw our future outlooks.

2 $f(R)$ gravity in Jordan and Einstein frame

Let us start with the general form of $f(R)$ gravity action

$$S = \int d^4x \sqrt{-g} \frac{f(R)}{2} + \int d^4x \mathcal{L}_M(g_{\mu\nu}, \Psi_M) \quad (2)$$

where g is the determinant of the metric $g_{\mu\nu}$, $R = g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar as gravitational sector and $\mathcal{L}_M$ is the Lagrangian of matter fields Ψ_M filling the universe as perfect fluid [101–103]. Also, here we suppose $\kappa^2 \equiv 8\pi G = 1$. By varying the action (2) respect to the metric, the Einstein field equation is given by

$$\mathcal{G}_{\mu\nu} \equiv FR_{\mu\nu} - \frac{g_{\mu\nu}f}{2} - \nabla_\mu \nabla_\nu F + g_{\mu\nu} \square F = T_{\mu\nu}^m \quad (3)$$

where F , d'Alembert $\square$ and the energy-momentum tensor of matter fields $T_{\mu\nu}^m$ are expressed by

$$F \equiv \frac{df(R)}{dR}, \quad \square = \frac{1}{\sqrt{-g}} \partial_\nu [\sqrt{-g} g^{\mu\nu} \partial_\mu], \quad T_{\mu\nu}^m = -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}}. \quad (4)$$

The conservation law of the energy-momentum tensor $\nabla^\mu T_{\mu\nu}^m = 0$ is valid when $\nabla^\mu \mathcal{G}_{\mu\nu} = 0$. Also, the field Eq. (3) can be rewritten in terms of the Einstein tensor $G_{\mu\nu}$ as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu}^m + T_{\mu\nu}^c \quad (5)$$

where $T_{\mu\nu}^c$ including the terms coming from $f(R)$ modifications takes the following form

$$T_{\mu\nu}^c \equiv (1 - F)R_{\mu\nu} + \frac{1}{2}(f - R)g_{\mu\nu} + \nabla_\mu \nabla_\nu F - g_{\mu\nu} \square F \quad (6)$$

and the conservation law of the energy-momentum tensor $\nabla^\mu T_{\mu\nu}^m = 0$ is valid as a consequence of the Bianchi identities $\nabla^\mu G_{\mu\nu} = 0$ if $\nabla^\mu T_{\mu\nu}^c = 0$. Obviously, in the case of $f(R) = R$, both approaches reduce to Einstein gravity. In the following, we consider a spatially flat universe described by a Friedmann–Robertson–Walker (FRW) metric as

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2) \quad (7)$$

where t and a depict to cosmic time and scale factor, respectively. By using the above metric and the definition of the energy-momentum tensor of perfect fluid as gravitational and matter sectors of Eq. (3), we obtain the dynamical equations as

$$3FH^2 = \frac{(FR - f)}{2} - 3H\dot{F} + \rho_m, \quad 2F\dot{H} = -\ddot{F} + H\dot{F} - (\rho_m + P_m) \quad (8)$$

where $H \equiv \frac{\dot{a}}{a}$ is Hubble parameter and dot represents time derivation. Moreover, ρ_m and P_m are energy density and pressure of the perfect fluid. In the rest of the paper, we confine ourselves to inflationary analysis in $f(R)$ gravity by removing the role of matter fields in the main action (2) since we don't require to consider any type of matter in order to push inflation. Hence, the inflationary action in $f(R)$ gravity takes the simple form

$$S = \int d^4x \sqrt{-g} \frac{f(R)}{2} \quad (9)$$

with the dynamical equations

$$3FH^2 = \frac{(FR - f)}{2} - 3H\dot{F}, \quad 2F\dot{H} = -\ddot{F} + H\dot{F}. \quad (10)$$

By using the conformal transformation as a useful mathematical tool, we can move from Jordan frame (main frame) to Einstein frame in order to escape from difficulties of $f(R)$ gravity. The metrics in the two frames are connected with

$$\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \quad \text{with} \quad \Omega^2 = F = e^{\sqrt{\frac{2}{3}}\varphi}. \quad (11)$$

Hence, the form of action in the Einstein frame takes the standard form as

$$S_E = \int d^4x \left(\hat{R} - \frac{1}{2}g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right) \quad (12)$$

where hat denotes to the parameters in the Einstein frame. Clearly, in the Einstein frame, we deal with a scalar field φ so that called scalaron with the potential

$$V(\varphi) = \frac{RF - f}{2F^2}. \quad (13)$$

Conversely, the Ricci scalar and the function of $f(R)$ in the Jordan frame can be connected to the potential in the Einstein frame by

$$R = e^{\sqrt{\frac{2}{3}}\varphi} \left(\sqrt{6} \frac{\partial V}{\partial \varphi} + 4V \right), \quad f(R) = e^{2\sqrt{\frac{2}{3}}\varphi} \left(\sqrt{6} \frac{\partial V}{\partial \varphi} + 2V \right). \quad (14)$$

Moreover, the dynamical equations in the Einstein frame are defined by

$$\hat{H}^2 = \frac{1}{3} \left(\frac{1}{2} \left(\frac{d\varphi}{d\hat{t}} \right)^2 + V(\varphi) \right), \quad \frac{d\hat{H}}{d\hat{t}} = -\frac{1}{2} \left(\frac{d\varphi}{d\hat{t}} \right)^2, \quad \frac{d^2\varphi}{d\hat{t}^2} + 3\hat{H} \left(\frac{d\varphi}{d\hat{t}} \right) + \frac{\partial V}{\partial \varphi} = 0. \quad (15)$$

Also, the parameters in the two frames are connected by

$$H = e^{\frac{\varphi}{\sqrt{6}}} \left(\hat{H} - \frac{1}{\sqrt{6}} \frac{d\varphi}{d\hat{t}} \right), \quad \frac{d\varphi}{dt} = e^{\frac{\varphi}{\sqrt{6}}} \frac{d\varphi}{d\hat{t}}, \quad \frac{d^2\varphi}{dt^2} = e^{\sqrt{\frac{2}{3}}\varphi} \left(\frac{d^2\varphi}{d\hat{t}^2} + \frac{1}{\sqrt{6}} \left(\frac{d\varphi}{d\hat{t}} \right)^2 \right). \quad (16)$$

Note that the above expressions are obtained due to $dt = e^{-\frac{\varphi}{\sqrt{6}}} d\hat{t}$ and $a = e^{-\frac{\varphi}{\sqrt{6}}} \hat{a}$.

3 Constant-roll $f(R)$ inflation

Deviations from slow-roll approximation can be found when we restrict our attention to ultra slow-roll inflationary models in which $\alpha = 0$ or even fast-roll models in which a fast-rolling period is considered at the beginning of inflation and then will be transited to the slow-roll stage just after a few e-folds. As a new and more complete approach, we can introduce constant-roll inflation in which inflation rolls down with a constant rate from top of the potential to the minimum point. This viewpoint has been proposed to provide non-Gaussianity properties in the single field models since these models do not predict any non-Gaussianity in the context of slow-roll approximation. Recently, diverse inflationary models have been investigated in the presence of the constant-roll condition. In this paper, we focus on some inflationary models motivated by $f(R)$ gravity. Hence, we consider a natural generalization of the constant-roll condition (1) in $f(R)$ gravity expressed in the Jordan frame as

$$\ddot{F} = \beta H \dot{F} \quad (17)$$

which is reduced to the standard slow-roll approximation for $\beta = 0$. Since the inflationary potential of $f(R)$ is defined in the Einstein frame, we follow the mechanism in the Einstein frame using the connecting relations between two frames (16). By the definition of F (11), the constant-roll condition (17) in the Einstein frame is rewritten by

$$\frac{d^2\varphi}{d\hat{t}^2} + \frac{3+\beta}{\sqrt{6}} \left(\frac{d\varphi}{d\hat{t}} \right)^2 - \beta \hat{H} \frac{d\varphi}{d\hat{t}} = 0. \quad (18)$$

Then, by using the Einstein Eq. (15), the above equation takes the following form

$$\frac{d\hat{H}}{d\varphi} \left(\frac{d^2\hat{H}}{d\varphi^2} + \frac{3+\beta}{\sqrt{6}} \frac{d\hat{H}}{d\varphi} + \frac{\beta}{2} \hat{H} \right) = 0 \quad (19)$$

which includes two separated cases. The first case $\frac{d\hat{H}}{d\varphi} = 0$ is associated to the case of $\hat{H} = \text{const.}$ which leads to $V = \text{const.}$ and then $f(R) = R - \text{const.}$ The second case is a second-order differential equation with a general solution

$$\hat{H}(\varphi) = C \left(\gamma F^{-\frac{3}{2}} + F^{-\frac{\beta}{2}} \right) \quad (20)$$

and by using the Friedmann Eq. (15), the potential and the evolution of the scalaron are obtained as

$$V(\varphi) = \frac{3-\beta}{3} C^2 \left(6\gamma F^{-\frac{(3+\beta)}{2}} + (3+\beta) F^{-\beta} \right), \quad \frac{d\varphi}{d\hat{t}} = -2 \frac{d\hat{H}}{d\varphi} = \frac{2}{\sqrt{6}} \left(3\gamma F^{-\frac{3}{2}} + \beta F^{-\frac{\beta}{2}} \right) \quad (21)$$

where $F = e^{\sqrt{\frac{2}{3}}\varphi}$. The parameters C and γ are the integration constants mass dimension 1 and dimensionless, respectively. The amplitude of C can be fixed by the CMB normalization and we follow the paper in the unit where $C = 1$. Also, the value of γ can be normalized by redefining C and φ so that it can be considered for three value $+1, 0, -1$. In ref. [50], the authors claimed that only the parameters region with $\beta = -0.02$ and $\gamma = -1$ are viable, cosmologically. In other words, these parameters are situated in a region where inflation shows an attractor-like behavior. By combination of the potential (21) and the connection relations (14), the Ricci scalar takes the following form

$$R = (\beta - 3) \left(2\gamma(\beta - 1) e^{-\frac{(1+\beta)\varphi}{\sqrt{6}}} + \frac{2}{3}(\beta - 2)(\beta + 3) e^{\frac{2(1-\beta)\varphi}{\sqrt{6}}} \right). \quad (22)$$

Now, we can calculate the slow-roll parameters as

$$\epsilon \equiv \frac{1}{2} \left(\frac{V'}{V} \right)^2 = \frac{(3+\beta)^2 F'^2 \left(3\gamma F^{-\frac{(5+\beta)}{2}} + \beta F^{-(\beta+1)} \right)^2}{2 \left(6\gamma F^{-\frac{(3+\beta)}{2}} + (3+\beta) F^{-\beta} \right)^2}, \quad (23)$$

$$\eta \equiv \frac{V''}{V} = \frac{-(3+\beta) \left(F''(3\gamma F^{-\frac{(5+\beta)}{2}} + \beta F^{-(\beta+1)}) - F'^2 \left(\frac{3\gamma(5+\beta)}{2} F^{-\frac{(7+\beta)}{2}} + \beta(1+\beta) F^{-(2+\beta)} \right) \right)}{\left(6\gamma F^{-\frac{(3+\beta)}{2}} + (3+\beta) F^{-\beta} \right)}, \quad (24)$$

$$\zeta^2 \equiv \frac{V'V'''}{V^2} = \frac{(3+\beta)^2 F' \left(3\gamma F^{-\frac{(5+\beta)}{2}} + \beta F^{-(\beta+1)} \right)}{\left(6\gamma F^{-\frac{(3+\beta)}{2}} + (3+\beta) F^{-\beta} \right)^2}$$

$$\begin{aligned} & \times \left(F''' \left(3\gamma F^{-\frac{(5+\beta)}{2}} + \beta F^{-(\beta+1)} \right) - 3F'' F' \left(\frac{3\gamma(5+\beta)}{2} F^{-\frac{(7+\beta)}{2}} + \beta(1+\beta) F^{-(2+\beta)} \right) \right. \\ & \left. + F'^3 \left(\frac{3\gamma(5+\beta)(7+\beta)}{4} F^{-\frac{(9+\beta)}{2}} + \beta(1+\beta)(2+\beta) F^{-(3+\beta)} \right) \right) \end{aligned} \quad (25)$$

where prime implies to the derivation with respect to the scalar field φ and inflation ends when the condition $\epsilon = 1$ or $\eta = 1$ is fulfilled. Also, we can apply the swampland conditions $\sqrt{2\epsilon} \geq c$ and $|\eta| \leq -c'$, first derived in [104] and then in [105], for the $f(R)$ models by

$$\frac{(3+\beta)F' \left(3\gamma F^{-\frac{(5+\beta)}{2}} + \beta F^{-(\beta+1)} \right)}{\left(6\gamma F^{-\frac{(3+\beta)}{2}} + (3+\beta)F^{-\beta} \right)} \geq c, \quad (26)$$

$$\left| \frac{-(3+\beta) \left(F'' \left(3\gamma F^{-\frac{(5+\beta)}{2}} + \beta F^{-(\beta+1)} \right) - F'^2 \left(\frac{3\gamma(5+\beta)}{2} F^{-\frac{(7+\beta)}{2}} + \beta(1+\beta) F^{-(2+\beta)} \right) \right)}{\left(6\gamma F^{-\frac{(3+\beta)}{2}} + (3+\beta)F^{-\beta} \right)} \right| \leq c' \quad (27)$$

where c and c' are unit orders. Based on our purpose, we need to calculate the number of e-folds in the Einstein frame using

$$N \equiv \int_{\varphi_f}^{\varphi_i} \frac{1}{\sqrt{2\epsilon}} d\varphi = \int_{\varphi_f}^{\varphi_i} \frac{\left(6\gamma F^{-\frac{(3+\beta)}{2}} + (3+\beta)F^{-\beta} \right)}{(3+\beta)F' \left(3\gamma F^{-\frac{(5+\beta)}{2}} + \beta F^{-(\beta+1)} \right)} d\varphi \quad (28)$$

where the subscribes “ i ” and “ f ” denote the value of the scalaron field at the beginning and the end of inflation, respectively. The spectral parameters, i.e., the first order of spectral index, the first order of running spectral index and the tensor-to-scalar ratio can be obtained by

$$n_s = 1 - 6\epsilon + 2\eta, \quad \alpha_s = \frac{dn_s}{d \ln k} = 16\epsilon\eta - 24\epsilon^2 - 2\zeta^2, \quad r = 16\epsilon. \quad (29)$$

4 $f(R)$ models

In this section, we shall examine the constant-roll inflation for some $f(R)$ models, i.e., Starobinsky R^2 , R^{2p} and logarithmic corrected models.

4.1 Starobinsky R^2 model

First, we start with the Starobinsky R^2 model as the most successful inflationary model which contains a quadratic term coming from higher-order curvature given by

$$f(R) = R + \lambda R^2 \quad (30)$$

where λ is an arbitrary constant. By using the form of the Ricci scalar (22), $F = \frac{df}{dR}$ for the Starobinsky model can be driven by

$$F_1 = 1 + 2\lambda(\beta - 3) \left(2\gamma(\beta - 1)e^{-\frac{(1+\beta)\varphi}{\sqrt{6}}} + \frac{2}{3}(\beta - 2)(\beta + 3)e^{\frac{2(1-\beta)\varphi}{\sqrt{6}}} \right). \quad (31)$$

Now the Hubble parameter (20), the potential of scalar field and the evolution of the scalar field (21) are given by

$$\hat{H}_1(\varphi) = \left(\gamma F_1^{-\frac{3}{2}} + F_1^{-\frac{\beta}{2}} \right), \quad V_1(\varphi) = \frac{3-\beta}{3} \left(6\gamma F_1^{-\frac{(3+\beta)}{2}} + (3+\beta)F_1^{-\beta} \right), \quad \frac{d\varphi_1}{d\hat{t}} = \frac{2}{\sqrt{6}} \left(3\gamma F_1^{-\frac{3}{2}} + \beta F_1^{-\frac{\beta}{2}} \right). \quad (32)$$

Now let's review the appropriate plots of the Starobinsky R^2 model. The panel (a) of Fig. 1 presents the behavior of scalaron potential (32) for three values of $\gamma = +1, 0, -1$ with $\beta = -0.02$ and $\lambda = 1$. For $\gamma = +1$, the potential rolls down to a minimum point and then tilts upwards. For $\gamma = 0$, the potential moves in a steady manner from negative points of scalaron and then gradually shows an increasing behavior. For $\gamma = -1$, the potential starts moving from the origin and tilts upward with a decreasing slope. Besides this information, the panel (a) also reveals that all three values of γ act the same with a constant rate at the last steps of inflation. The panels (b) and (c) of Fig. 1 show the phase diagram of scalaron and the behavior of the Hubble parameter (32) for the values $\gamma = +1, 0, -1$ with $\beta = -0.02$ and $\lambda = 1$, respectively. Notice that in the rest of paper, we study the inflationary paradigm in the

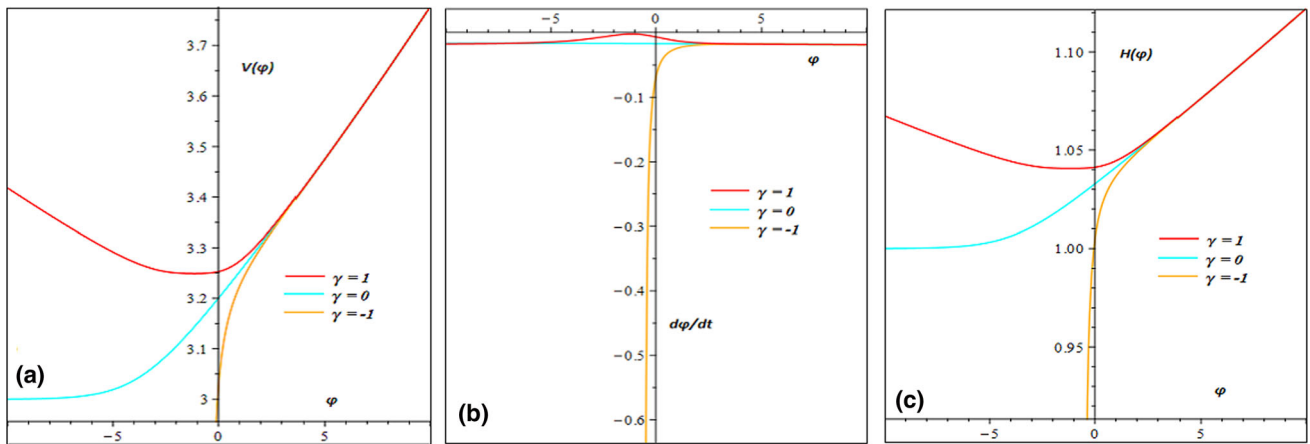


Fig. 1 The potential, evolution of scalaron and Hubble parameter (32) plotted versus φ for $\gamma = +1, 0, -1$ with $\beta = -0.02$ and $\lambda = 1$

case of $\gamma = 0$. Now let's find the slow-roll parameters (23–25) of the Starobinsky R^2 model for the case of $\gamma = 0$ by

$$\epsilon = \frac{\beta^2}{2} \left(\frac{F_1'}{F_1} \right)^2, \quad \eta = -\beta \left(\frac{F_1''}{F_1} - (\beta + 1) \left(\frac{F_1'}{F_1} \right)^2 \right), \quad \zeta^2 = \beta^2 \frac{F_1'}{F_1^2} \left(F_1''' - 3(\beta + 1) \frac{F_1'' F_1'}{F_1} + (\beta + 1)(\beta + 2) \frac{F_1'^3}{F_1^2} \right). \quad (33)$$

Also, the number of e-folds (28) of the model can be found as

$$N = \int_{\varphi_f}^{\varphi_i} \frac{1}{\beta} \frac{F_1}{F_1'} d\varphi = \frac{9}{8\lambda\beta(\beta - 1)^2(\beta - 2)(3 - \beta)(\beta + 3)} \left(e^{\frac{2(\beta - 1)\varphi_i}{\sqrt{6}}} - e^{\frac{2(\beta - 1)\varphi_f}{\sqrt{6}}} \right) + \frac{\sqrt{6}}{2\beta(1 - \beta)} (\varphi_i - \varphi_f) \quad (34)$$

and then by neglecting the value of scalar field at the end of inflation ($\varphi_f \ll \varphi_i$), the above expression is reduced to

$$N \simeq \frac{9}{8\lambda\beta(\beta - 1)^2(\beta - 2)(3 - \beta)(\beta + 3)} e^{\frac{2(\beta - 1)\varphi_i}{\sqrt{6}}}. \quad (35)$$

Now using the slow-roll parameters (33) and the above expression for the number of e-folds, the inflationary parameters (29) of the Starobinsky R^2 model can be calculated as shown in “Appendix (A1–A3)”.

4.2 R^{2p} model

As second case, we study R^{2p} model as a generalization of Starobinsky R^2 model which was first introduced in the context of higher derivative theories as

$$f(R) = R + \lambda R^{2p} \quad (36)$$

where λ and p are free parameters. In this case, F is obtained by

$$F_2 = 1 + 2p\lambda \left(2\gamma(\beta - 3)(\beta - 1)e^{-\frac{(1+\beta)\varphi}{\sqrt{6}}} + \frac{2}{3}(\beta - 3)(\beta - 2)(\beta + 3)e^{\frac{2(1-\beta)\varphi}{\sqrt{6}}} \right)^{2p-1} \quad (37)$$

and now the Hubble parameter (20), the potential of scalaron and also its evolution (21) can be expressed by

$$\hat{H}_2(\varphi) = \left(\gamma F_2^{-\frac{3}{2}} + F_2^{-\frac{\beta}{2}} \right), \quad V_2(\varphi) = \frac{3 - \beta}{3} \left(6\gamma F_2^{-\frac{(3+\beta)}{2}} + (3 + \beta) F_2^{-\beta} \right), \quad \frac{d\varphi_2}{d\hat{t}} = \frac{2}{\sqrt{6}} \left(3\gamma F_2^{-\frac{3}{2}} + \beta F_2^{-\frac{\beta}{2}} \right). \quad (38)$$

Figure 2 shows the potential, the evolution of scalaron and the Hubble parameter (38) versus φ for different values of p in R^{2p} model with $\gamma = -1$, $\beta = -0.02$ and $\lambda = 1$. From the panel (a), we recover the Starobinsky model in the case of $p = 1$. For the values $p = 1.05$ and $p = 0.95$, the potential behaves the same with $p = 1$. It seems that a tiny deviation from the Starobinsky model $p = 1$ does not provide a remarkable change when inflation is considered in the context of constant-roll approach. This fact also can be understood from the panels (b) and (c) which show the phase diagram of the scalaron and the Hubble parameter of the model, respectively. The slow-roll parameters (23–25) of the R^{2p} model for the case of $\gamma = 0$ are given by

$$\epsilon = \frac{\beta^2}{2} \left(\frac{F_2'}{F_2} \right)^2, \quad \eta = -\beta \left(\frac{F_2''}{F_2} - (\beta + 1) \left(\frac{F_2'}{F_2} \right)^2 \right), \quad \zeta^2 = \beta^2 \frac{F_2'}{F_2^2} \left(F_2''' - 3(\beta + 1) \frac{F_2'' F_2'}{F_2} + (\beta + 1)(\beta + 2) \frac{F_2'^3}{F_2^2} \right) \quad (39)$$

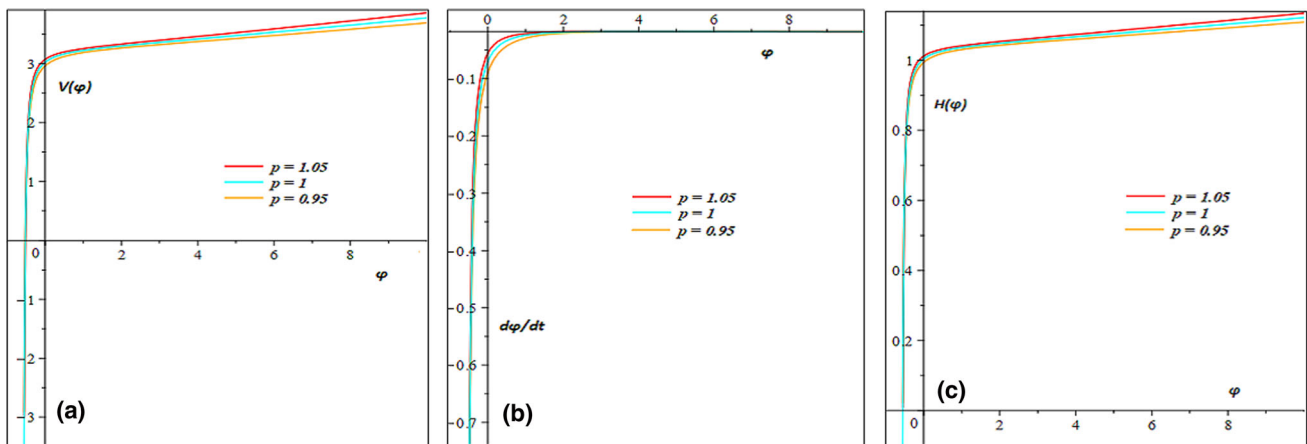


Fig. 2 The potential, evolution of scalaron and Hubble parameter (38) plotted versus φ for $p = 1.05, 1, 0.95$ with $\gamma = -1$, $\beta = -0.02$ and $\lambda = 1$

and the number of e-folds (28) of the R^{2p} model can be calculated as

$$N = \int_{\varphi_f}^{\varphi_i} \frac{1}{\beta} \frac{F_2}{F_2'} d\varphi = \frac{-3 \left(\frac{2}{3} (\beta - 2)(\beta - 3)(\beta + 3) \right)^{1-2p}}{4\lambda\beta p(2p-1)^2(1-\beta)^2} \left[\left(e^{\frac{2(1-\beta)\varphi_i}{\sqrt{6}}} \right)^{1-2p} - \left(e^{\frac{2(1-\beta)\varphi_f}{\sqrt{6}}} \right)^{1-2p} \right] + \frac{3(\varphi_i - \varphi_f)}{\sqrt{6}\beta(1-\beta)(2p-1)} \quad (40)$$

and by removing the role of scalar field when inflation ends ($\varphi_f \ll \varphi_i$), N takes the following form

$$N \simeq \frac{-3 \left(\frac{2}{3} (\beta - 2)(\beta - 3)(\beta + 3) e^{\frac{2(1-\beta)\varphi_i}{\sqrt{6}}} \right)^{1-2p}}{4\lambda\beta p(2p-1)^2(1-\beta)^2}. \quad (41)$$

By combining the slow-roll parameters (39) and the number of e-folds (41), the inflationary parameters (29) of the R^{2p} model are calculated as shown in (A4–A6).

4.3 Logarithmic corrected model

Let us consider now a logarithmic corrected $f(R)$ model

$$f(R) = R + \lambda R^2 + \nu R^2 \ln R \quad (42)$$

where the phenomenological parameters λ, ν will be fixed momentarily. In fact, the extra terms come from the leading quantum gravity corrections. See [106, 107] for details. For the model (42), F takes the following form

$$F_3 = 1 + \left(2\gamma(\beta - 3)(\beta - 1)e^{-\frac{(1+\beta)\varphi}{\sqrt{6}}} + \frac{2}{3}(\beta - 3)(\beta - 2)(\beta + 3)e^{\frac{2(1-\beta)\varphi}{\sqrt{6}}} \right) \times \left\{ 2\lambda + \nu + 2\nu \ln \left(2\gamma(\beta - 3)(\beta - 1)e^{-\frac{(1+\beta)\varphi}{\sqrt{6}}} + \frac{2}{3}(\beta - 3)(\beta - 2)(\beta + 3)e^{\frac{2(1-\beta)\varphi}{\sqrt{6}}} \right) \right\} \quad (43)$$

and now the Hubble parameter (20), then the potential and also the evolution of scalaron (21) are obtained by

$$\hat{H}_3(\varphi) = \left(\gamma F_3^{-\frac{3}{2}} + F_3^{-\frac{\beta}{2}} \right), \quad V_3(\varphi) = \frac{3-\beta}{3} \left(6\gamma F_3^{-\frac{(3+\beta)}{2}} + (3+\beta)F_3^{-\beta} \right), \quad \frac{d\varphi_3}{d\hat{t}} = \frac{2}{\sqrt{6}} \left(3\gamma F_3^{-\frac{3}{2}} + \beta F_3^{-\frac{\beta}{2}} \right). \quad (44)$$

The panel (a) of Fig. 3 exhibits the behavior of potential and also the phase diagram of the scalaron (44) for different signs of ν in the logarithmic $f(R)$ model. The potential behaves almost the same for different signs of ν . However, $\nu < 0$ shows a different behavior at the end of inflation compared to two other cases. Similar to the previous models, the potential shows a constant rate of rolling which is connected to the standard slow-roll. The slow-roll parameters (23–25) of the logarithmic corrected model for the case of $\gamma = 0$ are obtained as

$$\epsilon = \frac{\beta^2}{2} \left(\frac{F_3'}{F_3} \right)^2, \quad \eta = -\beta \left(\frac{F_3''}{F_3} - (\beta + 1) \left(\frac{F_3'}{F_3} \right)^2 \right), \quad \zeta^2 = \beta^2 \frac{F_3'}{F_3^2} \left(F_3''' - 3(\beta + 1) \frac{F_3'' F_3'}{F_3} + (\beta + 1)(\beta + 2) \frac{F_3'^3}{F_3^3} \right) \quad (45)$$

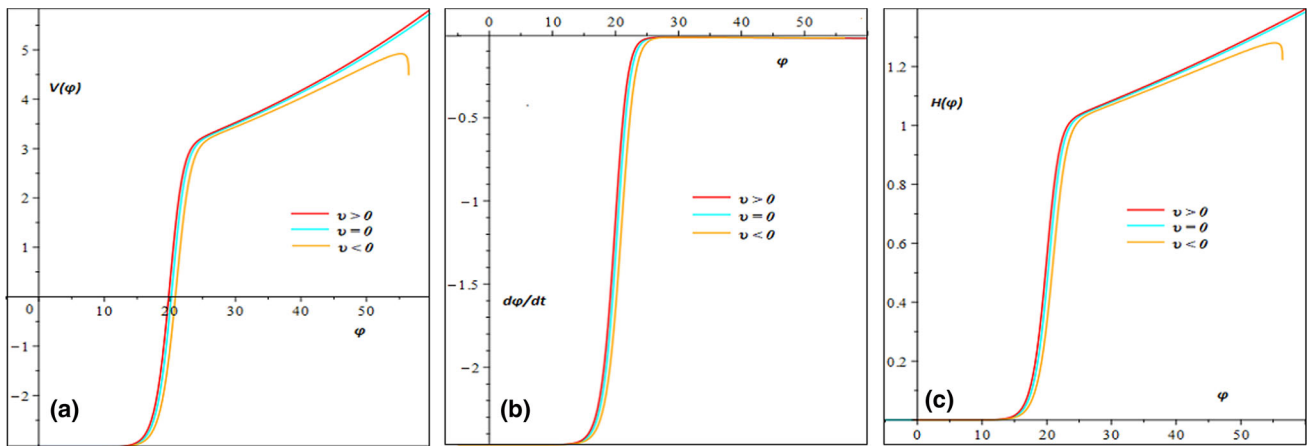


Fig. 3 The potential, evolution of scalaron and Hubble parameter (44) plotted versus φ for different signs of $\nu = 0, [0.02]\lambda$ with $\lambda = 1.2 \times 10^{-9}$, $\gamma = -1$ and $\beta = -0.02$

and the number of e-folds (28) of the logarithmic model discussed in this section is introduced by

$$N = \int_{\varphi_f}^{\varphi_i} \frac{1}{\beta} \frac{F_3}{F_3'} d\varphi = \frac{6e^{\frac{2\lambda+3\nu}{2\nu}}}{8\beta(1-\beta)^2\nu} \text{Ei} \left(-\frac{3\nu + 2\lambda + 2\nu \ln \left(\frac{2}{3}(\beta-1)(\beta-3)(\beta+3)e^{\frac{2(1-\beta)\varphi}{\sqrt{6}}} \right)}{2\nu} \right) \Big|_{\varphi_f}^{\varphi_i} \quad (46)$$

where Ei is the exponential integral that can be expressed by the Puiseux series

$$\text{Ei}(x) = \rho + \ln(x) + x + \frac{x^2}{4} + \frac{x^3}{18} + \frac{x^4}{96} + \dots \quad \text{with} \quad e^{2\rho} = 3.17221895... \quad (47)$$

along the positive real axis. Finally, by keeping the logarithmic term and also neglecting the value of scalaron at the end of inflation ($\varphi_f \ll \varphi_i$), N is written as

$$N \simeq \frac{6e^{\frac{2\lambda+3\nu}{2\nu}}}{8\beta(1-\beta)^2\nu} \ln \left\{ -\frac{3\nu + 2\lambda + 2\nu \ln \left(\frac{2}{3}(\beta-1)(\beta-3)(\beta+3)e^{\frac{2(1-\beta)\varphi_i}{\sqrt{6}}} \right)}{2\nu} \right\}. \quad (48)$$

By considering the slow-roll parameters (45) and the number of e-folds (48), the inflationary parameters (29) of the logarithmic corrected model are obtained as shown in (A8–A10).

5 Comparison with observations

In this section, we compare the obtained results of $f(R)$ models considered in the paper with the inflationary observations coming from temperature and polarization anisotropies of CMB.

In Fig. 4, we present the $n_s - r$ constraints coming from the marginalized joint 68% and 95% CL regions of the Planck 2018 in combination with BK15 and BAO data on the $f(R)$ inflationary models, i.e., the Starobinsky R^2 model, the R^{2p} model and the logarithmic corrected model described in the context of the constant-roll approach. The panels are drawn for some allowed values of β in the cases $N = 50$ (dashed line) and $N = 60$ (solid line). Panel (a) is belonged to the Starobinsky R^2 model (30) when $\gamma = 0$ in Eq. (20). As we can see, the obtained value of tensor-to-scalar ratio r for the allowed values of β is in good agreement with the Planck 2018 [51] constraint $r < 0.064$, in particular, for the cases of $\beta = 0.05$ and $\beta = 0.07$ which show $r = 0.047341$ and $r = 0.021806$, respectively. The negative cases of β predict a tiny tensor-to-scalar ratio $r = \mathcal{O}(10^{-3})$ which is beyond the reach of the current observations. Despite the mentioned success, the Starobinsky R^2 model does not show a scale-invariant spectrum since the value of the spectral index n_s for the allowed values of β is bigger than the unit. Panel (b) is dedicated to show the Planck constraints on the R^{2p} model (36) when $\gamma = 0$ in Eq. (20) and $\lambda = 1$ for two interesting cases of $p = 1.05$ and $p = 0.95$. The panel shows that by considering a tiny variation from the original Starobinsky model ($p = 1$), the obtained values of r are still in good agreement with the Planck constraint in exception the case of $\beta = 0.05$ in $p = 0.95$ which presents a disfavored value $r \simeq 0.077$. Also, the negative values of β reveal a tiny tensor-to-scalar ratio $r = \mathcal{O}(10^{-3})$ for both cases of p . Similar to the Starobinsky model, the R^{2p} model shows observationally disagreeable values of the spectral index $n_s > 1$ for both cases of p . In overall, by taking a look at the obtained values of r and n_s , we find that the case of $p = 0.95$ is more desirable than $p = 1.05$ since the values of

Fig. 4 The marginalized joint 68% and 95% CL regions for n_s and r at $k = 0.002 \text{ Mpc}^{-1}$ from Planck 2018 combining with BK15+BAO data [51] and the $n_s - r$ constraints on different $f(R)$ inflationary models described in the context of the constant-roll idea. The dashed and solid lines represent $N = 50$ and $N = 60$, respectively. The results are obtained for some allowed values of β when $\gamma = 0$ for Starobinsky R^2 model (a), $\gamma = 0$ and $\lambda = 1$ for R^{2p} model (b), $\gamma = 0$ and $\nu = 0.02\lambda$ for logarithmic corrected model (c)

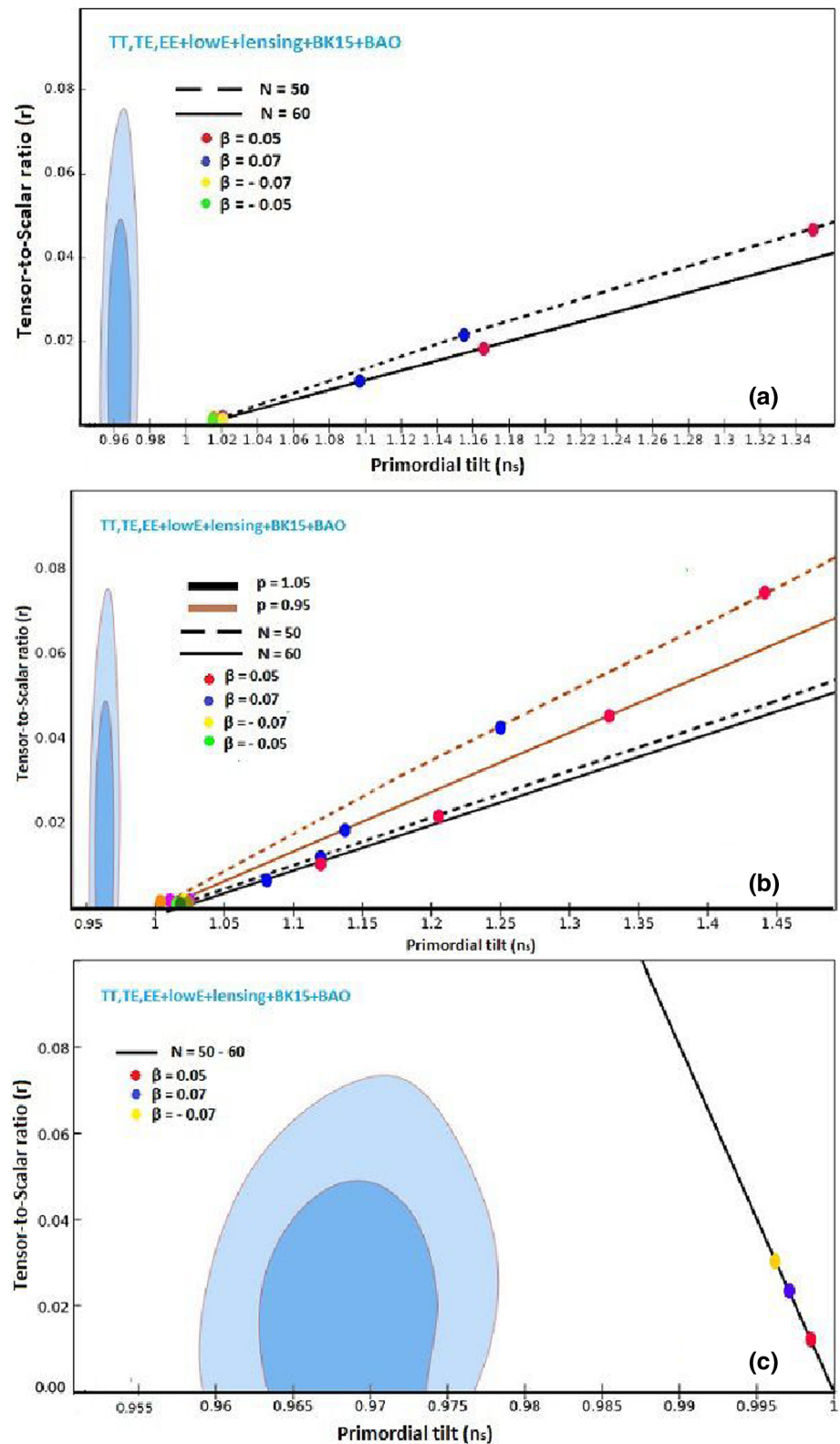


Table 1 The values of the swampland parameters c and c' coming from the swampland conditions (26) and (27) for different $f(R)$ inflationary models described in the context of the constant-roll approach

Model	β	N	$c \leq$	$c' \leq$
Starobinsky R^2	0.05	50–60	0.077065	−0.18394
	0.07	50–60	0.052303	−0.081309
	−0.07	50–60	0.016687	−0.010866
	−0.05	50–60	0.015134	−0.0086153
$R^{2p}(p = 1.05)$	0.05	50–60	0.052024	−0.101226
	0.07	50–60	0.04055	−0.059005
	−0.07	50–60	0.015865	−0.011918
	−0.05	50–60	0.014602	−0.0097295
$R^{2p}(p = 0.95)$	0.05	50–60	0.15984	−0.64812
	0.07	50–60	0.075377	−0.13836
	−0.07	50–60	0.017365	−0.0096579
	−0.05	50–60	0.015475	−0.0073956
Logarithmic corrected	0.05	50–60	0.039158	−0.0015361
	0.07	50–60	0.053666	−0.0028838
	−0.07	50–60	0.061746	−0.0038075

The results are obtained for some allowed values of β with $N = 50$ when $\gamma = 0$ for the Starobinsky R^2 model, $\gamma = 0$ and $\lambda = 1$ for the R^{2p} model, $\gamma = 0$ and $\nu = 0.02\lambda$ for the logarithmic corrected model

r are closer to the upper limit ($r < 0.064$) of the current observations. In contrast, by focusing on the n_s , one can find that the deviation from $n_s = 1$ in the case of $p = 1.05$ is remarkably less than the case of $p = 0.95$. Finally, panel (c) is corresponded to the interesting case of the logarithmic corrected model (42) when $\gamma = 0$ in Eq. (20) and $\nu = 0.02\lambda$. From the panel, we can see that the obtained values of r for the allowed values of β are compatible with the observational constraint coming from Planck 2018. This is valid for both positive and negative values of $\beta = 0.05, 0.07$ and -0.07 which show $r = 0.012266, 0.023041$ and 0.0305 , respectively. Concerning the spectral index, the panel tells us that the logarithmic corrected model shows a scale-invariant spectrum ($n_s < 1$) but still situated in a disfavorate region $0.996 \leq n_s \leq 0.999$ which is far from the Planck constraint $n_s = 0.9649 \pm 0.0042$. In comparison with the previous models, here the negative values of β predict more desirable results of r and n_s than the positive values of β .

On the other hand, we can study the issue from the viewpoint of the swampland criteria (26) and (27). In Table 1, we present the values of the swampland parameters c and c' for all three $f(R)$ constant-roll models. The results are obtained for some allowed values of β with N between 50 and 60 when $\gamma = 0$ for the Starobinsky R^2 model, $\gamma = 0$ and $\lambda = 1$ for the R^{2p} model and $\gamma = 0$ and $\nu = 0.02\lambda$ for the logarithmic corrected model. Also, we investigate the behavior of the swampland parameters c and c' versus the tensor-to-scalar ratio r for the considered models in Fig. 5. Now, let's review the swampland conditions for our $f(R)$ constant-roll inflationary models.

In the Starobinsky R^2 model, we find that the swampland condition of the parameter c shown in the table leads to some desirable values of the tensor-to-scalar ratio r , in particular, for the positive cases of β and this coincidence is distorted for tiny values of c since the corresponding values of r are not situated in the range of our current observations (see the first panel of Fig. 5). For another swampland parameter c' , the table tells us in the case of $\beta = 0.05$, c' takes the value -0.18394 which is connected to an observationally acceptable value $r \simeq 0.05$. By considering the swampland condition c' , the value of r approaches the observational upper limit and then reaches disfavored regions of r by crossing from the limit. For other values of β , we find that the values of c' are so tiny and lead to the small values of r . By setting the swampland condition c' , the situation becomes better since the corresponding value of r approaches the upper limit (see panel (a) of Fig. 5). In the R^{2p} model with $p = 1.05$, the situation of c is almost similar to the Starobinsky R^2 model while the parameter c' reveals some interesting features. The values of $\beta = 0.05, 0.07, -0.05$ show $r \simeq 0.02, 0.01, 0.02$ which are close to the lower limit of the observations and by considering the swampland condition, r approaches the upper limit. For $\beta = -0.07$, we find $r \simeq 0.7$ which is situated beyond the upper limit (see panel (b) of Fig. 5). In the case of $p = 0.95$, the parameter c behaves analogous to the Starobinsky R^2 model in exception the case of $\beta = 0.05$ which leads to r bigger than the upper limit ($r < 0.064$). Also, for $\beta = 0.07, -0.07, -0.05$, the parameter c' shows the values of $r \simeq 0.03, 0.05, 0.02$ so that by setting the swampland condition, they approach the upper limit. Note that for $\beta = 0.05$, the parameter c' predicts a very large r which is not compatible with the observations. In the logarithmic corrected model, the obtained values of c present r below the upper limit and again by considering the swampland condition, they approach the lower limit. For the parameter c' , $\beta = -0.07$ shows $r \simeq 0.04$ while $\beta = 0.05, 0.07$ predict tiny r (see panel (c) of Fig. 5).

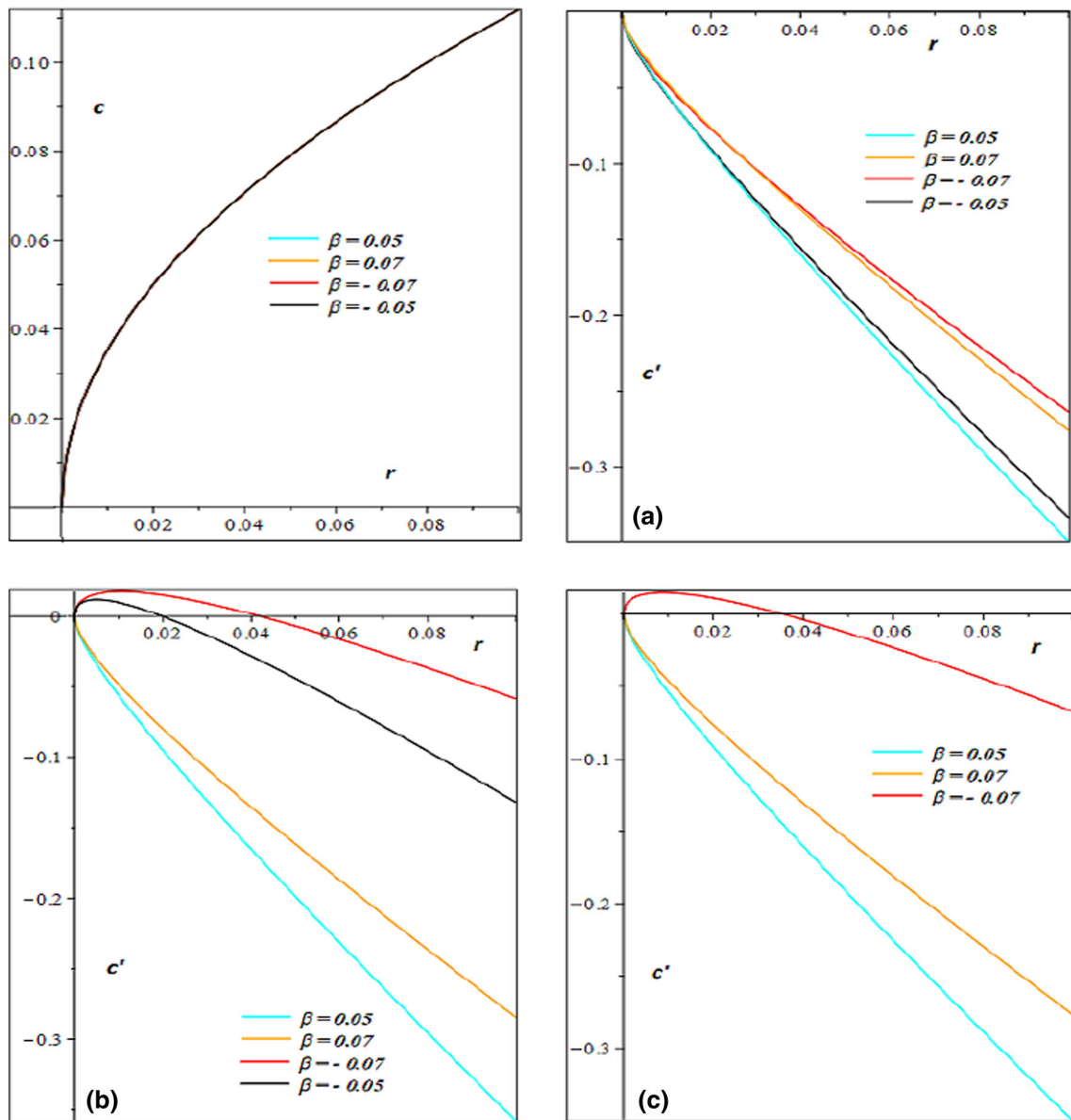


Fig. 5 The behavior of the swampland parameters c and c' (26) and (27) versus the tensor-to-scalar ratio r for different $f(R)$ inflationary models described in the context of the constant-roll approach. The results are obtained for some allowed values of β when $\gamma = 0$ for the Starobinsky R^2 model (a), $\gamma = 0$ and $\lambda = 1$ for the R^{2p} model (b), $\gamma = 0$ and $\nu = 0.02\lambda$ for the logarithmic corrected model (c)

6 Discussion and conclusions

In the present work, we focused on some $f(R)$ inflationary models, i.e., the Starobinsky R^2 , the R^{2p} and the logarithmic corrected models introduced in the context of the constant-roll where inflation rolls down with a constant rate as $\ddot{\phi} = \beta H \dot{\phi}$. We have investigated the inflationary dynamics for the three $f(R)$ models in presence of constant-roll condition. Specifically, we calculated the spectral parameters of the models, i.e., the spectral index, its running and the tensor-to-scalar ratio. Then, we compared the obtained results with Planck 2018 data combined with BK15+BAO data. Results can be summarized as follows :

- We studied the Starobinsky R^2 model for $\gamma = 0$ and found that, for positive values of β , r is in good agreement with observations while, for negative values of β , it shows a tiny $r = \mathcal{O}(10^{-3})$. Furthermore, the Starobinsky model does not present a scale-invariant spectrum since the obtained values of the spectral index n_s are bigger than the unit.
- In the R^{2p} model, we investigated the model for the two cases $p = 1.05$ and $p = 0.95$ when $\gamma = 0$ and $\lambda = 1$. We found that the obtained values of r are compatible with the observational constraint on r , except the case $\beta = 0.05$ in $p = 0.95$ which predicts $r \simeq 0.077$ bigger than the upper limit ($r < 0.064$). Also, a negative β presents a tiny $r = \mathcal{O}(10^{-3})$ for both cases of p . Moreover, the model shows the spectral index $n_s > 1$ which is not in good agreement with the Planck constraint. In summary,

we found the case of $p = 0.95$ more desirable than $p = 1.05$ since the values of r are closer to the upper limit ($r < 0.064$). In contrast, the deviation from $n_s = 1$ in the case of $p = 1.05$ is less than the case of $p = 0.95$.

- As a modified form of the Starobinsky model, we have considered the logarithmic corrected model when $\gamma = 0$ and $\nu = 0.02\lambda$. We have found that the obtained values of r for positive and negative values of β are in good agreement with the observations. Also, the model predicts the spectral index $n_s < 1$ but it is still situated in a disfavored region $0.996 < n_s < 0.999$.

According to the above results, one can conclude that although all three constant-roll $f(R)$ inflationary models, i.e., the Starobinsky R^2 model, the R^{2p} model and the logarithmic corrected model are observationally compatible with the values of r , they do not show acceptable values for the spectral index n_s . Consequently, the considered constant-roll $f(R)$ inflationary models are disfavored models when $\gamma = 0$ in Eq. (20). This result can be referred to Ref. [50] where the authors studied the behavior of constant-roll $f(R)$ inflation in a general approach. They found that the model can lead to an inflationary solution only for $-0.1 \leq \beta \leq 0$ when $\gamma = -1$ in Eq. (20).

Furthermore, we have studied the models from the viewpoint of the Weak Gravity Conjecture using the swampland criteria. By setting the condition of c , we found that the scalar-to-tensor ratio r starts from an observationally acceptable value and approaches the lower limit. Finally, by crossing from the lower limit, it tilts to the disfavored areas of r . Also, by setting the condition of c' , the value of r begins from a desirable value and finally, by crossing from the upper limit, it shifts to the disagreeable regions. Hence, we can conclude that our $f(R)$ inflationary models introduced in the context of the constant-roll idea are not fully compatible with the swampland criteria.

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Appendix A: The spectral parameters of $f(R)$ models

1. Starobinsky R^2 model

The spectral index of the Starobinsky R^2 model:

$$n_s = \frac{1}{(2N\beta^3 - 4N\beta^2 + 2N\beta - 3)^2} \left\{ (4N^2 + 8N)\beta^6 - (16N^2 + 32N)\beta^5 + (24N^2 + 48N - 6)\beta^4 + (-16N^2 - 44N + 12)\beta^3 + (4N^2 + 32N - 6)\beta^2 - 12\beta N + 9 \right\}. \quad (\text{A1})$$

The tensor-to-scalar ratio of the Starobinsky R^2 model:

$$r = \frac{48(\beta - 1)^2\beta^2}{(2N\beta^3 - 4N\beta^2 + 2N\beta - 3)^2}. \quad (\text{A2})$$

The running spectral index of the Starobinsky R^2 model:

$$\alpha_s = -\frac{32N(N\beta^2 - \beta N - \frac{3}{2})\beta^3(\beta - 1)^7}{(2N\beta^3 - 4N\beta^2 + 2N\beta - 3)^4}. \quad (\text{A3})$$

2. R^{2p} model

The spectral index of the R^{2p} model:

$$n_s = \frac{1}{\left(3p\lambda 4^p 9^{-p} \mathcal{W}^{-1} \left((\beta - 2)(\beta^2 - 9) \right)^{2p} + \beta^3 - 2\beta^2 - 9\beta + 18 \right)^2} \left\{ -3p^2 4^{2p} 9^{-2p} \lambda^2 \mathcal{W}^{-2} \right. \\ \times \left(-3 + \left(p - \frac{1}{2} \right)^2 (16\beta^3 - 32\beta^2 + 16\beta) \right) \left((\beta - 2)(\beta^2 - 9) \right)^{4p} - 16(\beta - 3)(\beta - 2) 4^p 9^{-p} \lambda p (\beta + 3) \mathcal{W}^{-1} \\ \times \left(-\frac{3}{8} + \left(p - \frac{1}{2} \right)^2 (\beta^3 - 2\beta^2 + \beta) \right) \left((\beta - 2)(\beta^2 - 9) \right)^{2p} - 24(\beta - 2) \left(\left(p - \frac{1}{2} \right)^2 \beta \mathcal{W}^{-2} \lambda^2 p^2 81^{-p} 16^p \right. \\ \left. \left. \times (\beta - 1)^2 \left((\beta - 2)(\beta^2 - 9) \right)^{4p} - \frac{(\beta - 2)(\beta - 3)^2 (\beta + 3)^2}{24} \right) \right\}. \quad (\text{A4})$$

The tensor-to-scalar ratio of the R^{2p} model:

$$r = \frac{192(p - \frac{1}{2})^2 \beta^2 \left((\beta - 2)(\beta^2 - 9) \right)^{4p} \mathcal{W}^{-2} \lambda^2 p^2 81^{-p} (\beta - 1)^2 16^p}{\left(3p\lambda 4^p 9^{-p} \mathcal{W}^{-1} \left((\beta - 2)(\beta^2 - 9) \right)^{2p} + \beta^3 - 2\beta^2 - 9\beta + 18 \right)^2}. \quad (\text{A5})$$

The running spectral index of the R^{2p} model:

$$\alpha_s = \frac{1}{\left(-3p\lambda 4^p \mathcal{W}^{-1} \left((\beta - 2)(\beta^2 - 9) \right)^{2p} + 3^{(2+2p)} \beta - (\beta^3 - 2\beta^2 + 18) 9^p \right)^4} \left\{ -128 \left(p - \frac{1}{2} \right)^4 \beta^2 p^2 \right. \\ \times (\beta - 2)(\beta - 3)(\beta + 3)(\beta - 1)^4 \lambda^2 \left(\mathcal{W}^{-2} 1296^p (\beta - 2)(\beta - 3)(\beta + 3) \left((\beta - 2)(\beta^2 - 9) \right)^{4p} + 3p 64^p 9^p \lambda \right. \\ \left. \left. \times \mathcal{W}^{-3} (\beta - 1) \left((\beta - 2)(\beta^2 - 9) \right)^{6p} \right) \right\} \quad (\text{A6})$$

where

$$\mathcal{W} = -\frac{2\beta p \lambda 9^{-p} (-1 + b)^2 (2p - 1)^2 N 4^p (\beta^3 - 2\beta^2 - 9\beta + 18)^{2p}}{(\beta - 2)(\beta^2 - 9)}. \quad (\text{A7})$$

3. Logarithmic corrected model

The spectral index of the Logarithmic corrected model:

$$n_s = \frac{1}{12 \left(e^{\mathcal{Z}} v (\beta - 2) \ln \left(\frac{(\beta - 2)e^{\mathcal{Z}}}{\beta - 1} \right) + \left(\lambda + \frac{v}{2} \right) (\beta - 2) e^{\mathcal{Z}} + \frac{(\beta - 1)}{2} \right)^2} \left\{ -8e^{2\mathcal{Z}} v^2 \ln \left(\frac{(\beta - 2)e^{\mathcal{Z}}}{\beta - 1} \right)^2 \right. \\ \times (\beta - 2)^2 (\beta^4 - 2\beta^3 + \beta^2 - \frac{3}{2}) - 16 \left(\left(\left(\lambda + \frac{3v}{2} \right) \beta^4 - (2\lambda + 3v) \beta^3 + \left(\lambda + \frac{3v}{2} \right) \beta^2 + -\frac{3}{4} (2\lambda + v) \right) (\beta - 2) e^{2\mathcal{Z}} \right. \\ \left. + \frac{(\beta^3 - 2\beta^2 + \beta - \frac{3}{2}) e^{\mathcal{Z}} (\beta - 1)}{2} \right) (\beta - 2) v \ln \left(\frac{(\beta - 2)e^{\mathcal{Z}}}{\beta - 1} \right) - 8 \left(\left(\lambda + \frac{3v}{2} \right)^2 \beta^4 - \left(2\lambda^2 + 6\lambda v + \frac{13}{2} v^2 \right) \beta^3 \right. \\ \left. + \left(\lambda^2 + 3\lambda v + \frac{25}{4} v^2 \right) \beta^2 - 2\beta v^2 - \frac{3 \left(\lambda + \frac{v}{2} \right)^2}{2} \right) (\beta - 2)^2 e^{2\mathcal{Z}} - 8 \left((\beta - 2) \left(\left(\lambda + \frac{5v}{2} \right) \beta^3 - (2\lambda + 5v) \beta^2 + \left(\lambda + \frac{5v}{2} \right) \beta \right. \right. \\ \left. \left. - \frac{3}{4} (2\lambda + v) e^{\mathcal{Z}} - \frac{3}{8} (\beta - 1) \right) (\beta - 1) \right) \right\}. \quad (\text{A8})$$

The tensor-to-scalar ratio of the Logarithmic corrected model:

$$r = \frac{16\beta^2 (\beta - 2)^2 \left(v \ln \left(\frac{(\beta - 2)e^{\mathcal{Z}}}{\beta - 1} \right) + \lambda + \frac{3v}{2} \right)^2 e^{2\mathcal{Z}} (\beta - 1)^2}{3 \left(e^{\mathcal{Z}} v (\beta - 2) \ln \left(\frac{(\beta - 2)e^{\mathcal{Z}}}{\beta - 1} \right) + \left(\lambda + \frac{v}{2} \right) (\beta - 2) e^{\mathcal{Z}} + \frac{\beta - 1}{2} \right)^2}. \quad (\text{A9})$$

The running spectral index of the Logarithmic corrected model:

$$\alpha_s = -\frac{1}{9 \left(e^{\mathcal{Z}} v (\beta - 2) \ln \left(\frac{(\beta - 2)e^{\mathcal{Z}}}{\beta - 1} \right) + \left(\lambda + \frac{v}{2} \right) (\beta - 2) e^{\mathcal{Z}} + \frac{(\beta - 1)}{2} \right)^4} \left\{ 4\beta^2 \left(e^{\mathcal{Z}} v^2 (\beta - 2) (\beta - 1)^2 \ln \left(\frac{(\beta - 2)e^{\mathcal{Z}}}{(\beta - 1)} \right)^2 \right. \right. \\ \left. + 2 \left(-\beta v^2 (\beta - 2)^2 e^{2\mathcal{Z}} + \left(\frac{1}{4} + (\lambda + 2v) (\beta - 2) e^{\mathcal{Z}} \right) (\beta - 1)^2 \right) v \ln \left(\frac{(\beta - 2)e^{\mathcal{Z}}}{\beta - 1} \right) - 2 \left(\left(\frac{3\beta}{2} - 2 \right) n + \beta \lambda \right) v^2 (\beta - 2)^2 e^{2\mathcal{Z}} \right. \\ \left. + (\beta - 1) \left(\left(\left(\frac{15\beta}{4} - \frac{31}{4} \right) v^2 + 4\lambda v (\beta - 1) + \lambda^2 (\beta - 1) \right) (\beta - 2) e^{\mathcal{Z}} + \frac{(\lambda + \frac{7v}{2}) (\beta - 1)}{2} \right) \right) (\beta - 2)^2 \left(v \ln \left(\frac{(\beta - 2)e^{\mathcal{Z}}}{\beta - 1} \right) \right. \\ \left. \left. + \lambda + \frac{3v}{2} \right) e^{2\mathcal{Z}} (\beta - 1)^4 \right\} \quad (\text{A10})$$

where

$$\mathcal{Z} = \frac{2\nu \exp\left(\frac{8(1-\beta)^2 \beta \nu N}{6e^{\frac{2\lambda+3\nu}{2\nu}}}\right) + 3\nu + 2\lambda}{2\nu}. \quad (\text{A11})$$

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