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Review

Structure of gauge theories

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Abstract Elementary interactions are formulated according to the principle of minimal interaction although paying special attention to symmetries. In fact, we aim at rewriting any field theory on the framework of Lie groups, so that, any basic and fundamental physical theory can be quantized on the grounds of a group approach to quantization. In this way, connection theory, although here presented in detail, can be replaced by "jet-gauge groups" and "jet-diffeomorphism groups." In other words, objects like vector potentials or vierbeins can be given the character of group parameters in extended gauge groups or diffeomorphism groups. As a natural consequence of vector potentials in electroweak interactions being group variables, a typically experimental parameter like the Weinberg angle ϑ_W is algebraically fixed. But more general remarkable examples of success of the present framework could be the possibility of properly quantizing massive Yang–Mills theories, on the basis of a generalized Non-Abelian Stueckelberg formalism where gauge symmetry is preserved, in contrast to the canonical quantization approach, which only provides either renormalizability or unitarity, but not both. It proves also remarkable the actual fixing of the Einstein Lagrangian in the vacuum by generalized symmetry requirements, in contrast to the standard gauge (diffeomorphism) symmetry, which only fixes the arguments of the possible Lagrangians.

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A la gata caramanchón (To the garret she-cat).

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1 The idea of gauge principle

The idea of formulating the basic interactions among elementary particles in terms of vector potentials, generalizing electromagnetism, is traced back to the pioneers papers by Yang and Mills [1], Utiyama [2] and Kibble [3].

Starting from a free matter Lagrangian, let us think for instance of the Lagrangian corresponding to the free Dirac field,

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi,$$

which is invariant under the rigid (or global) group $U(1)$, that is, under the transformation

$$\psi(x) \;\rightarrow\; \psi'(x) = e^{-i\alpha}\psi(x), \quad x \in M \text{ the space-time,}$$

we require $\mathcal{L}$ to be minimally modified, to $\hat{\mathcal{L}}$ so as to be invariant under the corresponding gauge (or local) transformation

$$\psi \;\rightarrow\; \psi' = e^{-i\alpha(x)}\psi.$$

Note that the term in the original Lagrangian $\bar{\psi}\gamma^\mu\partial_\mu\psi$, due to the derivative acting on the local parameter $\alpha(x)$, transforms as

$$\bar{\psi}\gamma^\mu\partial_\mu\psi \;\rightarrow\; \bar{\psi}\gamma^\mu\partial_\mu\psi - i\bar{\psi}\gamma^\mu\partial_\mu\alpha(x)\psi$$

so that we should require an extra field that includes a derivative of the local coefficient in its transformation law under $U(1)$, that is

$$A_\mu(x) \;\rightarrow\; A'_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x)$$

and replacing

$$\partial_\mu\psi \;\text{ with }\; D_\mu\psi \equiv (\partial_\mu + ieA_\mu)\psi.$$

In the same way, for non-Abelian symmetries, associated with a (let us say) compact group G, we generalize the discussion above:

$$\psi \;\rightarrow\; \psi'(x) = e^{i\phi^a T_a}\psi \equiv U(\phi)\psi$$

with Lie algebra generators satisfying

$$[T_a,\, T_b] = C^c_{ab}T_c$$

and modifying the usual derivative with the covariant derivative

$$A^{(a)}_\mu \;\rightarrow\; A^{(a)}_\mu + C^{ab}_c\phi_b A^{(c)}_\mu - \frac{1}{g}{''}\partial''_\mu\phi^a$$

where by ${''}\partial''_\mu\phi^a$ we mean something like $\theta^{(a)}_b(\phi)\partial_\mu\phi^b$, associated with the canonical (left or right) 1-form on the Lie group G.

2 Basics on differential geometry

This first section is devoted to a presentation from scratch of those mathematical ingredients that are required to a sound understanding of a general setting of basic physical interactions in Nature. Here, we follow rather standard textbooks on Differential Geometry [4–10] and Lie Groups [11–13].

2.1 Differentiable manifolds

Let S be a set. A local chart on S is a pair $(U, \varphi)/$

$U \subset S$
φ is a bijection $U \leftrightarrow V$, an open subset of some vector space F.

An atlas is a family $\mathcal{A}$ of local charts $(U_i,\ \varphi_i\ i \in I)$

$a)$ $S = \cup\{U_i : i \in I\}$
$b)$ $\forall\,(U_i, \varphi_i)\,, \left(U_j, \varphi_j\right)$ in $\mathcal{A}$ with $U_i \cap U_j \neq \emptyset$, then $\varphi_i\left(U_i \cap U_j\right)$ is an open set $\subset F$ and $\varphi_{ji} \equiv \varphi_j \circ \varphi_i^{-1}\,|_{\varphi_i(U_i\cap U_j)}$ is a diffeomorfism (C^∞) (compatibility, Fig. 1).

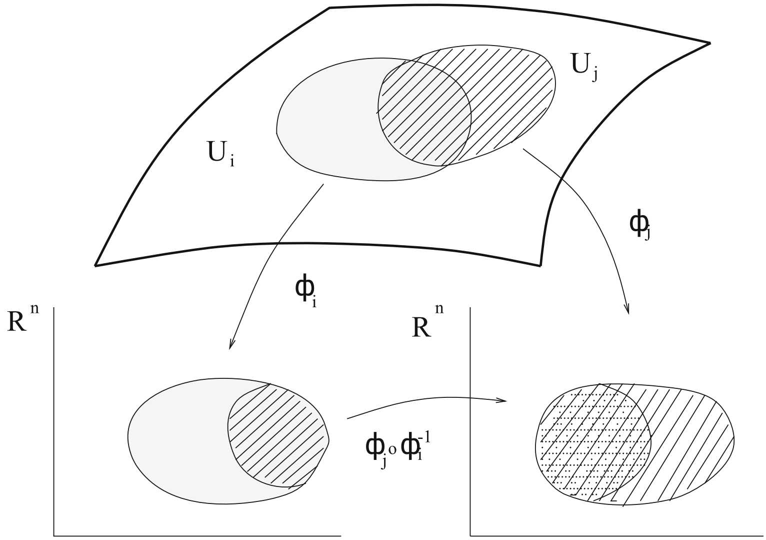


Fig. 1 Local chart compatibility

Two atlases are equivalent, $\mathcal{A}_1 \sim \mathcal{A}_2$, if $\mathcal{A}_1 \cup \mathcal{A}_2$ is an atlas.

A differentiable structure on S, $\mathcal{S}$, is an equivalence class of atlases on S.

A differentiable manifold M is a pair $(S, \mathcal{S}) \equiv M$. If $\varphi_i(U_i) \subset R^n \quad \forall i \in I$, we say that M has dimension n.

Topology on M : A topology on M can be defined by means of a family of open sets:

$$\begin{aligned}&A \subset M \text{ is open if}\\&\forall a \in A, \text{ there exists } (U, \varphi) \text{ local chart } / \ a \in U, \ U \subset A.\end{aligned}$$

Differentiable map: Let f be a mapping $M \to N$ where M and N are differentiable manifolds. We say that f is differentiable if (Fig. 2):

$$\begin{aligned}&\forall m \in M, \ \forall (V, \psi) \text{ local chart on } M, \ f(m) \in V\\&\text{there exists } (U, \varphi), \text{ local chart on } M, \ m \in U, \ f(U) \subset V \ /\\&f_{\varphi\Psi} \equiv \Psi \circ f \circ \varphi^{-1} \text{ is differentiable as a map from } R^n \text{ to } R^n\\&f_{\varphi\Psi} \equiv \text{ local representative of } f\end{aligned}$$

Local coordinates: Given a local chart on M and a coordinate system on R^n, $\{x^i\}, \ i = 1, ...n$, the set of composition functions on M, $\{u^i \equiv x^i \circ \varphi\}$ constitutes a local coordinate system on M (Fig. 3).

Simplest non-trivial example: Let as mention that a non-trivial differentiable manifold (with non-trivial topology) might have relevant consequences in solving a given physical problem. We usually solve a certain equation in local coordinates and we have to be aware that not all solutions obtained locally must be kept as true solutions. In fact, we must restrict ourselves to those solutions that are globally defined on the full manifold. This can be easily exemplified in the simplest situation of the S^1 (radius 1) manifold in which we analyze which analytic functions in a local chart can be kept, as such, when considering the compatibility condition with the other chart.

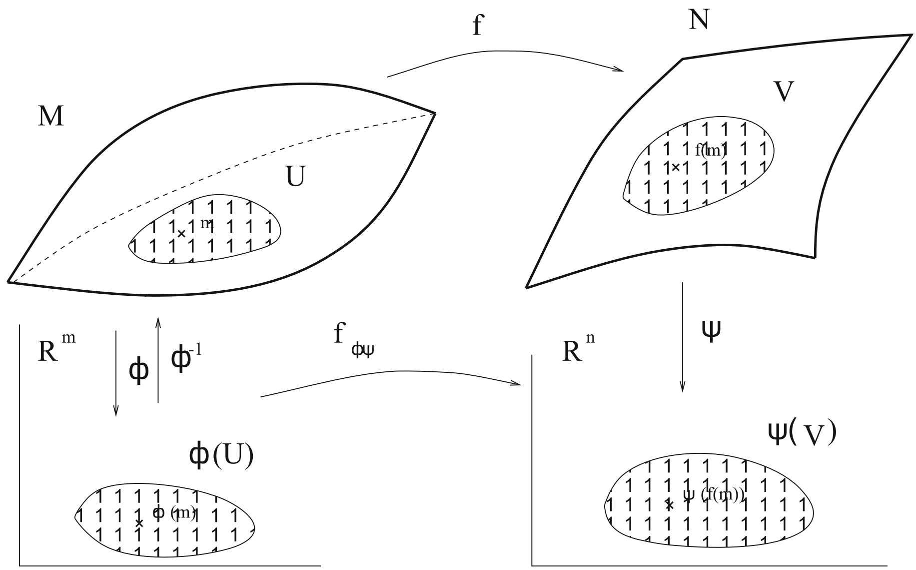


Fig. 2 Local representation of f

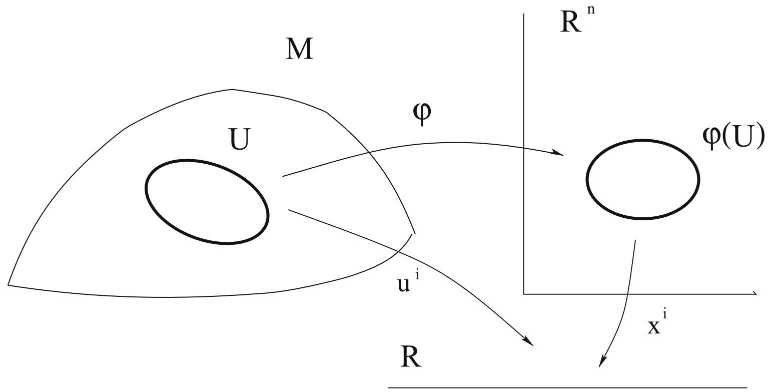


Fig. 3 Local coordinates

The two chosen charts correspond to the stereographic projection from both north and south poles. Coordinates from the south pole corresponding to the point $\zeta \in S^1$ will be noted x, whereas those obtained from the same point through the north projection will be y. Looking at Fig. 4, the point ζ is characterized by the angle ϕ, $\zeta = e^{i\phi}$, or by the projection angles ψ and θ corresponding to the two coordinate systems. The relationship among the three angles is:

$$\mathrm{tg}\theta = \sqrt{\frac{1-\sin\phi}{1+\sin\phi}} \ , \ \ \mathrm{tg}\psi = \sqrt{\frac{1+\sin\phi}{1-\sin\phi}} \, . \tag{1}$$

Writing x, for instance, in terms of ϕ, that is, $\sin\phi = \frac{4-x^2}{4+x^2}$, we can express the relation between both local coordinates as:

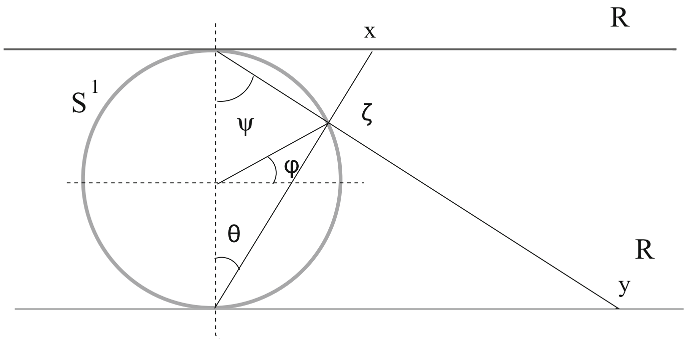


Fig. 4 Global analyticity

$$y = 2\mathrm{tg}\psi = 2\sqrt{\frac{1+\sin\phi}{1-\sin\phi}} = 2\sqrt{\frac{1+\frac{4-x^2}{4+x^2}}{1-\frac{4-x^2}{4+x^2}}} = \frac{4}{x}\,. \tag{2}$$

Clearly, polynomial functions are not, in general, allowed, since positive powers in y lead to negative ones in x. However, we can find rational functions which are analytic as seen from both local charts, for instance

$$\frac{4x}{4+x^2} \quad\longleftrightarrow\quad \frac{4y}{4+y^2} \tag{3}$$

and, more generally, the Chebyshev Polynomials $T_n(\frac{4x}{4+x^2})$, and 2^{nd}-class Chebyshev Polynomials $\frac{-4+x^2}{4+x^2}U_n(\frac{4x}{4+x^2})$, which constitute a basis for the analytical functions that are well defined on the manifold S^1 (Fig. 4).

Tangent Space

Tangent curves: A (differentiable) curve c at $m \in M$ is a (differentiable) application from $I \subset R$ to M such that $c(0) = m$ (Fig. 5). We say that two curves $c_1,\ c_2$ at $m \in U \subset M$ are equivalent, $c_1 \sim c_2$, if $\varphi \circ c_1$ and $\varphi \circ c_2$ are tangent at $\varphi(m)$ in the sense of R^n, i.e.,

$$D(\varphi \circ c_1)(0) \cdot 1 = D(\varphi \circ c_2)(0) \cdot 1. \tag{4}$$

This equivalence condition is independent of the local chart (U, φ).

We define the *Tangent Space at $m \in M$* as the space of equivalence classes of tangent curves at m, that is,

$$T_m(M) \equiv \{[c]_m/c \ \text{ is a curve at } m\}$$

and the (total) Tangent Space to M as

$$T(M) \equiv \cup_{m\in M} T_m(M).$$

Note that there is a natural projection: $T(M) \xrightarrow{\pi} M, \quad [c]_m \mapsto m$ (Fig. 5).
The triplet $(T(M), \pi, M)$ constitutes an example of *vector bundle* .

Also note that in R^n there is a natural representative for each $[c]_n$; that is to say, $[\varphi \circ c]_{\varphi(m)}$ has a preferred member:

$$c_{e,m} = \varphi(m) + te, \quad e \equiv \frac{\mathrm{d}}{\mathrm{d}t}(\varphi \circ c)|_{t=0} \equiv D(\varphi \circ c)(0) \cdot 1 \equiv v_m\,. \tag{5}$$

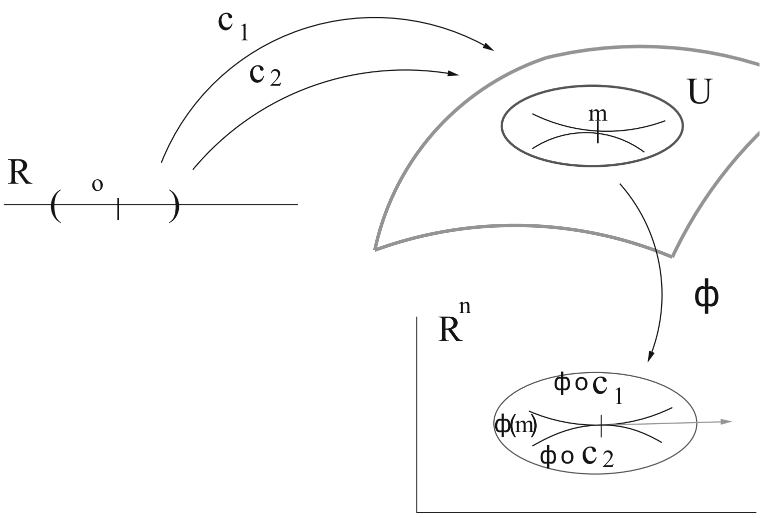


Fig. 5 Tangent space

This allows us to define a vector space structure on $T_m(M)$:

$$\begin{aligned} \lambda[c]_m &= [\varphi^{-1} \circ c_{\lambda e,m}] \\ [c]_m + \left[c'\right]_m &= \left[\varphi^{-1} \circ c_{e+e',m}\right] . \end{aligned} \tag{6}$$

Tangent map: Given a differentiable map $f : M \longrightarrow N$, we define the corresponding tangent map as follows (Fig. 6):

$$\begin{aligned} Tf \equiv f^T : T(M) &\longrightarrow T(N) \; / \\ [c]_m &\mapsto [f \circ c]_{f(m)} . \end{aligned} \tag{7}$$

Composition Theorem: Given the applications f, g, h, among manifolds, $M \overset{f}{\to} N \overset{g}{\to} P, \quad M \overset{h}{\to} M$, we have:

$$\begin{aligned} &(a) \;\; T(g \circ f) = Tg \circ Tf \\ &(b) \;\; h : M \to M \text{ identity } \Rightarrow Th : T(M) \to T(N) \text{ identity} \\ &(c) \;\; f \text{ diffeomorphism } \Rightarrow Tf \text{ bijection, } T(f^{-1}) = (Tf)-1 . \end{aligned}$$

Locally the following expressions are correct:

$$\begin{aligned} f^T &: (m, v_m) \mapsto (f(m), \; D(f)(m) \cdot v_m) \\ df &: (m, \; v_m) \mapsto D(f)(m) \cdot v_m \quad (2^{nd} \text{ component of } f^T) \\ & T(U) \approx U \times T_m(M) . \end{aligned}$$

Coordinates at $T(U) \subset T(M)$:

$$\begin{aligned} &\{u^i, \, \xi^j\} \equiv \{x^i \circ \varphi, \, \epsilon^j \circ \mathrm{d}\varphi\} \\ &\{x^i, \, \epsilon^j\} \text{ are coordinates at } \varphi(U) \times R^n . \end{aligned}$$

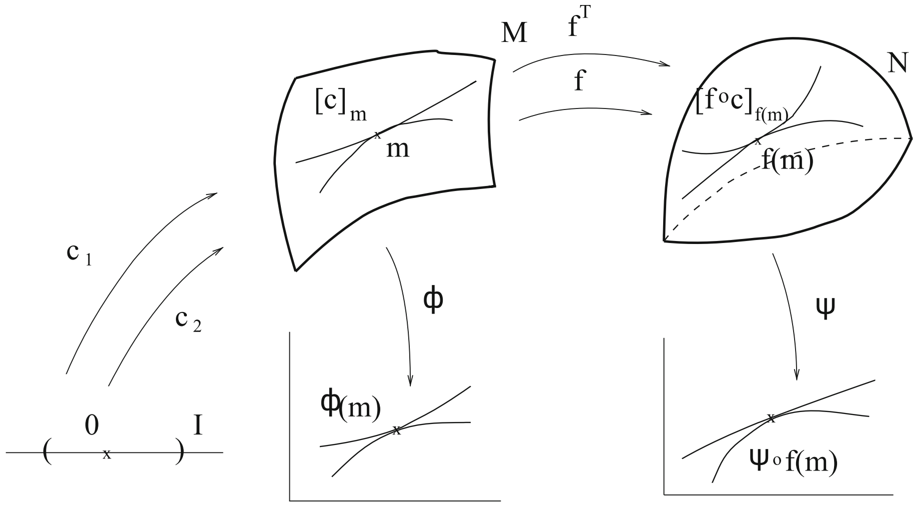


Fig. 6 Local representative of f^T

Here, $\{\epsilon^j\}$ is a linear coordinate system in R^n: given a basis $\{e_i\}$ in R^n as a vector space, an arbitrary vector v is written as $v = v^j e_j$, and the dual basis provides the linear functions $\epsilon^j(v) = v^j$.

Derivations at$m \in M$, $\mathcal{D}_m(M)$: Let us consider the algebra $\mathcal{F}(U)$ of differentiable functions defined on the open subset of a local chart (U, φ) (in case we desire to consider a ring structure, we must be restricted to the germs of differentiable functions; that is to say, the set of equivalence classes of functions that coincide in some open subset of U).
A *derivation at m* is a linear map:

$$\begin{array}{l} D_m \ : \ \mathcal{F}(U) \ \longrightarrow \ R \\ \qquad\qquad f \ \mapsto \ D_m(f) \ / \\ a) \ \ D_m(fg) = D_m(f)g(m) + f(m)D_m(g) \\ b) \ \ D_m(f) = 0 \quad \text{if} \quad f \quad \text{is a constant} . \end{array} \tag{8}$$

$\mathcal{D}_m(M)$ is a vector space isomorphic to $T_m(M)$. In fact, the correspondence is as follows:

$$D_m \ \rightarrow \ D_m u^i \equiv \xi^i \ \rightarrow \ c_{\xi,m} = \varphi(m) + t\xi \quad \varphi^{-1} \circ c_{\xi,m} \equiv [c]_m .$$

A basis on $\mathcal{D}_m(M) \approx T_m(M)$, $\{(\frac{\partial}{\partial u^i})\}$, is constructed in the form:

$$\left(\frac{\partial}{\partial u^i}\right)_m \quad f \ \mapsto \ D\left(f \circ \varphi^{-1})(\varphi(m)\right) \cdot e_i , \tag{9}$$

so that, a tangent vector is written as:

$$X_m \equiv D_m = X^i_m (\frac{\partial}{\partial u^i})_m . \tag{10}$$

Vector fields on M, $\mathcal{X}(M)$: They are mappings associating a tangent vector on the tangent space to any point on the manifold, that is:

$$X \ : \ M \ \longrightarrow \ T(M) \ / \ \pi \circ X = I_M .$$

Analogously, we define

<u>Derivations on M, $\mathcal{D}(M)$</u>: They are R-linear maps

$$\begin{aligned} &D \ : \ \mathcal{F}(M) \ \longrightarrow \ \mathcal{F}(M) \ / \\ &D(fg) = f D(g) + g D(f) \\ &D(f) = 0 \ \text{ if } \ f \quad \text{is a constant}. \end{aligned}$$

$\mathcal{X}(M)$ is isomorphic to $\mathcal{D}(M)$. The **proof** makes use of the isomorphism $T_m(M) \approx \mathcal{D}(M)$ running on m:

$$\cup_m \left[T_m(M) \approx \mathcal{D}(M)\right].$$

<u>Algebra of Derivations $\mathcal{D}(M) \approx \mathcal{X}(M)$</u>: Given two derivations (or vector fields, although thinking of them as derivations), the product

$$\begin{aligned} (D_1,\ D_2) \ \mapsto \ [D_1,\ D_2] \ / \\ [D_1,\ D_2]f = D_1 D_2 f - D_2 D_1 f\,, \quad f \in \mathcal{F}(M) \end{aligned}$$

is internal (although none of the summands, separately, are a derivation), bilinear (R-linear) and anti-symmetric, and it is named Lie bracket, mainly when acting on $\mathcal{X}(M)$. In local coordinates, the Lie bracket is written as:

$$[X,\ Y] = \left(X^i \frac{\partial Y^j}{\partial u^i} - Y^i \frac{\partial X^j}{\partial u^i} \right) \frac{\partial}{\partial u^j}\,. \tag{11}$$

It satisfies the following four properties characterizing a *Lie Algebra*:

$$\begin{aligned} &[X,\ Y + Z] = [X,\ Y] + [X,\ Z] \\ &[X,\ fY] = (X.f)Y + f\,[X,\ Y] \quad f \in \mathcal{F}(M) \\ &[X,\ Y] = -\,[Y,\ X] \\ &[X,\ [Y,\ Z]] + [Y,\ [Z,\ X]] + [Z,\ [X,\ Y]] = 0 \ \ \text{(Jacobi identity)} \end{aligned} \tag{12}$$

Given a basis $\{X_{(i)}\}$, we have:

$$\left[X_{(i)},\ X_{(j)}\right] = C_{ij}^k X_{(k)}\,, \tag{13}$$

where C_{ij}^k are constants called structure constants.
The tangent map is a Lie algebra homomorphism: That is,

$$f^T \circ [X_1,\ X_2] \circ f^{-1} = \left[f^T \circ X_1 \circ f^{-1},\ f^T \circ X_2 \circ f^{-1} \right]$$

$$\begin{array}{ccc} T\,(M) & \xrightarrow{f^T} & T\,(N) \\ X_{1,2} \uparrow\downarrow \pi_M & & \pi_N \downarrow\uparrow Y_{1,2}\,, \qquad Y \equiv f^T \circ X \circ f^{-1} \\ M & \xrightarrow{f} & N\,. \end{array}$$

Here, we have assumed that f has an inverse, **but** even if it is not invertible, Y can be defined so that $Y \circ f = f^T \circ X$ and still named the transformed vector field of X. In this case, it also holds that

$$[Y_1,\ Y_2] \circ f = f^T \circ [X_1,\ X_2]\,.$$

Example: Lie algebra of rotations in R^3

$$J_{(i)} = \eta_{ij.}^{\ \ k} x^j \frac{\partial}{\partial x^k} \quad \Rightarrow \quad \left[J_{(i)},\ J_{(j)}\right] = \eta_{ij.}^{\ \ k} J_{(k)}\,. \tag{14}$$

Example: Lie algebra of the Galilei group in $R^3 \times R$

$$\left|\begin{array}{l} H = \frac{\partial}{\partial t} \\ \\ P_{(i)} = \frac{\partial}{\partial x^i} \\ \\ J_{(i)} = \eta_{ij.}^{\ \ k} x^j \frac{\partial}{\partial x^k} \\ \\ K_{(i)} = t \frac{\partial}{\partial x^i} \end{array}\right. \quad \Rightarrow \quad \begin{cases} \left[H,\ P_{(i)}\right] = 0, \ \left[H,\ J_{(i)}\right] = 0, \ \left[H,\ K_{(i)}\right] = P_{(i)} \\ \\ \left[P_{(i)},\ J_{(j)}\right] = \eta_{ij.}^{\ \ k} J_{(k)}, \ \left[P_{(i)},\ K_{(j)}\right] = 0 \\ \\ \left[P_{(i)},\ P_{(j)}\right] = 0 \\ \\ \left[J_{(i)},\ J_{(j)}\right] = \eta_{ij.}^{\ \ k} J_{(k)}, \ \left[J_{(i)},\ K_{(j)}\right] = \eta_{ij.}^{\ \ k} K_{(k)} . \end{cases} \tag{15}$$

Example: Lie algebra of the Poincaré group in $R^3 \times R$: $(x_i = -x^i, \ \ x_0 = x^0 \equiv ct)$ (we just write the differences with the Galilei group)

$$P_{(0)} = \frac{\partial}{\partial x^0}, \ \ K_{(i)} = x^0 \frac{\partial}{\partial x^i} + x_i \frac{\partial}{\partial x^0}, \ \ \left[P_{(i)},\ K_{(j)}\right] = \delta_{ij} P_0 . \tag{16}$$

Example: The diffeomorphism algebra of R

$$\begin{gathered} L_{-1} = \frac{\partial}{\partial x}, \ \ L_0 = x \frac{\partial}{\partial x}, \ \ L_1 = x^2 \frac{\partial}{\partial x}, \ldots\ L_n = x^{n+1} \frac{\partial}{\partial x} \\ \Rightarrow \end{gathered} \tag{17}$$

$$\begin{aligned} [L_n,\ L_m] = \left[x^{n+1} \frac{\partial}{\partial x},\ x^{m+1} \frac{\partial}{\partial x}\right] &= x^{n+1}(m+1)x^m \frac{\partial}{\partial x} \\ -x^{m+1}(n+1)x^n \frac{\partial}{\partial x} &= (m-n)x^{n+m+1} \frac{\partial}{\partial x} = (m-n)L_{n+m} . \end{aligned} \tag{18}$$

Example: The diffeomorphism algebra of S^1
The difference is just the way we wright the generators,

$$\zeta \in C \ \ / \ \ |\zeta| = 1, \quad L_n = \zeta^{n+1} \frac{\partial}{\partial \zeta}, \quad \forall n \in Z. \tag{19}$$

<u>Tensor fields on M</u>:

Associated with the vector space $T_m(M)$ on any $m \in M$, it is possible to construct the entire tensor space $T_m(M)^r_s \equiv T^r_s(T_m(M))$, that is, the space of the tensor of $\{^r_s\}$-type (r-times contravariant, s-times covariant):

$$T_m(M)^r_s \equiv T_m(M) \otimes T_m(M) \otimes \overset{r}{\ldots}\ T_m(M) \otimes T^*_m(M) \otimes T^*_m(M) \otimes \overset{s}{\ldots}\ T^*_m(M) . \tag{20}$$

Tensor fields are then defined in an analogous manner to the vector fields:

$$\begin{array}{ll} t^r_s \ : \ M \ \longrightarrow \ T(M)^r_s \equiv \cup_m T_m(M)^r_s \ \ / \qquad\qquad & \quad T(M)^r_s \\ \qquad \pi^r_s \circ t^r_s = I_M & t^r_s \uparrow \downarrow \ \pi^r_s \\ & \quad M \end{array}$$

In particular, we shall consider very frequently $\Lambda(M)^p \equiv \{t^0_p\}$ and $\Lambda(M) \equiv \otimes_p \Lambda(M)^p$. Locally, $\lambda = \lambda_{i_1 i_2 \ldots i_p} du^{i_1} \wedge du^{i_2} \wedge \ldots \wedge du^{i_p} \quad (a \wedge b \equiv a \otimes b - b \otimes a)$.

2.2 Differential calculus

<u>Interior product i_X</u> : Given a vector field on M, $X \in \mathcal{X}(M)$, the interior product by X is defined as the following endomorphism of $\Lambda(M)$:

$$\begin{aligned} i_X &: \Lambda(M) \to \Lambda(M) \quad / \\ &\alpha \in \Lambda(M)^p \longmapsto (i_X\alpha) \in \Lambda(M)^{p-1} \\ &(i_X\alpha)\left(X_1, ..., X_{p-1}\right) = \alpha\left(X, X_1, ..., X_{p-1}\right) . \end{aligned} \tag{21}$$

Properties:

$$\left| \begin{array}{l} i_{X+Y} = i_X + i_Y \\ i_{fX} = f i_X \\ \text{If } h\ M \to N, \quad h^*(i_Y\alpha) = i_X(h^*\alpha) \end{array} \right| \begin{array}{l} \mathcal{F}(M) - \text{linear} \\ \\ \Rightarrow \end{array}$$

in particular, h may be the restriction to an open set, $h = |_U$, $\Rightarrow$ the interior product commutes with the restriction to U, that is, i_X is a *local operator*; note that $h^*\alpha(X) = \alpha(h^T X)$.

i_X is characterized by $\left| \begin{array}{l} i_X f = 0 \\ i_X df = df(X) = X.f \end{array} \right.$ This is a consequence of i_X being local. In fact, locally, any differential form can be written as a product of functions, $\alpha_{i_1 i_2,\dots i_p}$, and differentials of functions, du^{i_k}.

Exterior Differential D : Let $\alpha \in \Lambda(M)^p$ with $p \geq 1$. We define the exterior differential $D\alpha \in \Lambda(M)^{p+1}$ as:

$$\begin{aligned} D\alpha : \left(X_1, ..., X_{p+1}\right) \longmapsto & \sum_{i=1}^{p+1} (-)^{i-1} X_i \alpha\left(X_1, ..., \widehat{X}_i, ..., X_{p+1}\right) . \\ & + \sum_{i<j} (-)^{i+j} \alpha\left(\left[X_i, X_j\right], X_1, ..., \widehat{X}_i, ..., \widehat{X}_j, ..., X_{p+1}\right) . \end{aligned} \tag{22}$$

For $p = 1$, the expression (22) reduces to

$$D\alpha\,(X, Y) = X\alpha\,(Y) - Y\alpha\,(X) - \alpha\,([X, Y]) \ . \tag{23}$$

In particular, if $\alpha = df \quad , \quad f \in \Lambda(M)^0 \equiv \mathcal{F}(M)$,

$$\begin{aligned} Ddf\,(X, Y) &= Xdf\,(Y) - Ydf\,(X) - df\,([X, Y]) \\ &= XYf - YXf - [X, Y]\,f = 0. \end{aligned}$$

If $h \ : \ M \ \longrightarrow \ N, \quad h^* \circ D = D \circ h^* \Rightarrow D$ is a local operator.

D is characterized by $\left| \begin{array}{l} Df = df \\ D(df) = 0, \quad \forall f \in \mathcal{F}(M) \quad \Rightarrow \quad D^2 = 0 \,. \end{array} \right.$

From now on, D will be named d since it extends the ordinary differential.

Note: De Rham Cohomology. In the vector space (Abelian group) $\Lambda(M)$, a quotient space can be established:

Closed forms $Z^p(M)$: α such that $d\alpha = 0, \quad \alpha \in \Lambda(M)^p$

Exact forms $B^p(M)$: α such that $\alpha = d\beta \quad \beta \in \Lambda(M)^{p-1}$.

Obviously, Exact $\Rightarrow$ Closed but not the other way round. The quotient

$$H^p(M) \equiv \frac{Z^p(M)}{B^p(M)} \tag{24}$$

is called the p^{th}-cohomology group of M.

Lie Derivative L_X : Combining the interior product and the exterior differential, we define the Lie derivative by the vector field X as the following endomorphism in $\Lambda(M)$ preserving the order of the differential forms:

$$L_X \equiv i_X d + d i_X \,. \tag{25}$$

This operator is local so that

$$L_X \ \text{ is characterized by } \left| \begin{array}{l} L_X f = X.f \\ L_X df = d(X.f)\,. \end{array} \right.$$

Exercise: Prove that if $\alpha \in \Lambda^p(M)\,, \quad p \geq 1$,

$$(L_X\alpha)\left(Y_1, Y_2, ..., Y_p\right) = X.\{\alpha\left(Y_1, ..., Y_p\right)\} - \sum_i \alpha\left(Y_1, ..., [X, Y_i], ..., Y_p\right)\,. \tag{26}$$

In particular, if $p = 1$:

$$(L_X\alpha)(Y) = X.\alpha(Y) - \alpha([X, Y]) \tag{27}$$

(A more general definition of L_X, nextly).

2.3 Integration of vector fields

Let $X \in \mathcal{X}(M)$ be a vector field. Then, there exists an open set $V \subset R \times M \ni \{0\} \times M$ and a differentiable mapping Φ /

$$\begin{array}{rcl} \Phi : V & \longrightarrow & M \\ (t, u) & \mapsto & \varphi_t(u) \quad \text{satisfying:} \end{array}$$

$$\text{one parameter group} \left\{ \begin{array}{ll} \text{(a)} & t \ \mapsto \ \varphi_t(u) \ \text{ is an integral curve of } X, \ \text{ that is,} \\ & \quad \frac{\mathrm{d}\varphi_t(u)}{\mathrm{d}t} = X(\phi_t(u)) \quad \left(\frac{\mathrm{d}c(t)}{\mathrm{d}t} = X(c(t))\right) \\ \text{(b)} & \quad \varphi_0(u) = u \\ \text{(c)} & \text{if } \ (t', u), \ \ (t' + t, u) \ \text{ and } \ (t, \varphi_{t'}(u)) \in V \\ & \quad \varphi_{t+t'}(u) = \varphi_t(\varphi_{t'}(u)). \end{array} \right.$$

$\varphi_t(u) \equiv \Phi(t, u)$ is the one-parameter group generated by X. We usually call $e^{tX} \equiv \varphi_t$ and say that X is the infinitesimal generator of Φ.

Formally, $\frac{\mathrm{d}}{\mathrm{d}t} e^{tX}|_{t=0} = X$.

Exercise:

$$L_X\alpha = \lim_{t\to 0} \frac{\varphi_t^*\alpha - \alpha}{t} \tag{28}$$

$$L_X Y = \lim_{t\to 0} \frac{\varphi_{-t}^T Y \varphi_t - Y}{t} = [X, Y]\,. \tag{29}$$

Proposition (Frobenius Lemma): Given $X \in \mathcal{X}(M)$, written in a coordinate system $\{u^i\}$ around $m \in U \subset R^n$ / $X(m) \neq 0$, there exist new coordinates $\{u'^1, u'^2, ...u'^n\}$ such that

$$X = X^i \frac{\partial}{\partial u^i} = \frac{\partial}{\partial u'^1}\,. \tag{30}$$

Before going to a general proof, let us give an instructive simple example.

Example: $M = R \times R^3 \times R^3$, coordinates $\{t, x^i, p^j\}$, vector field

$$X = \frac{\partial}{\partial t} + p^i \frac{\partial}{\partial x^i}\,. \tag{31}$$

Integral curves $\begin{cases} t = \tau \\ x^i = K^i + \frac{P^i}{m}\tau \\ p^i = P^i \end{cases}$ $\{K^i, P^j\}$ constants of motion.

We perform the change of variables in $R \times R^3 \times R^3$:

$$(t, x^i, p^j) \quad \longleftrightarrow \quad (\tau, K^i, P^j) \quad \left| \begin{array}{l} \tau = t \\ K^i = x^i - \frac{p^i}{m}t \\ P^i = p^i \end{array} \right. \Rightarrow$$

$$\left| \begin{array}{ll} \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial t} + \frac{\partial}{\partial K^i}\frac{\partial K^i}{\partial t} + \frac{\partial}{\partial P^i}\frac{\partial P^i}{\partial t} & = \frac{\partial}{\partial \tau} - \frac{P^i}{m}\frac{\partial}{\partial K^i} \\ \frac{\partial}{\partial x^i} = ... & = \frac{\partial}{\partial K^i} \qquad \Rightarrow \\ \frac{\partial}{\partial p^i} = ... & = \frac{\partial}{\partial P^i} - \frac{\tau}{m}\frac{\partial}{\partial K^i} \end{array} \right.$$

$$X = \frac{\partial}{\partial t} + p^i \frac{\partial}{\partial x^i} = \frac{\partial}{\partial \tau} - \frac{P^i}{m}\frac{\partial}{\partial K^i} + \frac{P^i}{m}\frac{\partial}{\partial K^i} = \frac{\partial}{\partial \tau} \qquad !!$$

Proof We shall proceed to a constructive proof in Physical terms (Mechanics à la Cartan) leaving a more formal proof to the seasoned reader.

In Mechanical terms, Frobenius Lemma would say that a vector field (associated with a dynamical system)

$$X = \frac{\partial}{\partial t} + X^{x^i}\frac{\partial}{\partial x^i} + X^{p^i}\frac{\partial}{\partial p^i} \quad \text{goes to} \quad X = \frac{\partial}{\partial \tau} \tag{32}$$

under the change of variables, constituting the

$$\underline{\text{Hamilton–Jacobi} \quad \text{Transformation}} \qquad \begin{cases} t \longleftrightarrow \tau \\ x^i \longleftrightarrow K^i \\ p^i \longleftrightarrow P^i \end{cases} \tag{33}$$

for the Principal Hamilton function (in the language of canonical transformations in Analytical Mechanics [14,15]). After this transformation, the new variables $\{K^i, P^j\}$ behave as constant Canonical Coordinates and Momenta. In fact: The vector field $X = \frac{\partial}{\partial t} + X^x \frac{\partial}{\partial x} + X^p \frac{\partial}{\partial p}$ (we have omitted the index i) provides the uniparametric group φ in terms of which we construct, explicitly, the change of variables

$$\begin{array}{ll} x = \varphi^x(K, P, \tau) & \frac{d\varphi}{d\tau} = X^x(x, p, t) \\ p = \varphi^p(K, P, \tau) & \frac{d\varphi^*}{d\tau} = X^p(x, p, t) \\ t = \varphi^0(K, P, \tau) = \tau & \frac{d\varphi^0}{d\tau} = 1\,, \end{array} \tag{34}$$

where we have assumed that the possible component of X in $\frac{\partial}{\partial t}$, X^0, does not vanish in those local coordinates and the entire vector field has been divided by X^0. Applying the tangent coordinate transformation we arrive at:

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial x}{\partial K}\frac{dK}{dt} + \frac{\partial x}{\partial P}\frac{dP}{dt} + \frac{\partial x}{\partial \tau}\frac{d\tau}{dt} = \frac{\partial x}{\partial K} \times 0 + \frac{\partial x}{\partial P} \times 0 + \frac{\partial x}{\partial \tau} \times 1 = X^x \\ \frac{dp}{dt} &= \frac{\partial p}{\partial K}\frac{dK}{dt} + \frac{\partial p}{\partial P}\frac{dP}{dt} + \frac{\partial p}{\partial \tau}\frac{d\tau}{dt} = \frac{\partial p}{\partial K} \times 0 + \frac{\partial p}{\partial P} \times 0 + \frac{\partial p}{\partial \tau} \times 1 = X^p \\ \Rightarrow \quad \frac{\partial}{\partial \tau} &= \frac{\partial t}{\partial \tau}\frac{\partial}{\partial t} + \frac{\partial x}{\partial \tau}\frac{\partial}{\partial x} + \frac{\partial p}{\partial \tau}\frac{\partial}{\partial p} = \frac{\partial}{\partial t} + X^x\frac{\partial}{\partial x} + X^p\frac{\partial}{\partial p}\,. \end{aligned} \tag{35}$$

□

3 Lie groups

A group is a composition law on a set G /
$a * b \in G$
$a * (b * c) = (a * b) * c$
$a * e = e * a = a$
$\forall a \in G\ \exists a^{-1} \in G \quad / \quad a * a^{-1} = a^{-1} * a = e\,.$
A Lie Group is a par $(G, \mathcal{S})$ where G is a group (with the composition law $*$) and $\mathcal{S}$ is a differentiable structure on G, respect to which the mappings

$$\begin{array}{lcl} * : G \times G \to G & \text{and} & -1 : G \to G \\ \quad (a, b) \longmapsto a * b & & \quad a \longmapsto a^{-1} \end{array}$$

are differentiable.

Equivalently, the maps $L_g \equiv *|_{\{e\}\times G}$, $\;R_g \equiv *|_{G\times\{g\}}$ are required to be differentiable.

The transformations $L_g\, a \;\mapsto\; g * a$; $\;R_g\, a \;\mapsto\; a * g$ are called left-translation; right-translation and **do commute**:

$$L_{g'} R_g = R_g L_{g'} \quad !!$$

In addition, $L_{g^{-1}} = L_g^{-1}$; $\;R_{g^{-1}} = R_g^{-1}$.

The tangent space at the identity, $T_e(G)$, is called the **Lie Algebra**.

G is an ordinary manifold, so that we may define any object as in M. In particular, vector fields $X \;:\; G \;\longrightarrow\; T(G)$.

$X \in \mathcal{X}(G)$ is left-invariant if $L_g^* X = X$, that is

$$L_g^T \circ X \circ L_{g^{-1}} = X\,. \tag{36}$$

In the same way, $X \in \mathcal{X}(G)$ is right-invariant if $R_g^* X = X$, that is,

$$R_g^T \circ X \circ R_g^{-1} = X\,. \tag{37}$$

The set of left-invariant vector fields will be named $\mathcal{X}^L(M)$
" " " right-invariant " " " " " $\mathcal{X}^R(M)$.

Proposition *$\mathcal{X}^{L,R}(G)$ is a finite-dimensional subalgebra of $\mathcal{X}(G)$ isomorphic to $T_e(G) \equiv \mathcal{G}$, so that*

$$\mathcal{X}^L(G) \approx \mathcal{G} \approx \mathcal{X}^R(G)$$

Proof Given an element in $T_e(G)$, X_e, we construct X on G in the form $X(g) = L_g^T X_e$. This vector field, so built, is in $\mathcal{X}^L(G)$. In fact,

$$\begin{aligned} L_a^* X(g) &= L_a^T \circ X \circ L_a^{-1}(g) = L_a^T X(a^{-1}g) = L_a^T \circ L_{a^{-1} * g} X_e \\ &= L_a^T \circ L_{a^{-1}}^T \circ L_g^T X_e = L_g^T X_e = X(g) \end{aligned} \tag{38}$$

Lie algebra structure: $\mathcal{X}^{L,R}$ where subalgebras of $\mathcal{X}(G)$ and isomorphic to $\mathcal{G} \equiv T_e(G)$. Let us denote χ this isomorphism. We can translate the Lie bracket from $\mathcal{X}^{L,R}(G)$ to $\mathcal{G}$:

$$\forall Z, Z' \in \mathcal{G}, \quad [Z',\, Z] \equiv \chi^{-1}[Z_G,\, Z_G]\,, \tag{39}$$

where $Z_G^{L,R}$ is the translated of Z by L_g, R_g, respectively.

This way, we have $\;[Z',\, Z]_G^{L,R} = [Z'^{L,R}_G,\, Z_G^{L,R}]$. Given a basis of $\mathcal{G}$, $\{Z_{(i)}\}$,

$$[Z_{(i)},\, Z_{(j)}] = C_{ij}^k Z_{(k)}\,. \tag{40}$$

Note that in terms of vector fields, from L to R there is a global minus sign on C^i_{jk}: $[Z^L_{(i)}, Z^L_{(j)}] = -[Z^R_{(i)}, Z^R_{(j)}] = C^k_{ij} Z_{(k)}$.

In practice,

$$\begin{aligned} Z^L_G(g) &= L^T_g Z \equiv D(L_g)(e) \cdot Z \\ Z^R_G(g) &= R^T_g Z \equiv D(R_g)(e) \cdot Z \,, \end{aligned} \tag{41}$$

where L_g is the mapping

$$\begin{array}{cccc} L_g & : & G \longrightarrow & G \\ & a & \mapsto & g * a \end{array};$$

here a plays the role of x in a function $f(x)$ and g that of f. Similar comment holds for $L \leftrightarrow R$.

If we write $g'' = g' * g$, $\begin{array}{l} X^L(g') = \frac{\partial g''}{\partial g}|_{g=e} \frac{\partial}{\partial g'} \\ X^R(g) = \frac{\partial g''}{\partial g'}|_{g'=e} \frac{\partial}{\partial g} \end{array}$ (after writing $X^L(g')$ we can rename g' by g). □

3.1 Some examples

Example I $G \equiv SU(2)$

The group $SU(2)$ is a double covering of the group $SO(3)$ of rotations in the space R^3. We shall parameterize the group by the components of a vector in the direction of the rotation axis and a module related to the rotation angle; that is, $g \equiv \{\epsilon^i\}$, $|\boldsymbol{\epsilon}| = 2\sin\frac{\phi}{2}$. A rotation in R^3 with this parameterization is written as:

$$R(\boldsymbol{\epsilon})^i_j = \left(1 - \frac{\boldsymbol{\epsilon}^2}{2}\right)\delta^i_j + \sqrt{1 - \frac{\boldsymbol{\epsilon}^2}{4}}\,\eta^i_{.jk}\epsilon^k + \frac{1}{2}\epsilon^i\epsilon_j \,. \tag{42}$$

From the product of two rotations $R(\boldsymbol{\epsilon}')R(\boldsymbol{\epsilon}) = R(\boldsymbol{\epsilon}'')$, we deduce the composition law:

$$\boldsymbol{\epsilon}'' = \sqrt{1 - \frac{\boldsymbol{\epsilon}'^2}{4}}\,\boldsymbol{\epsilon} + \sqrt{1 - \frac{\boldsymbol{\epsilon}^2}{4}}\,\boldsymbol{\epsilon}' - \frac{1}{2}\boldsymbol{\epsilon}' \wedge \boldsymbol{\epsilon} \,. \tag{43}$$

Now we proceed to compute the left and right generators:

$$X^L_{(i)} = \left[\sqrt{1 - \frac{\boldsymbol{\epsilon}^2}{4}}\,\delta^j_i - \frac{1}{2}\eta^j_{.ki}\epsilon^k\right]\frac{\partial}{\partial\epsilon^j} \tag{44}$$

$$X^R_{(i)} = \left[\sqrt{1 - \frac{\boldsymbol{\epsilon}^2}{4}}\,\delta^j_i + \frac{1}{2}\eta^j_{.ki}\epsilon^k\right]\frac{\partial}{\partial\epsilon^j} \,. \tag{45}$$

Example II $G \equiv$ Galilei Group

Galilei transformations [16,17] relate inertial reference systems, that is, reference systems where the Newton Laws hold. We shall write the transformations in $R \times R^3 \times R^3$, parameterized by the ten parameters $\{B, \mathbf{A}, \mathbf{V}, R(\boldsymbol{\epsilon})\}$ corresponding to a translation in time, translation in space, change in velocity and rotation of the axis. By composing two of them, a composition law is obtained from which we compute left and right generators:

$$\left|\begin{array}{l} t' = t + B \\ \mathbf{x}' = R(\boldsymbol{\epsilon})\mathbf{x} + \mathbf{V}t + \mathbf{A} \\ \mathbf{v}' = R(\boldsymbol{\epsilon})\mathbf{v} + \mathbf{V} \\ \end{array}\right. \quad \stackrel{*}{\Rightarrow} \quad \left|\begin{array}{l} B'' = B' + B \\ \mathbf{A}'' = \mathbf{A}' + R'\mathbf{A} + \mathbf{V}'B \\ \mathbf{V}'' = \mathbf{V}' + R'\mathbf{V} \\ R'' = R'R \end{array}\right. \tag{46}$$

$$\left|\begin{array}{l} X^L_B = \frac{\partial}{\partial B} + \mathbf{V}\cdot\frac{\partial}{\partial \mathbf{A}} \\ \\ X^L_{A^i} = R^j_i(\boldsymbol{\epsilon})\frac{\partial}{\partial A^j} \\ \\ X^L_{V^i} = R^j_i(\boldsymbol{\epsilon})\frac{\partial}{\partial V^j} \\ \\ X^L_{\epsilon^i} = \left[\sqrt{1-\frac{\boldsymbol{\epsilon}^2}{4}}\,\delta^j_i - \frac{1}{2}\eta^{\ j}_{.ki}\epsilon^k\right]\frac{\partial}{\partial \epsilon^j} \end{array}\right| \begin{array}{l} X^R_B = \frac{\partial}{\partial B} \\ \\ X^R_{A^i} = \frac{\partial}{\partial A^i} \\ \\ X^R_{A^i} = \frac{\partial}{\partial A^i} + \frac{\partial}{\partial V^j} \\ \\ X^R_{(i)} = \left[\sqrt{1-\frac{\boldsymbol{\epsilon}^2}{4}}\,\delta^j_i + \frac{1}{2}\eta^{\ j}_{.ki}\epsilon^k\right]\frac{\partial}{\partial \epsilon^j} + \eta_{ij.}^{\ \ k}A^j\frac{\partial}{\partial A^k} + \eta_{ij.}^{\ \ k}V^j\frac{\partial}{\partial V^k} \end{array} \tag{47}$$

3.2 The adjoint representation: Killing form

An action of G on a manifold M is a Lie group homomorphism from G to the group of diffeomorphisms of M:

$$\Phi \ : \ G \ \longrightarrow \ GD(M)\,,$$

such that the mapping $\Phi \ : \ G \times M \ \longrightarrow \ M$
$\qquad (g, m) \ \mapsto \ \Phi_g(m) \equiv \Phi(g, m) \quad \text{is } C^\infty.$
The tangent mapping $\Phi^T|_e \ : \ T_e(G) \ \longrightarrow \ T_I(GD(M))$

provides a Lie algebra homomorphism: $\mathcal{G} \ \longrightarrow \ \mathcal{G}D(M) \equiv \mathcal{X}(M)$
$\qquad Z \ \mapsto \ Z_M\,.$
As a particular case, M can be G, $\mathcal{G}$ or $\mathcal{G}^*$

With a given $g \in G$, we associate the (nonlinear, in general) mapping on G,

$$\text{ad}g \ \ a \ \mapsto \ g * a * g^{-1} \equiv L_g R_{g^{-1}} a \equiv R_{g^{-1}} L_g a\,.$$

Now, we take the tangent of ad$e$ at e:

$$(\text{ad}g)^T_e\, X_e = L^T_g R_{g^{-1}}\, X_e \equiv R^T_{g^{-1}} L^T_g\, X_e.$$

It defines an action of G on $\mathcal{G}$ named *Adjoint Representation of* G:

$$\left|\begin{array}{lcl} Ad \ : \ G & \longrightarrow & GD(\mathcal{G}) \\ \qquad\quad g & \mapsto & (\text{ad}g)^T_e\,. \end{array}\right. \tag{48}$$

The tangent of Ad at the identity $g = e$ is called *Adjoint Representation of* $\mathcal{G}$ and noted *ad*. ad is a Lie algebra homomorphism that turns out to be

$$\text{ad}(X)\cdot = [X, \cdot] \quad (\text{ad}(X)(Y) = [X,\ Y])\,. \tag{49}$$

The **Killing form** is defined as:

$$\begin{array}{rcl} k \ : \ \mathcal{G}\times\mathcal{G} & \longrightarrow & R \\ (X, Y) & \mapsto & \text{Tr}\,(\text{ad}(X)\text{ad}(Y))\,. \end{array} \tag{50}$$

It is bilinear, symmetric and satisfies:

$$k\left([X,\ Y],\ Z\right) = k\left(X,\ [Y,\ Z]\right), \tag{51}$$

which means, somehow, that Ad is unitary with respect to the scalar product k (ad is Hermitian). In coordinates, $k_{ij} = C^k_{im} C^m_{jk}$.

To be precise, k is a scalar product only when $|k| \equiv \det k \neq 0$, which happens iff G is semi-simple, that is, it contains no Abelian invariant subgroup. If $|k| = -1$, G is also compact.
Invariant Forms: They are dual to left- and right-invariant vector fields. If $\{X^{L,R}_{(i)}\}$ is a basis of $\mathcal{X}^{L,R}$, $\{\theta^{(i)L,R}\}$ will be the dual basis, that is, $\theta^{(i)L}(X^L_{(j)}) = \delta^i_j$. They are explicitly calculated as:

$$\theta^{(i)L}(g) = D(L_g)(e)^*\theta^{(i)}, \quad \theta^{(i)} = du^i(e,\cdot). \tag{52}$$

Invariance properties:

$$\left|\begin{array}{l} (L_a)^*\theta^L = \theta^L \\ (R_a)^*\theta^R = \theta^R \\ L_{X^R}\theta^L = 0 \\ L_{X^L}\theta^R = 0 \end{array}\right. \qquad \left|\begin{array}{l} (R_a)^*\theta^L = \theta^L \cdot Ad(a^{-1}) \\ (L_a)^*\theta^R = \theta^R \cdot Ad(a) \\ L_{X^L}\theta^L = \theta^L \cdot \mathrm{ad}(-X) \\ L_{X^R}\theta^R = \theta^R \cdot \mathrm{ad}\,. \end{array}\right. \tag{53}$$

The set of invariant forms are codified by a single 1-form, that is, the
<u>Canonical</u>1**<u>-form</u>**: $\theta^{L,R} = \theta^{(i)L,R} \circ Z_{(i)}$ or $\theta^{L,R} = \theta^{(i)L,R} \circ Z^{L,R}_{G(i)}$.

Note that $\theta^{L,R}$ is a $\mathcal{G}$-valued 1-form, the $\theta^{(i)L,R}$ being ordinary R-valued 1-forms.

Note also that $\theta^{L,R}(Z^{L,R}_G) = Z \approx Z^{L,R}_G$, that is to say,
$\theta^{L,R}$ is the $\mathcal{G}$-valued 1-form that is the identity on $\mathcal{X}^{L,R}(G)$.

Exercise: Compute k_{ij} and $\theta^{L,R}$ for $G = SU(2)$ and realize that $\theta^{(i)L,R}(X^{L,R}_{(j)}) = \delta^i_j$ and that

$$k_{ij} = -\delta_{ij}.$$

3.3 Central extensions of Lie groups

We say that $\tilde{G}$ is an <u>extension</u> of the Lie group G by H if H is a normal subgroup (that is, invariant under conjugation: ghg^{-1}) and

$$\tilde{G}/H = G\,.$$

Note that G is not necessarily subgroup of $\tilde{G}$.
$\tilde{G}$ is a <u>central extension</u> of G by H, if H is Abelian and is in the <u>center</u> of $\tilde{G}$ (that is, the elements in H commute with all the elements in $\tilde{G}$). Very special situation appears when $H = U(1)$ [18].

<u>Central Extensions of G by $U(1)$</u>: In that case, the group law for $\tilde{G}$ can be written as follows:

$$\begin{array}{ll} g'' = g'g & g \in G \\ \zeta'' = \zeta'\zeta\,\Delta(g',g) \equiv \zeta'\zeta e^{i\xi(g',g)}\,, & \zeta \in U(1) \end{array} \tag{54}$$

where the local exponent $\xi(g',g)$ is named 2-*cocycle of G valued on* $U(1)$. The properties which establish the 2-cocycle definition can be derived from the condition of the expression above being a group law for $\tilde{G}$:

$$\xi \;:\; G \times G \longrightarrow R \;\; / \;\; \left| \begin{array}{l} \xi(g'', g') + \xi(g''g', g) = \xi(g'', g'g) + \xi(g', g) \\ \xi(e, g) = 0 = \xi(g', e)\,. \end{array} \right. \tag{55}$$

Coboundaries: A cocycle ξ_{cob} satisfying

$$\xi_{cob}(g', g) = (\delta\eta)(g', g) \equiv \eta(g'g) - \eta(g') - \eta(g) \tag{56}$$

is called coboundary. Coboundaries define trivial extensions. In fact, a change of variables

$$\begin{array}{lcl} \hat{g} = g & & \hat{g}'' = \hat{g}'\hat{g} \\ & \Rightarrow & \\ \hat{\zeta} = e^{-i\eta}\zeta & & \hat{\zeta}'' = \hat{\zeta}'\hat{\zeta} \end{array}$$

destroys the central extension turning $\tilde{G}$ into $G \times U(1)$. The function η is the **generating function** of the coboundary.
The name cocycle comes from the fact that the set of central extensions of G by $U(1)$ are parameterized by the 2^{nd}-cohomology group of G with values on $U(1)$ (according to Bargmann):

$$H^2(G, U(1)) = Z^2/B^2 \quad \left| \begin{array}{ll} Z \equiv & \text{cocycles} \\ B \equiv & \text{coboundaries} \end{array} \right.$$

that is to say, cocycles that are not a coboundary.

"Pseudo-cohomology": However, there are coboundaries which are generated by a linear function on G and they do modify the structure constants of the Lie algebra, as if they were "true" cocycles [19,20]. This subset of coboundaries (in fact a subgroup of B^2) defines a (true) cohomology group $H^2(G_C, U(1))$ of a contracted group G_C of G.

The typical situation could be that of a family of generating functions η on G that **go badly** under a certain lie group contraction, that is, $\eta \to \infty$ in a contraction limit, but $\xi_{cob} \equiv \delta\eta$ has a **well-defined** limit.

Paradigmatic **Example:** The Poincaré group with $\eta = mcx^0$. G_C is then the Galilei group and $\delta\eta$ a non-coboundary cocycle for $c \to \infty$.

4 Principal bundles

In this subsection, we shall follow the presentation of principal bundles given by Koszul [21] (see also [9]).

A principal bundle is a differentiable manifold P on which a Lie group G acts from the right, along with a differentiable mapping p from P onto a differentiable manifold M such that:

$$\begin{array}{l} \forall m \in M, \quad \text{there exists} \quad U \ni m \quad \text{and a diffeomorphism} \quad \gamma \quad \text{satisfying:} \\ \gamma \;:\; U \times G \quad \longrightarrow \quad p^{-1}(U) \\ \qquad p \circ \gamma(m, s) = m \\ \qquad \gamma(m, st) = \gamma(m, s)t\,, \qquad s, t \in G\,. \end{array} \tag{57}$$

The application p is called projection, M base, G structure group, p^{-1} fiber over m (Figs. 7, 8).
Properties:

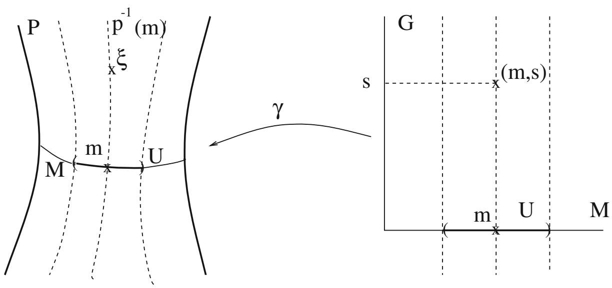


Fig. 7 Principal-bundle local chart

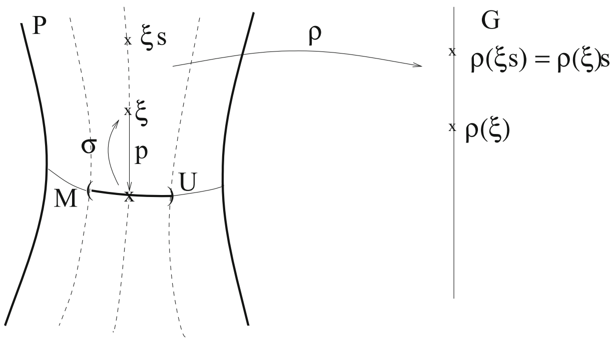


Fig. 8 Local mappings σ and ρ

(a) Each fiber is stable under G, G acts without fixed points on P, that is, $\xi s = \xi$ for some $\xi \in P \quad \Rightarrow \quad s = e$
(b) G acts transitively on the fiber
(c) $\forall m_0 \in M$ there exists $U \ni m_0$ and $\sigma \; : \; U \to P \;\; / \;\; p \circ \sigma\,(m) = m \; , \; \forall m \in U$
(d) $\forall m_0 \in M$, there exits $U \ni m_0$ and $\rho \; : \; p^{-1}\,(U) \to G \quad / \quad \rho\,(\xi s) = \rho\,(\xi)\, s \; , \quad \forall \xi \in p^{-1}\,(U) \; , s \in G$.

Exercise: Prove (a)–(b)

Example (a) Trivial bundle: $P = M \times G \; , \quad p = \pi_1$ (projection over the first factor)
(b) $G \to G/H \, : \; G$ being a Lie group and H a closed subgroup
(c) Reference Bundle: Let M be a manifold, $T_m\,(M)$ the tangent space at $m \in M$, $P_m \equiv \{\text{basis in } T_m\,(M)\}$. We define $P \equiv \cup_{m \in M} P_m$.

In all cases, $P/G = M$.

Homomorphism between principal bundles: A homomorphism H between two principal bundles $P, \; P'$ with the same structure group G is a (differentiable) mapping

$$H \;\; : \;\; P \;\; \longrightarrow \;\; P' \;\; / \;\; H(\xi s) = H(\xi) s \, . \tag{58}$$

It is clear that H takes fibers into fibers defining $h \; : \; M \; \to \; M'$:

$$\begin{array}{lll} P \xrightarrow{H} P' & & \\ \downarrow p \qquad \downarrow p' & & \text{(commutative diagram)} \\ M \xrightarrow{h} M' & & h \equiv \text{projection of } H \end{array}$$

Proposition *If h is a diffeomorphism, H is an isomorphism.*

Conditions for triviality: We say that $P \longrightarrow M$ is trivial if P is isomorphic to $M \times G$.

Proposition *The following conditions are equivalent*

(1) *P is trivial*
(2) *There exists a differentiable (global) section σ of P over M*
(3) *There exists a differentiable map $\rho : P \to G \quad \rho(\xi s) = \rho(\xi)s$.*

Proof left as an **Exercise**
Transition functions: **Reconstruction Theorem**
Let $P \overset{p}{\to} M$ be a principal bundle with structure group G and $\{U_\alpha\}$ an open covering of M, with the corresponding

$$\begin{aligned} &\gamma_\alpha : U_\alpha \times G \longrightarrow p^{-1}(U_\alpha) \quad \text{(local chart)} \\ &\rho_\alpha : p^{-1}(U_\alpha) \longrightarrow G \quad \text{(those satisfying } \rho_\alpha(\xi s) = \rho(\xi)s) \end{aligned} \tag{59}$$

If $\xi \in p^{-1}(U_\alpha \cap U_\beta)$, we have

$$\rho_\alpha(\xi s)[\rho_\beta(\xi s)]^{-1} = \rho_\alpha(\xi)ss^{-1}\rho_\beta(\xi)^{-1} = \rho_\alpha(\xi)\rho_\beta(\xi)^{-1} \Rightarrow g_{\alpha\beta}(\xi) \equiv \rho_\alpha(\xi)\rho_\beta(\xi)^{-1} \tag{60}$$

do not depend on the particular element ξ taken on $p^{-1}(p(\xi))$, that is, they define

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G, \tag{61}$$

the transition functions on $P \overset{p}{\to} M$ relative to the covering $\{U_\alpha\}$ of M. □

Proposition The transition functions satisfy

$$\begin{aligned} g_{\alpha\gamma}(m) &= g_{\alpha\beta}(m)g_{\beta\gamma}(m) \\ g_{\alpha\alpha}(m) &= e \qquad\qquad \forall m \in U_\alpha \cap U_\beta \cap U_\gamma\ , \\ g_{\alpha\beta}(m) &= g_{\beta\alpha}(m)^{-1} \end{aligned} \tag{62}$$

properties that characterize the so-called 1-cocycle on $\{U_\alpha\}$ valued on G.

The name transition functions comes from the fact that they address the change between local charts: in $p^{-1}(U_\alpha \cap U_\beta)$ we have

$$\begin{aligned} &\sigma_\alpha(m)\rho_\alpha(\xi) = \xi = \sigma_\beta(m)\rho_\beta(\xi) \left| \begin{aligned} &\sigma_\alpha(m) = \gamma_\alpha(m, e) \\ &\xi = \gamma_\alpha(m, \rho_\alpha(\xi)) \end{aligned} \right. \Rightarrow \\ &\sigma_\alpha(m) = \sigma_\beta(m)\rho_\beta(\xi)\rho_\alpha(\xi)^{-1} \equiv \sigma_\beta(m)g_{\alpha\beta}(m) \end{aligned} \tag{63}$$

as well as

$$\gamma_\beta^{-1}\gamma_\alpha(m, s) = (m, g_{\beta\alpha}(m)s) \quad \text{(the change of local charts)}.$$

Local expression of a homomorphism: Given $H : P \to P'$ and open coverings $\{U_\alpha\}, \{U_{\alpha'}\}$ of M, M', for $m \in U_\alpha \cap h^{-1}(U_{\beta'})$, we have:

$$\begin{aligned} \gamma_{\beta'}^{-1} \circ H \circ \gamma_\alpha(m, s) &= \big(h(m), \rho_{\beta'}(H(\sigma_\alpha(m)s))\big) = \big(h(m), \rho_{\beta'}(H(\sigma_\alpha(m)))\, s\big) \\ &= \big(h(m), \rho_{\beta'}(H(\sigma_\alpha(m)))\, s\big)\ . \end{aligned}$$

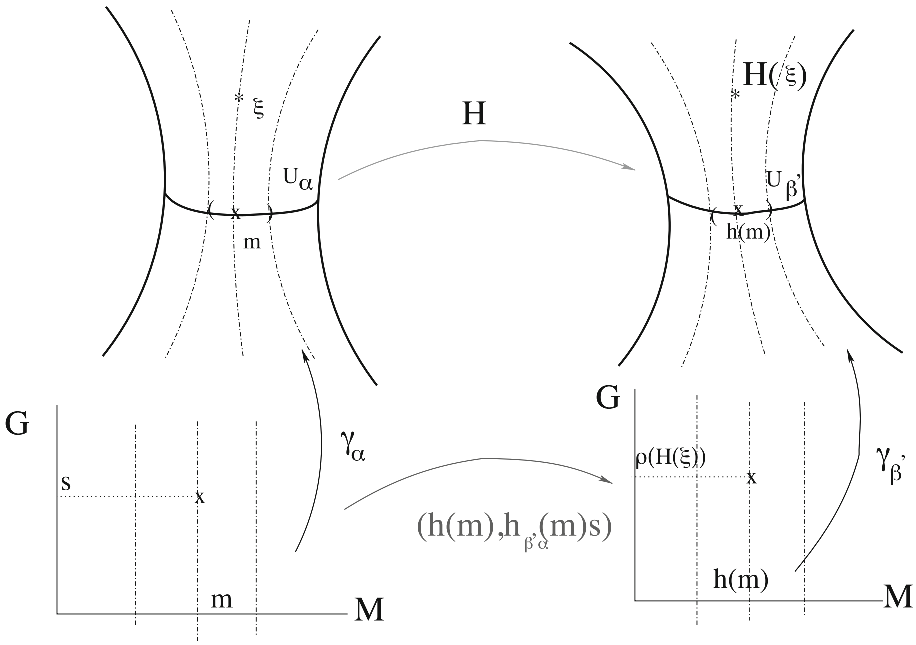


Fig. 9 Local expression of H

Denoting $h_{\beta'\alpha}(m) \equiv \rho_{\beta'}\,(H(\sigma_\alpha(m)))$, we have:

$$\gamma_{\beta'}^{-1} \circ H \circ \gamma_\alpha(m, s) = \big(h(m),\ h_{\beta'\alpha}(m)s\big)\,, \tag{64}$$

to be compared with $f^T_{\varphi\psi}(x, e) = \big(f_{\varphi\psi}(x),\ D(\psi \circ f \circ \varphi^{-1})(\varphi(m)) \cdot e\big)$ (Fig. 9).
The functions $h_{\beta'\alpha}\ :\ U_\alpha \cap h^{-1}(U_{\beta'})\ \rightarrow\ G$ satisfy:

$$\begin{aligned} h_{\beta'\alpha}(m) &= h_{\beta'\gamma}(m)g_{\gamma\alpha}(m) \quad \forall m \in U_\alpha \cap U_\gamma \cap h^{-1}(U_{\beta'}) \\ h_{\beta'\alpha}(m) &= g_{\beta'\gamma'}(h(m))h_{\gamma'\alpha}(m) \quad \forall m \in U_\alpha \cap h^{-1}(U_{\gamma'} \cap U_{\beta'})\,. \end{aligned} \tag{65}$$

The pair $(h,\ \{h_{\beta'\alpha}\})$ defines H globally on P.

Remark If H is an isomorphism of P such that h is the identity on M, the expressions above reduce to:

$$\left| \begin{array}{l} h_{\gamma'\alpha}(m) = h_{\gamma'\beta}(m)g_{\beta\alpha}(m)\ \ m \in U_{\gamma'} \cap U_\beta \cap U_\alpha \\ h_{\delta'\alpha}(m) = g_{\delta'\gamma'}(m)h_{\gamma'\alpha}(m)\ m \in U_{\delta'} \cap U_{\gamma'} \cap U_\alpha\,, \end{array} \right. \tag{66}$$

which express the relationship between the transition functions corresponding to two isomorphic principal bundles, $(\{U_\alpha\},\ g_{\alpha\beta})$ and $(\{U_{\gamma'}\},\ g_{\gamma'\delta'})$.

As a **Corollary**, the transition functions $g_{\alpha\beta}$ and $g'_{\gamma\delta}$ corresponding to two isomorphic bundles, subordinated to the same covering $\{U_\alpha\}$, are related by a family of functions $\{h_\alpha\ :\ U_\alpha\ \rightarrow\ G\}$ such that:

$$g'_{\beta\alpha}(m)h_\alpha(m) = h_\beta(m)g_{\beta\alpha}(m)\,. \tag{67}$$

In fact: it suffices to define $h_\alpha \equiv h_{\alpha\alpha}$.

We come from motivating the following cohomological characterization of Principal Bundles on M, that is, the $\check{H}^1(M, \{U_\alpha\}; G\}$:

Non-equivalent principal bundles on M, with structure group G, are characterized by 1-cocycles $g_{\alpha\beta}$, that is, satisfying:

$$(\delta g)_{\alpha\beta\gamma}(m) \equiv g_{\alpha\beta}(m) g_{\beta\gamma}(m) g^{-1}_{\alpha\gamma}(m) = e \in G \tag{68}$$

which are not coboundaries, that is,

$$g_{\alpha\beta}(m) \neq (\delta h)_{\alpha\beta}(m) \equiv h_\alpha(m) g_{\alpha\beta}(m) h_\beta(m)^{-1} \quad \text{for some family } \{h_\alpha\}\,.$$

In the limit of *refinement* of $\{U_\alpha\}$, with a minimum of elements and minimal intersection, it defines the <u>Čech Cohomology Space</u> $\check{H}^1(M; G)$.

Reconstruction Theorem: Let M be a manifold, $\{U_\alpha\}$ an open covering and G a Lie group. Given a cocycle $\{g_{\alpha\beta}\}$ relative to $\{U_\alpha\}$, valued on G, there exists a unique (except for an isomorphism) principal bundle $P \to M$, with structure group G having $\{g_{\alpha\beta}\}$ as transition functions.

Proof (just sketched): We construct $\Sigma \equiv \cup_\alpha U_\alpha \times G$ and take quotient by the following equivalence relation $\sim$:

$$\begin{aligned} (\alpha, m, a) &\sim (\beta, m', b) \quad \text{if} \\ m &= m' \in U_\alpha \cap U_\beta \\ a &= g_{\alpha\beta}(m) b\,. \end{aligned} \tag{69}$$

The quotient $\Sigma/\sim$ is P, the projection being $p(\alpha, m, a) = m$. □

4.1 Associated vector bundles

Let $P \overset{p}{\to} M$ be a principal bundle characterized by $(\{U_\alpha\}, g_{\alpha\beta})$, and $\lambda : G \longrightarrow GL(F)$ a linear representation of the structure group G on a vector space F.
The set $(\{U_\alpha\}, \lambda \circ g_{\alpha\beta} \equiv \bar{g}_{\alpha\beta})$ constitutes a 1-cocycle relative to $\{U_\alpha\}$ and valued on $GL(F)$.

The quotient $\cup_\alpha U_\alpha \times F/\bar{\sim}$, where now $\bar{\sim}$ is defined as:

$$\begin{aligned} (\alpha, m, v) &\,\bar{\sim}\, (\beta, m', v') \quad \text{if} \\ m &= m' \in U_\alpha \cap U_\beta \\ v' &= \bar{g}_{\beta\alpha}(m) v\,, \end{aligned} \tag{70}$$

defines a *vector bundle* $E \overset{\pi}{\to} M$, with fiber F associated with P trough the representation λ.

<u>G-functions on P</u>: Let $E \overset{\pi_E}{\to} M$ be an associated bundle with fiber L by means of the representation λ, and let $\Gamma(E)$ be the linear space of sections of E, that is, mapping from M to E such that $\pi_E \circ \sigma = I_M$. The following commutative diagram corresponds to a homomorphism between vector bundles:

$$\begin{array}{ccc} P \times L & \xrightarrow{q} & E \\ \sigma' \uparrow\downarrow p_1 & & \pi_E \downarrow\uparrow \sigma \\ P & \xrightarrow{\pi} & M \end{array}$$

q is the natural projection on the equivalence classes defined by $(\xi, v) \sim (\xi s, \lambda(s^{-1} v))$

In fact, q defines E !!

Given $\sigma : M \to E$, there exists $\sigma' : P \to P \times L \;/\; q \circ \sigma' = \sigma \circ \pi$ and, therefore, $\tilde{\sigma} : P \to L \;/\; \sigma'(\xi) = (\xi, \tilde{\sigma}(\xi))$.

The mapping $\beta : \Gamma(E) \to \mathcal{L}(P) \equiv \{\tilde{\sigma} : P \to L\}$ is a homomorphism from the $\mathcal{F}(M)$-module $\Gamma(E)$ to the $\mathcal{F}(P)$-module $\mathcal{L}(P)$.

Definition A differentiable function on P with values on L satisfying the condition

$$\psi(\xi s) = \lambda(s^{-1})\psi(\xi) \tag{71}$$

is called *G-function* and we say that $\psi \in \mathcal{L}_G(P)$.

Proposition *The application $\beta : \Gamma(E) \to \mathcal{L}(P)$ is injective and verifies $\mathcal{L}_G(P) = Img\beta$, that is*

$$\Gamma(E) \approx \mathcal{L}_G(P)\,.$$

From now on, we shall identify sections of a vector bundle E with G-functions on the principal bundle P from which E is an associated vector bundle.

<u>**Vector fields on a Principal Bundle**</u>. The different structures of the base manifold M and the fiber G of a principal bundle P manifest themselves in the behavior of the components of a vector field, in a (principal-bundle) local chart, under a change of coordinates. As we shall see, the components along the fiber keep some identity as vertical components, whereas those along the base cannot be considered as horizontal since this character changes in going from one chart to other. The reason for that lies in the expression of the

<u>change of chart</u>: Denoting the local coordinates as (x^μ, s^a), we have:

$$(x^\mu, s^a) \;\mapsto\; (x^{\nu'}, s^{b'}) \equiv \left(x^{\nu'}(x),\, s^{b'}(x,s)\right)$$
$$s^{b'}(x,s) = g(x)^{b'}_a s^a\,.$$

Then, a vector field X on P, will be written alternatively as

$$X = X^\mu \frac{\partial}{\partial x^\mu} + X^a \frac{\partial}{\partial s^a} = X^{\nu'} \frac{\partial}{\partial x^{\nu'}} + X^{b'} \frac{\partial}{\partial s^{b'}} \tag{72}$$

and the tangent application to the change of coordinates above reads

$$\left.\begin{array}{l} \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x^{\nu'}}\frac{\partial x^{\nu'}}{\partial x^\mu} + \frac{\partial}{\partial s^{b'}}\frac{\partial s^{b'}}{\partial x^\mu} = \frac{\partial}{\partial x^{\nu'}}\frac{\partial x^{\nu'}}{\partial x^\mu} + \frac{\partial}{\partial s^{b'}}\frac{g^{b'}_a}{\partial x^\mu}s^a \\ \frac{\partial}{\partial s^a} = \frac{\partial}{\partial x^{\nu'}}\frac{\partial x^{\nu'}}{\partial s^a} + \frac{\partial}{\partial s^{b'}}\frac{\partial s^{b'}}{\partial s^a} = g^{b'}_a(x)\frac{\partial}{\partial s^{b'}} \end{array}\right| \;\Rightarrow$$

$$X^\mu \frac{\partial}{\partial x^\mu} + X^a \frac{\partial}{\partial s^a} = X^\mu \left(\frac{\partial x^{\nu'}}{\partial x^\mu}\frac{\partial}{\partial x^{\nu'}} + \frac{\partial g^{b'}_a}{\partial x^\mu} s^a \frac{\partial}{\partial s^{b'}} \right) + X^a g^{b'}_a(x) \frac{\partial}{\partial s^{b'}} \;\;\Rightarrow$$

$$\left| \begin{array}{l} X^{\nu'} = \frac{\partial x^{\nu'}}{\partial x^\mu} X^\mu \\ X^{b'} = g^{b'}_a(x) X^a + \frac{\partial g^{b'}_a}{\partial x^\mu} s^a X^\mu\,. \end{array}\right. \tag{73}$$

This way, even though $X^a = 0$, in the new basis X acquires a non-null vertical component $X^{b'} = \frac{\partial g^{b'}_a}{\partial x^\mu} s^a X^\mu$!!.

In other words, the property of X being "horizontal" is not preserved under a change of coordinates.

Only vertical vector fields preserve their structure in changing coordinates.

Therefore, it makes sense to define the vertical subspace $T^v_\xi(P)$ at a point $\xi \in P$.

The submodule $\mathcal{X}^v(P)$ admits a basis made of generators of the action of G on P:

$$\mathcal{X}^v(P) \quad \text{is generated by} \quad \left\{ Z^L_{P(a)}, \quad a = 1, ..., \dim G \right\}$$

where by $Z^L_{P(a)}$ we mean the generator of G associated with $Z_{(a)} \in \mathcal{G}$. Note that $\mathcal{G} \approx T^v_\xi(P)$. $Z^L_{P(a)}$ are called principal vector fields, although not all vector fields in $\mathcal{X}^v(P)$ are principal. $\mathcal{X}^v(P)$ is a free module (it admits a basis) of $\mathcal{F}(P)$-dimension dim$G$.

4.2 Connections on principal bundles

In the last section, we motivated the need for some extra structure in order to define properly the notion of horizontality as regards the components of the vector fields on a Principal Bundle. This extra structure corresponds to a connection.

A **connection** on a principal bundle is a 1-form on P, Γ, $\mathcal{X}^v(P)$-valued, such that:

$$\begin{aligned} &(1) \;\; \Gamma(X) = X \quad \text{if} \;\; X \in \mathcal{X}^v(P) \\ &(2) \;\; \Gamma(a^T_P X) = a^T_P \Gamma(X) \;, \quad a \in G \;, \;\; a_P \;\; \text{the action of} \;\; a \;\; \text{on} \;\; P \;. \end{aligned} \tag{74}$$

That is to say, Γ is a projection of $\mathcal{X}(P)$ onto $\mathcal{X}^v(P)$ invariant under G.

This allows us to define a horizontal submodule $\mathcal{X}^h(P)$ such that:

$$\mathcal{X}(P) = \mathcal{X}^v(P) \oplus \mathcal{X}^h(P) \,. \tag{75}$$

In fact, $\mathcal{X}^h(P) \equiv \text{Ker}\,\Gamma$ and $X^h \equiv X - \Gamma(X)$. Note that $\Gamma(X - \Gamma(X)) = \Gamma(X) - \Gamma(X) = 0$.

Connection 1-form: Denoting χ the isomorphism $T^v_\xi(P) \approx \mathcal{G}$ we define

$$\gamma \equiv \chi \circ \Gamma \,. \tag{76}$$

It is a $\mathcal{G}$-valued 1-form on P with the properties:

$$\begin{aligned} &(1) \;\; \gamma(Z_P) = Z \\ &(2) \;\; a_{P*}\gamma = Ad(a^{-1})\gamma \,. \end{aligned} \tag{77}$$

Curvature 2-form: $K \equiv \mathrm{d}\gamma + [\gamma, \; \gamma]$

Transformation properties ofγ : γ is a 1-form on P, $\mathcal{G}$-valued. Locally we may characterize γ by means of a set $\{\gamma_\alpha\}$ of 1-forms on M. In fact, given $\{U_\alpha\} \ni M$, we define:

$$\begin{aligned} \gamma_\alpha &\equiv \sigma^*_\alpha \gamma \qquad \text{on} \;\; U_\alpha \\ \theta_{\alpha\beta} &\equiv g^*_{\alpha\beta}\theta \quad\;\; \text{on} \;\; U_\alpha \cap U_\beta \,. \end{aligned}$$

Proposition *On the intersection $U_\alpha \cap U_\beta$, we have:*

$$\gamma_\beta = Ad(g^{-1}_{\beta\alpha})\gamma_\alpha + \theta_{\beta\alpha} \,. \tag{78}$$

Example 1 Case of $G = GL(n)$ as the structure group of the Reference Bundle. We shall use the matrix elements as coordinates, so that we have:

$$\{u^i\} \to \{s^i_j\}$$

$$\{e_{(i)}\} \to \{e^{(j)}_{(i)}\}$$

$$\{\epsilon^{(i)}\} \to \{\epsilon^{(i)}_{(j)}\} = \{ds^i_j(e, \cdot)\}$$

We compute the explicit expression of the left translation and its tangent:

$$\begin{aligned}
\left(L_g s\right)^i_j &= s(g)^i_k s^k_j \equiv g^i_k s^k_j \equiv s^{i'}_{j'} \\
\left(L_{g^{-1}} s\right)^i_j &= {g^{-1}}^i_{\ k} s^k_j \\
\left(L^T_{g^{-1}}\right)^{i'm}_{j'n} &= \frac{\partial s^{i'}_{j'}}{\partial s^n_m} = {g^{-1}}^i_{\ k}\delta^k_n\delta^m_j = {g^{-1}}^i_{\ n}\delta^m_j \\
&\equiv s(g^{-1})^i_n\delta^m_j \equiv (s^{-1})^i_n\delta^m_j\,,
\end{aligned}$$

where we have relaxed the notation so as to identify s^i_j and $s(g)^i_j$, as usual. Then, we obtain:

$$\begin{aligned}
\theta &= \theta^{(i)}_{(j)} \circ e^{(j)}_{(i)} \equiv \theta^{(i)m}_{(j)n} ds^n_m \circ e^{(j)}_{(i)} \\
\theta^{(i)m}_{(j)n} &= {s^{-1}}^i_{\ n}\delta^m_j \\
\theta^{(i)}_{(j)} &= {s^{-1}}^i_{\ n}\delta^m_j ds^n_m = {s^{-1}}^i_{\ n} ds^n_j
\end{aligned}$$

and finally, and in global and symbolic form, though rather standard,

$$\theta^L = g^{-1}dg \quad \text{and analogously} \quad \theta^R = dgg^{-1}\,. \tag{79}$$

Let us compute $g^*_{\alpha\beta}\theta$ on the intersection $U_\alpha \cap U_\beta$. For simplicity, we denote $\{x^i\}$ the coordinates on U_α and $\{\bar{x}^j\}$ those on U_β, then

$$\begin{aligned}
\left[g_{\alpha\beta}(x)\right]^m_n &= \frac{\partial x^m}{\partial \bar{x}^n}\,, \quad \left[g_{\beta\alpha}(x)\right]^i_j = \frac{\partial \bar{x}^i}{\partial x^j}\,, \quad \left[g^T_{\alpha\beta}(x)\right]^i_{jk} = \frac{\partial^2 x^i}{\partial \bar{x}^j \partial \bar{x}^k} \\
X &= X^i\frac{\partial}{\partial x^i} = \bar{X}^k\frac{\partial}{\partial \bar{x}^k} \quad | \quad g^T_{\alpha\beta}(x)X = \frac{\partial^2 x^i}{\partial \bar{x}^j \partial \bar{x}^k}\bar{X}^k\frac{\partial}{\partial s^i_j} \\
(g^*_{\beta\alpha}\theta)(X) &= \theta(g^T_{\beta\alpha}(X)) = \theta\left(\frac{\partial^2 x^i}{\partial \bar{x}^j \partial \bar{x}^k}\bar{X}^k\frac{\partial}{\partial s^i_j}\right) \\
&= (s^{-1})^i_j(g_{\beta\alpha}(x))ds^j_k \circ e^{(k)}_{(i)}\left(\frac{\partial^2 x^m}{\partial \bar{x}^p \partial \bar{x}^n}\bar{X}^n\frac{\partial}{\partial s^m_p}\right) \\
&= \frac{\partial \bar{x}^i}{\partial x^j}\frac{\partial^2 x^m}{\partial \bar{x}^p \partial \bar{x}^n}\bar{X}^n\delta^j_m\delta^p_k \circ e^{(k)}_{(i)} = \frac{\partial \bar{x}^i}{\partial x^j}\frac{\partial^2 x^j}{\partial \bar{x}^k \partial \bar{x}^n}\bar{X}^n \circ e^{(k)}_{(i)} \\
&= \frac{\partial \bar{x}^i}{\partial x^j}\frac{\partial^2 x^j}{\partial \bar{x}^k \partial \bar{x}^n}d\bar{x}^n(X) \circ e^{(k)}_{(i)} \\
&\Rightarrow \left[g^*_{\beta\alpha}\theta\right]^{(i)}_{(k)} = \frac{\partial \bar{x}^i}{\partial x^j}\frac{\partial^2 x^j}{\partial \bar{x}^k \partial \bar{x}^n}d\bar{x}^n\,.
\end{aligned}$$

Denoting Γ^k_{ij} the components of γ:

$$\gamma_\alpha = \gamma^{(k)}_{(i)} \circ e^{(i)}_{(k)} = \Gamma^k_{ij}\mathrm{d}x^j \circ e^{(i)}_{(k)}\,, \quad \gamma_\beta = \bar{\gamma}^{(t)}_{(r)} \circ e^{(r)}_{(t)} = \bar{\Gamma}^t_{rn}\mathrm{d}x^j \circ e^{(r)}_{(t)}\,.$$

The transformation property

$$\gamma_\beta(X) = Ad(g^{-1}_{\beta\alpha})\gamma_\alpha(X) + \theta_{\beta\alpha}(X) \quad \Rightarrow \tag{80}$$

$$\gamma_\beta(X) \equiv \bar{\Gamma}^t_{rn}d\bar{x}^n(X) \circ e^{(r)}_{(t)} = \Gamma^s_{mn}\mathrm{d}x^n(X) \circ \overbrace{\frac{\partial x^m}{\partial \bar{x}^r}e^{(r)}_{(t)}\frac{\partial \bar{x}^t}{\partial x^s}}^{g^{-1}_{\beta\alpha}Zg_{\beta\alpha}} + \frac{\partial \bar{x}^t}{\partial x^j}\frac{\partial^2 x^j}{\partial \bar{x}^r \partial \bar{x}^n}d\bar{x}^n(X) \circ e^{(r)}_{(t)}$$

$$= \Gamma^s_{mu} \frac{\partial x^u}{\partial \bar{x}^n} d\bar{x}^n(X) \frac{\partial x^m}{\partial \bar{x}^r} \frac{\partial \bar{x}^t}{\partial x^s} \circ e^{(r)}_{(t)} + \frac{\partial \bar{x}^t}{\partial x^j} \frac{\partial^2 x^j}{\partial \bar{x}^r \partial \bar{x}^n} d\bar{x}^n(X) \circ e^{(r)}_{(t)} \quad \Rightarrow$$

$$\bar{\Gamma}^k_{ij} = \frac{\partial x^n}{\partial \bar{x}^i} \frac{\partial x^s}{\partial \bar{x}^j} \frac{\partial \bar{x}^k}{\partial x^m} \Gamma^m_{ns} + \frac{\partial^2 x^m}{\partial \bar{x}^i \partial \bar{x}^j} \frac{\partial \bar{x}^k}{\partial x^m}, \tag{81}$$

where we have computed $Ad(g^{-1})Z$ as $g^{-1}Zg$, as corresponding to the action of a linear group. The symbols Γ^k_{ij} are non-tensorial (due to the affine term in the transformation law) and are called *Christoffel Symbols*.

Example 2 Case of structure group $G = U(1)$. This is a very special, though simple, case relevant in both gauge theory and quantization. The elements of the group are parameterized globally by $\zeta \in C, \quad / \quad |\zeta| = 1$, and locally by $\zeta = e^{i\phi}$. The canonical 1-form and the transition one are:

$$\theta = \frac{\mathrm{d}\zeta}{i\zeta}, \quad \theta_{\beta\alpha} = \frac{dg_{\beta\alpha}}{ig_{\beta\alpha}} \quad (Ad = I)$$

and the transformation rule,

$$\gamma_\beta = \gamma_\alpha + \frac{dg_{\beta\alpha}}{ig_{\beta\alpha}}.$$

4.3 Derivation law on associated vector bundles

Derivation law on an A-module $\mathcal{M}$: We provide the more general (algebraic) definition and then specify the more relevant cases.
Let

$$\begin{aligned} &K \;\text{ be a commutative ring} \\ &A \;\text{ be an algebra over the ring } \; K \\ &\mathcal{M} \;\text{ be a module over } \; A \;\; (A - \text{module}). \end{aligned}$$

A derivation law on $\mathcal{M}$ is a mapping

$$\nabla \in \mathrm{Hom}_A\left(\mathrm{Der}\, A,\; \mathrm{Hom}_K\left(\mathcal{M},\, \mathcal{M}\right)\right). \tag{82}$$

Two derivation laws differ in an element of

$$\mathrm{Hom}_A\left(\mathrm{Der}\, A,\; \mathrm{Hom}_A\left(\mathcal{M},\, \mathcal{M}\right)\right). \tag{83}$$

It must be remarked that a derivation law is not tensorial since the elements in $\mathrm{Hom}_K(\mathcal{M}, \mathcal{M})$ are only linear with respect to the scalars in K. Conversely, the elements in $\mathrm{Hom}_A(\mathcal{M}, \mathcal{M})$ are linear with respect to "scalars" in A, so that the difference of two derivations laws is a tensor. This extent will be nitid in the following

Example

$$\left| \begin{array}{l} K \equiv R \quad (M \equiv \text{differentiable manifold}) \\ A \equiv \mathcal{F}(M) \quad \Rightarrow \\ \mathrm{Der} A = \mathcal{X}(M) \\ \mathcal{M} \equiv \mathcal{X}(M). \end{array} \right.$$

A derivation law then turns out to be a derivation law for vector fields (usually referred to as "connection"):

$$\nabla \in \mathrm{Hom}_{\mathcal{F}(M)}\left(\mathcal{X}(M),\ \mathrm{Hom}_R(\mathcal{X}(M),\ \mathcal{X}(M))\right)$$
$$\nabla\ :\ X\ \mapsto\ \nabla_X\ /$$
$$\nabla_X\ :\ Y\ \mapsto\ \nabla_X Y\,. \tag{84}$$

Taking a local basis in $\mathcal{X}(U)$, $\left\{X_{(i)}\right\}$, we have:

$$\begin{aligned}
&\nabla_{X_{(i)}} X_{(j)} \equiv \Gamma^k_{ij} X_{(k)} \qquad X_{(i)} \equiv e_i \\
&\nabla_{X^i e_i}(Y^j e_j) = X^i \nabla_{e_i}(Y^j e_j) = X^i \frac{\partial Y^j}{\partial x^i} + X^i Y^k \Gamma^j_{ik} e_j \quad \Rightarrow \\
&\nabla_i Y^j = \frac{\partial Y^j}{\partial x^i} + \Gamma^j_{ik} Y^k\,.
\end{aligned} \tag{85}$$

More generally, $\mathcal{M}$ is the module of sections, $\Gamma(E)$, of a vector bundle E over M, with basis $\{\chi_\alpha\}$: $\psi = \psi^\alpha \chi_\alpha \in \Gamma(E)$

$$\begin{aligned}
\nabla_{e_i} \chi_\alpha &\equiv \Gamma^\beta_{i\alpha} \chi_\beta \quad \Rightarrow \\
\nabla_i \psi^\alpha &= \frac{\partial \psi^\alpha}{\partial x^i} + \Gamma^\alpha_{i\beta} \psi^\beta\,.
\end{aligned} \tag{86}$$

If now P is a principal bundle over M with structure group G, ρ is a linear representation of G on $GL(F)$, and E is a vector bundle associated with P, through the representation ρ, with any connection γ on P we may associate the following derivation law on $\Gamma(E)$, ∇^γ:

$$\nabla^\gamma_X \psi = X.\psi + \rho(\gamma(X))\psi \tag{87}$$

$$\Gamma^\alpha_{i\beta} = \gamma^{(a)}_i \rho(Z_{(a)})^\alpha_\beta \qquad \Bigg|\ \gamma = \overbrace{\gamma^{(a)}_i \mathrm{d}x^i}^{(A^{(a)}_\mu \mathrm{d}x^\mu)} \circ Z_{(a)} \tag{88}$$

$A^{(a)}_\mu \equiv$ vector potentials or *Yang–Mills fields*.

5 Variational calculus

After a more traditional exposition of variational calculus as in standard textbooks [22], we recommend intermediate texts as [23,24] and, finally, more formal papers as [25] and references therein.

5.1 Jet bundles

Let $E \xrightarrow{\pi} M$ be a vector bundle on M, $x \in M$, and $\Gamma_x(E)$ the space of all local (differentiable) sections about x. In $\Gamma_x(E)$, we define the following equivalence relation $\overset{1}{\sim}$:

$$\psi \overset{1}{\sim} \psi' \quad \Leftrightarrow \quad \left|\begin{array}{l} \psi'(x) = \psi(x) \\ \partial_\mu \psi'(x) = \partial_\mu \psi(x) \end{array}\right., \tag{89}$$

and consider the quotient space $J^1_x(E) \equiv \Gamma_x(E)/\overset{1}{\sim}$, and the natural projection

$$\begin{aligned}
\pi^1\ :\ J^1_x(E) &\longrightarrow M \\
(\psi, x) &\mapsto\ x\,.
\end{aligned}$$

The union $J^1(E) \equiv \cup_x J^1_x(E) \xrightarrow{\pi^1} M$ is called the bundle of 1-jets of $\Gamma(E)$ (the space of sections of E).
$J^1(E) \xrightarrow{\pi^1} M$ is parameterized locally by $\{x^\mu,\ \psi^\alpha,\ \psi^\beta_\nu\}$.
Given a section $\psi : M \rightarrow E$, we can define its 1-jet extension

$$j^1(\psi)(x) = (\psi^\alpha(x),\ \psi^\beta_\mu(x) = \partial_\mu \psi^\beta(x))\,, \tag{90}$$

which is an immersion of $\Gamma(E) \hookrightarrow \Gamma(J^1(E))$.
The structure 1-forms $\{\theta^\alpha\}$,

$$\theta^\alpha = \mathrm{d}\psi^\alpha - \psi^\alpha_\mu \mathrm{d}x^\mu\,, \tag{91}$$

characterize the jet extension of sections and vector fields:

$$\theta^\alpha|_{j^1(\psi)(M)} = 0\,.$$

In the same way, given $X \in \mathcal{X}(M)$, $j^1(X) \in \mathcal{X}(J^1(E))$, is the only field that projects on X and preserves the 1-form system $\{\theta^\alpha\}$:

$$j^1(X) \equiv \bar{X},\quad \bar{X} = X + \bar{X}^\alpha_\mu \frac{\partial}{\partial \psi^\alpha_\mu}\,, \tag{92}$$

$$L_{\bar{X}}\theta^\alpha = C^\alpha_\beta \theta^\beta \quad \Rightarrow \quad \left| \begin{array}{l} C^\alpha_\beta = \frac{\partial X^\alpha}{\partial \psi^\beta} - \psi^\alpha_\mu \frac{\partial X^\mu}{\partial \psi^\beta} \\ \bar{X}^\alpha_\mu = \frac{\partial X^\alpha}{\partial x^\mu} - \psi^\alpha_\nu \frac{\partial X^\nu}{\partial x^\mu} + \left(\frac{\partial X^\alpha}{\partial \psi^\beta} - \psi^\alpha_\nu \frac{\partial X^\nu}{\partial \psi^\beta} \right) \psi^\beta_\mu\,. \end{array} \right. \tag{93}$$

The jet extension is a Lie algebra homomorphism:

$$j^1([X,\ Y]) = [j^1(X),\ j^1(Y)]\,. \tag{94}$$

Lagrangian (density): A Lagrangian density is a real function $\mathcal{L}$ on $J^1(E)$. Then, we define the Action functional

$$\begin{array}{c} \mathcal{S} : \Gamma(E) \longrightarrow R\ / \\ \mathcal{S}(\psi) = \int_{j^1(\psi)(M)} \mathcal{L}(j^1(\psi))\pi^{1*}\omega\,, \end{array} \tag{95}$$

where ω is a volume n-form on M and π^{1*} is its pull-back to $J^1(E)$. (Usually, M is the Minkowski space-time and $\omega = \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$).

5.2 Hamilton principle

The Ordinary Hamilton Principle establishes that the critical sections of the variational problem are the points of $\Gamma(E)$ where $\delta\mathcal{S}$, the "differential" of $\mathcal{S}$, is zero, that is:

$$\psi \in \Gamma(E)\ /\ \delta\mathcal{S}_{(\psi)}(X) \equiv \int_{j^1(\psi)(M)} L_{\bar{X}}\left(\mathcal{L}(j^1(\psi))\omega\right) = 0\,,\quad \forall\ X \in \mathcal{X}(E)\,. \tag{96}$$

As is well known, critical sections satisfy the Euler–Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \psi^\alpha_\mu}\right) - \frac{\partial \mathcal{L}}{\partial \psi^\alpha} = 0\,, \tag{97}$$

where $\frac{\mathrm{d}}{\mathrm{d}x^\mu}$ stands for "derivative with respect to x^μ along the section ψ."
Exercise: Derive the Euler–Lagrange equations!!

Hint: Realize that the variations δx^μ, $\delta\psi^\alpha$ and $\delta\partial_\mu\psi^\alpha$ correspond to the components X^μ, X^α, X^α_μ of $\bar{X}$. The fact that ψ^α_μ behaves as $\partial_\mu\psi^\alpha$ under variation is related to the fact that $\bar{X}$ is $j^1(X)$.

5.3 Modified Hamilton principle: the Poincaré–Cartan form

The Modified Hamilton Principle assumes the independent variation of ψ^α and ψ^β_ν. That means that we look for critical sections in the module $\Gamma(J^1(E))$, rather than $\Gamma(E)$, where the variations are caused by arbitrary $X^1 \in \mathcal{X}(J^1(E))$ that are no longer jet extensions.

The Modified Hamilton Action $\mathcal{L}^1 \ : \ \Gamma(J^1(E)) \longrightarrow R$ is defined as the integral

$$\mathcal{S}^1(\psi^1) = \int_{\psi^1(M)} \Theta_{PC}\,, \tag{98}$$

where the *Poincaré–Cartan(–Hilbert)* form is a (n =dim M)-form defined by

$$\Theta_{PC} = \frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}(\mathrm{d}\psi^\alpha - \psi^\alpha_\nu \mathrm{d}x^\nu) \wedge \theta_\mu + \mathcal{L}\omega\,, \tag{99}$$

where $\theta_\mu \equiv i_{\frac{\partial}{\partial x^\mu}}\omega$.

The Poincaré–Cartan n-form can also be written as

$$\Theta_{PC} = \frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\mathrm{d}\psi^\alpha \wedge \theta_\mu - \left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\psi^\alpha_\mu - \mathcal{L}\right)\omega\,. \tag{100}$$

When $\mathcal{L}$ is regular, that is, det $\left(\frac{\partial^2\mathcal{L}}{\partial\psi^\alpha_\mu\psi^\beta_\nu}\right) \neq 0$, we may define the covariant Hamiltonian

$$\mathcal{H} \equiv \pi^\mu_\alpha\psi^\alpha_\mu - \mathcal{L}\,, \tag{101}$$

where $\pi^\mu_\alpha \equiv \frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}$ are the covariant momenta and the form Θ_{PC} can be written as

$$\Theta_{PC} = \pi^\mu_\alpha \mathrm{d}\psi^\alpha \wedge \theta_\mu - \mathcal{H}\omega\,. \tag{102}$$

Remark The Poincaré–Cartan form might be redefined as

$$\Theta_{PC} = \left[\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}(\mathrm{d}\psi^\alpha - \psi^\alpha_\nu \mathrm{d}x^\nu) + \frac{1}{4}\mathcal{L}\mathrm{d}x^\mu\right] \wedge \theta_\mu \equiv \mathcal{T}^\mu_{PC} \wedge \theta_\mu \tag{103}$$

for future relationships.

The Modified Hamilton Principle defines critical sections as those sections $\psi^1 \in \Gamma(J^1(E))$ on which the functional derivative of $\mathcal{S}^1$, $\delta\mathcal{S}^1$, is zero:

$$\begin{aligned}&(\delta\mathcal{S}^1)_{\psi^1}(X^1) \equiv \int_{\psi^1(M)} L_{X^1}\Theta_{PC} = 0 \quad \forall X^1 \in \mathcal{X}(J^1(E)) \quad \Rightarrow \\ &i_{X^1}\mathrm{d}\Theta_{PC}|_{\psi^1} = 0\,.\end{aligned} \tag{104}$$

The equations of motion above generalize the Euler–Lagrange ones in the sense that if $\mathcal{L}$ is regular

$$i_{X^1}\mathrm{d}\Theta_{PC}|_{\psi^1} = 0 \ \ \forall X^1 \quad \Rightarrow \quad \left|\begin{array}{l}\text{Euler–Lagrange equation}\\ \psi^1 = j^1(\psi)\,.\end{array}\right. \tag{105}$$

Let us remark that Θ_{PC} reduces to $\mathcal{L}\omega$ on jet extensions since $\Theta_{PC} = \pi^{\mu}_{\alpha}\theta^{\alpha} \wedge \theta_{\mu} + \mathcal{L}\omega$ and $\theta^{\alpha}|_{j^1(\psi)} = 0$.

In the regular case, $i_{X^1}\mathrm{d}\Theta_{PC}|_{\psi^1} = 0$ can be taken into the Hamiltonian form:

$$\frac{\partial \mathcal{H}}{\partial \pi^{\mu}_{\alpha}} = \frac{\partial \psi^{\alpha}}{\partial x^{\mu}}\,, \quad \frac{\partial \mathcal{H}}{\partial \psi^{\alpha}} = -\frac{\partial \pi^{\mu}_{\alpha}}{\partial x^{\mu}}\,. \tag{106}$$

We shall remark that in the Ordinary Variational Calculus people define only $\pi_{\alpha} \equiv \pi^{0}_{\alpha}$, the time component, and the <u>non-covariant</u> Hamiltonian $H = \pi_{\alpha}\psi^{\alpha}_{0} - \mathcal{L} \quad (\pi_{\alpha}\dot{\psi}^{\alpha} - \mathcal{L})$. The extra Hamiltonian equations we have, simply provide the definition of covariant momenta. **Note:** The non-covariant Hamiltonian H will be obtained in our scheme as the time component of the conserved current associated with the invariance under time translations (see later).

5.4 Symmetries and the Noether Theorem: Hamilton–Jacobi transformation and Solution Manifold

A symmetry of the variational problem is a vector field $Y^1 \in \mathcal{X}(J^1(E))$ such that

$$L_{Y^1}\Theta_{PC} = \mathrm{d}\alpha_{Y^1}\,, \quad \alpha_{Y^1} \quad (n-1) - \text{form}\,. \tag{107}$$

We actually say that Θ_{PC} is semi-invariant if $\alpha_{Y^1} \neq 0$.

Theorem (Noether): If Y^1 is a symmetry of Θ_{PC}, the quantity $\mathcal{J}_{Y^1} \equiv i_{Y^1}\Theta_{PC} - \alpha_{Y^1}$ is conserved <u>along the solutions</u>.

Proof $L_{Y^1}\Theta_{PC} \equiv i_{Y^1}\mathrm{d}\Theta_{PC} + d i_{Y^1}\Theta_{PC} = \mathrm{d}\alpha_{Y^1}$. Restricting this expression to solutions, we have:

$$i_{Y^1}\mathrm{d}\Theta_{PC}|_{sol.} = 0 = \mathrm{d}(\alpha_{Y^1} - i_{Y^1}\Theta_{PC})|_{sol}\,.$$

The quantity $\mathcal{J}_{Y^1} \equiv i_{Y^1}\Theta_{PC} - \alpha_{Y^1}$ is the Noether Invariant.

Note that $\mathcal{J}_{Y^1}$ is an $(n-1)$-form and we can define the dual current (we shall omit the subscript) $J \equiv (i_{Y^1}\Theta_{PC} - \alpha_{Y^1})$.

If we denote $j \equiv i_{Y^1}\Theta_{PC} = \frac{\partial \mathcal{L}}{\partial \psi^{\alpha}_{\mu}}(Y^{\alpha} - \partial_{\nu}\psi^{\alpha}Y^{\nu})\theta_{\mu} + \mathcal{L}Y^{\mu}\theta_{\mu}$, that is, the conserved current for strict invariance, we can write the vector current

$$j^{\mu} = \frac{\partial \mathcal{L}}{\partial \psi^{\alpha}_{\mu}}(Y^{\alpha} - \partial_{\nu}\psi^{\alpha}Y^{\nu}) + \mathcal{L}Y^{\mu}\,. \tag{108}$$

In terms of J, the constancy of $i_{Y^1}\Theta_{PC} - \alpha_{Y^1}$ along solutions becomes

$$\partial_{\mu}J^{\mu}|_{sol} = 0 \quad \Rightarrow \quad \int_{\Sigma} \mathrm{d}\sigma_{\mu}J^{\mu} \equiv Q\,, \tag{109}$$

where Σ is a Cauchy surface, is a constant. It is named *conserved charge* associated with the symmetry. □

5.5 Examples

The free Galilean particle:

$$\left|\begin{array}{ll} E = R \times R^3 \to R & , \; J^1(E) = R \times R^3 \times R^3 \to R \\ \{t, x^i\} & \{t, x^i, \dot{x}^j\} \\ \theta^i = \mathrm{d}x^i - \dot{x}^i \mathrm{d}t & \mathcal{L} = \frac{1}{2} m \dot{\mathbf{x}}^2 \end{array}\right.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^i}\right) - \frac{\partial \mathcal{L}}{\partial x^i} = 0 \;\Rightarrow\; \frac{\mathrm{d}}{\mathrm{d}t} m \dot{x}^i = 0\,,\; \dot{x}^i = \frac{\mathrm{d}x^i}{\mathrm{d}t} \;\Rightarrow$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} x^i = 0 \;\Rightarrow\; \begin{cases} x^i = K^i + \frac{P^i}{m}\tau \\ p_i = P_i \\ t = \tau \end{cases}$$

$$\left(p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}^i}, \quad H = p_i \dot{x}^i - \mathcal{L} = \frac{\boldsymbol{p}^2}{2m}\right). \tag{110}$$

The expression above concerning the trajectories of the free particle can be read as an invertible transformation in $R \times R^3 \times R^3$ to be referred to as the

$$\textit{Hamilton–Jacobi transformation} \quad \Leftrightarrow \quad (x^i, p_j, t) \leftrightarrow (K^i, P_j, \tau)\,. \tag{111}$$

This Hamilton–Jacobi transformation permits the pass to the Solution Manifold parameterized by the basic constants of motion.

After this (H–J) transformation, Θ_{PC} comes down to the Solution Manifold, except for a total differential:

$$\begin{aligned} \Theta_{PC} &= \frac{\partial \mathcal{L}}{\partial \dot{x}^i}(\mathrm{d}x^i - \dot{x}^i \mathrm{d}t) + \mathcal{L}\mathrm{d}t = p_i \mathrm{d}x^i - \frac{\boldsymbol{p}^2}{2m}\mathrm{d}t = P_i d\left(K^i + \frac{P^i}{m}\tau\right) - \frac{\boldsymbol{P}^2}{2m}\mathrm{d}\tau \\ &= P_i \mathrm{d}K^i + \frac{\boldsymbol{P}^2}{m}\mathrm{d}\tau + \frac{P_i \tau}{m}\mathrm{d}P^i - \frac{\boldsymbol{P}^2}{2m}\mathrm{d}\tau = P_i dK^i + d\left(\frac{\boldsymbol{P}^2}{2m}\tau\right). \end{aligned} \tag{112}$$

Its differential is

$$\mathrm{d}\Theta_{PC} = dP_i \wedge dK^i \equiv \omega = \mathrm{d}\Lambda\,, \tag{113}$$

that is, the symplectic form on the Solution Manifold.
$\Lambda \equiv P_i dK^i$ is the Potential 1-form or Liouville 1-form.

Reminder: At this moment, we must remind the reader some few words on Symplectic Manifold (to be completed with traditional references like Ref. [5] and/or Ref. [4]):
Let (S, ω) be a symplectic manifold.
X on S is locally Hamiltonian if $i_X\omega = \alpha$, a closed 1-form.
X on S is globally Hamiltonian if $i_X\omega = -df$, (an exact 1-form).
Since $\det(\omega) \neq 0$, given $f : S \to R$, the equation

$$i_{X_f}\omega = -df \quad \text{determines} \quad X_f\,.$$

The correspondence $f \mapsto X_f$ is a homomorphism with kernel R.
Poisson Bracket: $f, g \;\mapsto\; \{f, g\} \;/\; i_{[X_f, X_g]}\omega = -d\{f, g\}$.

Symmetries of the free particle:

$$X_{(B)} \equiv \frac{\partial}{\partial t}\,,\ X_{(\mathbf{A})} \equiv \frac{\partial}{\partial \mathbf{x}}\,,\ X_{(\mathbf{V})} \equiv t\frac{\partial}{\partial \mathbf{x}}\,,\ X_{(\boldsymbol{\epsilon})} \equiv \mathbf{x}\wedge\frac{\partial}{\partial \mathbf{x}}$$
$$\bar{X}_{(B)} = X_{(B)}\,,\ \bar{X}_{(\mathbf{A})} = X_{(\mathbf{A})}\,,\ \bar{X}_{(\mathbf{V})} = t\frac{\partial}{\partial \mathbf{x}} + \frac{\partial}{\partial \dot{\mathbf{x}}}\ \ \bar{X}_{(\boldsymbol{\epsilon})} = \mathbf{x}\wedge\frac{\partial}{\partial \mathbf{x}} + \dot{\mathbf{x}}\wedge\frac{\partial}{\partial \dot{\mathbf{x}}} \tag{114}$$

$$\begin{aligned}
L_{\bar{X}_{(B)}}\Theta_{PC} = 0 \quad &\Rightarrow \quad \mathcal{J}_{(B)} = i_{\bar{X}_{(B)}}\Theta_{PC} = \frac{1}{2}m\dot{x}^2 = \frac{\mathbf{P}^2}{2m}\\
L_{\bar{X}_{(\mathbf{A})}}\Theta_{PC} = 0 \quad &\Rightarrow \quad \mathcal{J}_{(\mathbf{A})} = i_{\bar{X}_{(\mathbf{A})}}\Theta_{PC} = m\dot{\mathbf{x}} \equiv \mathbf{P}\\
L_{\bar{X}_{(\mathbf{V})}}\Theta_{PC} = \mathrm{d}(m\mathbf{x}) \quad &\Rightarrow \quad \mathcal{J}_{(\mathbf{V})} = m\mathbf{x} - i_{\bar{X}_{(\mathbf{V})}}\Theta_{PC} = m\mathbf{x} - \mathbf{p}t \equiv \mathbf{K}\\
L_{\bar{X}_{(\boldsymbol{\epsilon})}}\Theta_{PC} = 0 \quad &\Rightarrow \quad \mathcal{J}_{(\boldsymbol{\epsilon})} = i_{\bar{X}_{(\boldsymbol{\epsilon})}}\Theta_{PC} = \mathbf{x}\wedge\mathbf{p} = \mathbf{K}\wedge\mathbf{P}\,.
\end{aligned} \tag{115}$$

Note that all Noether invariants are written in terms of the basic ones **K**, **P**.

The free scalar field (Klein–Gordon)

Klein–Gordon fields are sections ϕ of the line (R for real fields, C for charged ones) vector bundle over Minkowski space-time M [24] (see for instance [26–28] for a more physically minded presentation)

$$\text{Klein–Gordon}\ \left|\begin{array}{l} E = R\times M \ \rightarrow\ M\,,\ \ \phi\in\Gamma(E)\\ \text{Lagrangian}\ \ \mathcal{L} = \frac{1}{2}\phi_\mu\phi^\mu - \frac{1}{2}m^2\phi\phi \end{array}\right. .$$

The Euler–Lagrange equations lead to:

$$\frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi_\mu}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \quad\Rightarrow\quad \Box\phi + m^2\phi = 0\,,\quad \text{with solutions:}$$
$$\phi(x) = \int\frac{\mathrm{d}^3k}{2k^0}\{a(\mathbf{k})e^{-ik\cdot x} + a^*(\mathbf{k})e^{ik\cdot x}\}\,,\quad k^0 = \sqrt{\mathbf{k}^2+m^2} \tag{116}$$
$$a(\mathbf{k}) = i\int_\Sigma \mathrm{d}s^\mu e^{ik\cdot x}\overset{\leftrightarrow}{\partial_\mu}\phi(x)\,,\quad a^*(\mathbf{k}) = -i\int_\Sigma \mathrm{d}s^\mu e^{-ik\cdot x}\overset{\leftrightarrow}{\partial_\mu}\phi(x)\,, \tag{117}$$

where Σ is a Cauchy surface, usually R^3 $(x^0=0)$. The "constants" $a(\mathbf{k})$, $a^*(\mathbf{k})$ parameterize the Solution Manifold.

The Poincaré–Cartan form can be written as

$$\Theta_{PC} = \pi^\mu\mathrm{d}\phi\wedge\theta_\mu - \mathcal{H}\omega\,, \tag{118}$$

where $\mathcal{H} = \frac{1}{2}\pi_\mu\pi^\mu + \frac{1}{2}m^2\phi\phi$ is the covariant Hamiltonian.

Space-time symmetry: The K–G Lagrangian is invariant under the Poincaré group generated by

$$X_{(\mu)} = \frac{\partial}{\partial x^\mu}\ ;\quad X_{(\mu\nu)} = \delta^{\sigma\rho}_{\mu\nu}x_\sigma\frac{\partial}{\partial x^\rho} \tag{119}$$

with jet extension:

$$\bar{X}_{(\mu)} = \frac{\partial}{\partial x^\mu}\ ;\quad \bar{X}_{(\mu\nu)} = \delta^{\sigma\rho}_{\mu\nu}x_\sigma\frac{\partial}{\partial x^\rho} + \delta_{\mu\nu,}{}^{\sigma\cdot}{}_{\rho}\phi_\sigma\frac{\partial}{\partial\phi_\rho}\,.$$
$$L_{\bar{X}_{(\mu)}}\Theta_{PC} = 0\,,\ \ i_{X_{(\mu)}}\Theta_{PC} = \left(\mathcal{L}X^\sigma_{(\mu)} - \frac{\partial\mathcal{L}}{\partial\phi_\sigma}\partial_\nu\phi X^\nu_{(\mu)}\right)\theta_\sigma\quad \left|\begin{array}{l}\theta_\mu = i_{\frac{\partial}{\partial x^\mu}}\omega \equiv \mathrm{d}\sigma_\mu\\ \omega\equiv\ \text{volume on}\ \ M\end{array}\right.$$

$$\Rightarrow \quad j^{\sigma}_{(\mu)} \equiv \delta^{\sigma}_{\mu}\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \phi_{\sigma}}\partial_{\mu}\phi = \left(\frac{1}{2}\partial_{\nu}\phi\partial^{\nu}\phi - \frac{1}{2}m^2\phi^2\right)\delta^{\sigma}_{\mu} - \partial^{\sigma}\phi\partial_{\mu}\phi \;\; \text{conserved}$$
$$\Rightarrow \quad Q_{(\mu)} \equiv \int_{\Sigma} \mathrm{d}\sigma_{\nu} j^{\nu}_{(\mu)} \equiv P_{\mu} \;\; \text{constant}$$

In particular

$$\left| \begin{array}{l} H = \int \mathrm{d}^3x \left(\frac{1}{2}\dot{\phi}\dot{\phi} + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{1}{2}m^2\phi^2\right) \\ \mathbf{P} = \int \mathrm{d}^3x\dot{\phi}\nabla\phi\,. \end{array} \right. \tag{120}$$

In the same way

$$L_{\bar{X}_{(\mu\nu)}}\Theta_{PC} = 0 \quad \Rightarrow \quad j^{\sigma}_{(\mu\nu)} \equiv -\delta^{\epsilon\kappa}_{\mu\nu}x_{\epsilon}\partial^{\sigma}\phi\partial_{\kappa}\phi + \frac{1}{2}\delta^{\epsilon\sigma}_{\mu\nu}x_{\epsilon}\partial_{\kappa}\phi\partial^{\kappa}\phi \;\; \text{conserved}$$
$$\Rightarrow \quad M_{(\mu\nu)} \equiv \int \mathrm{d}^3x(j^0_{(\mu)}x_{\nu} - j^0_{(\nu)}x_{\mu}) \;\; \text{constant}\,. \tag{121}$$

The space-time symmetries play the analogous role of time translations in Mechanics and the corresponding Noether invariants do not contribute to the Solution Manifold, that is to say: SM cannot be parameterized by Noether invariants associated with space-time symmetries.
"Internal" symmetries: (Such symmetries are rarely reported in Literature and considered as "hidden symmetries" [27])

The following vector fields on the bundle E are non-trivial symmetries:

$$X_{a^*(\mathbf{k})} \equiv ie^{ikx}\frac{\partial}{\partial\phi}\,, \quad X_{a(\mathbf{k})} \equiv -ie^{-ikx}\frac{\partial}{\partial\phi}$$
$$\bar{X}_{a^*(\mathbf{k})} = X_{a^*(\mathbf{k})} - ik_{\nu}e^{ikx}\frac{\partial}{\partial\phi_{\nu}}\,, \quad \bar{X}_{a(\mathbf{k})} = \bar{X}^*_{a^*(\mathbf{k})} \tag{122}$$

with Noether invariants

$$\left| \begin{array}{ll} Q_{a^*(\mathbf{k})} = \int \mathrm{d}^3x j^0_{a^*(\mathbf{k})} = i\int \mathrm{d}^3xe^{ikx}(\dot{\phi} - ik^0\phi) = a(\mathbf{k}) \\ Q_{a(\mathbf{k})} \;\; = \ldots \qquad\qquad\qquad\qquad \ldots = a^*(\mathbf{k})\,. \end{array} \right. \tag{123}$$

Alternatively, the "configuration space" counterparts are

$$X_{\pi(\mathbf{y})} \equiv i\int\frac{\mathrm{d}^3k}{2k^0}\left\{e^{i\mathbf{k}\cdot\mathbf{x}}e^{ikx}\frac{\partial}{\partial\phi} - h.c.\right\} = \mathrm{Cos}[x^0\sqrt{m^2-\nabla^2}]\delta(\mathbf{y}-\mathbf{x})\frac{\partial}{\partial\phi}$$
$$X_{\varphi(\mathbf{y})} \equiv -\int\frac{\mathrm{d}^3k}{2k^0}k^0\left\{e^{i\mathbf{k}\cdot\mathbf{x}}e^{ikx}\frac{\partial}{\partial\phi} + h.c.\right\} = \frac{\mathrm{Sin}[x^0\sqrt{m^2-\nabla^2}]}{\sqrt{m^2-\nabla^2}}\delta(\mathbf{y}-\mathbf{x})\frac{\partial}{\partial\phi} \tag{124}$$

with Noether invariants

$$\left| \begin{array}{l} Q_{\pi(\mathbf{y})} = \varphi(\mathbf{y}) \\ Q_{\varphi(\mathbf{y})} = \pi(\mathbf{y})\,. \end{array} \right. \tag{125}$$

Note that $\pi(\mathbf{x}) = \dot{\phi}(x^0 = 0, \mathbf{x})$, $\varphi(\mathbf{x}) = \phi(x^0 = 0, \mathbf{x})$ and that $\mathbf{k}$ and $\mathbf{y}$ in the subscript are indices, whereas x^{μ} is the variable in the base manifold M of E.

The Hamilton–Jacobi transformation: Passing to the Solution Manifold

By writing the Klein–Gordon solutions in a proper way, and adding the trivial transformation $x^0 = \chi^0$, the following transformation $(H-J)$ has an inverse $((H-J)^{-1})$:

$$\left|\begin{array}{l} \phi(x) = \mathrm{Cos}\left(\chi^0\sqrt{m^2-\nabla^2}\right)\varphi(\mathbf{x}) + \frac{\mathrm{Sin}\left(\chi^0\sqrt{m^2-\nabla^2}\right)}{\sqrt{m^2-\nabla^2}}\dot{\varphi}(\mathbf{x}) \\ \dot{\phi}(x) = \mathrm{Cos}\left(\chi^0\sqrt{m^2-\nabla^2}\right)\dot{\varphi}(\mathbf{x}) - \sqrt{m^2-\nabla^2}\,\mathrm{Sin}\left(\chi^0\sqrt{m^2-\nabla^2}\right)\varphi(\mathbf{x}) \\ x^0 = \chi^0 \end{array}\right. \tag{126}$$

$$\left|\begin{array}{l} \varphi(\mathbf{x}) = \mathrm{Cos}\left(x^0\sqrt{m^2-\nabla^2}\right)\phi(x^0,\mathbf{x}) + \frac{\mathrm{Sin}\left(x^0\sqrt{m^2-\nabla^2}\right)}{\sqrt{m^2-\nabla^2}}\dot{\phi}(x^0,\mathbf{x}) \\ \dot{\varphi}(\mathbf{x}) = \mathrm{Cos}\left(x^0\sqrt{m^2-\nabla^2}\right)\dot{\phi}(x^0,\mathbf{x}) - \sqrt{m^2-\nabla^2}\,\mathrm{Sin}\left(x^0\sqrt{m^2-\nabla^2}\right)\phi(x^0,\mathbf{x}) \\ \chi^0 = x^0 \end{array}\right. \tag{127}$$

The tangent H–J transformation becomes:

$$\left|\begin{array}{l} \frac{\partial}{\partial\varphi(\mathbf{x})} = \mathrm{Cos}\left(x^0\sqrt{m^2-\nabla^2}\right)\frac{\partial}{\partial\phi(x)} - \sqrt{m^2-\nabla^2}\,\mathrm{Sin}\left(x^0\sqrt{m^2-\nabla^2}\right)\frac{\partial}{\partial\dot{\phi}(x)} \\ \frac{\partial}{\partial\dot{\varphi}(\mathbf{x})} = \frac{\mathrm{Sin}\left(x^0\sqrt{m^2-\nabla^2}\right)}{\sqrt{m^2-\nabla^2}}\frac{\partial}{\partial\phi(x)} + \mathrm{Cos}\left(x^0\sqrt{m^2-\nabla^2}\right)\frac{\partial}{\partial\dot{\phi}(x)} \end{array}\right. \tag{128}$$

$$\left|\begin{array}{l} \frac{\partial}{\partial\phi(x)} = \mathrm{Cos}\left(x^0\sqrt{m^2-\nabla^2}\right)\frac{\partial}{\partial\varphi(\mathbf{x})} + \sqrt{m^2-\nabla^2}\,\mathrm{Sin}\left(x^0\sqrt{m^2-\nabla^2}\right)\frac{\partial}{\partial\dot{\varphi}(\mathbf{x})} \\ \frac{\partial}{\partial\dot{\phi}(x)} = -\frac{\mathrm{Sin}\left(x^0\sqrt{m^2-\nabla^2}\right)}{\sqrt{m^2-\nabla^2}}\frac{\partial}{\partial\varphi(\mathbf{x})} + \mathrm{Cos}\left(x^0\sqrt{m^2-\nabla^2}\right)\frac{\partial}{\partial\dot{\varphi}(\mathbf{x})}\,. \end{array}\right. \tag{129}$$

Acting with H–J on the objects on $\Gamma(E)$ we arrive at the Solution Manifold endowed with a symplectic structure and Hamiltonian symmetries. In fact, the "integral on the Cauchy surface" of Θ_{PC} comes down to the SM except for a total differential after applying the H–J transformation:

$$\begin{aligned} \vartheta_{PC} &\equiv \int_\Sigma \mathrm{d}\sigma_\mu\left\{\frac{\partial\mathcal{L}}{\partial\phi_\mu}(\mathrm{d}\phi - \phi_\nu\mathrm{d}x^\nu) + \mathcal{L}\mathrm{d}x^\mu\right\} \\ &= \int \mathrm{d}^3x\{\dot{\phi}\mathrm{d}\phi - \frac{1}{2}(\dot{\phi}^2 + \nabla\phi\cdot\nabla\phi + m^2\phi^2)\mathrm{d}x^0\} \\ &\equiv \int \mathrm{d}^3x\{\dot{\phi}\mathrm{d}\phi - H\mathrm{d}x^0\} = \overbrace{\int \mathrm{d}^3x\dot{\varphi}(\mathbf{x})\delta\varphi(\mathbf{x})}^{\text{Liouville form }\Lambda} + \text{total differential} \end{aligned} \tag{130}$$

and the differential $\mathrm{d}\vartheta_{PC}$ actually comes down defining the Symplectic form: $\Omega \equiv \delta\Lambda$

Hamiltonian vector fields: $\quad i_{X_f}\Omega = -\delta f$

$$\left.\begin{array}{l} f = \varphi(\mathbf{x}) \;\Rightarrow\; X_{\varphi(\mathbf{x})} = \frac{\delta}{\delta\dot{\varphi}(\mathbf{x})} \\ f = \dot{\varphi}(\mathbf{x}) \;\Rightarrow\; X_{\dot{\varphi}(\mathbf{x})} = \frac{\delta}{\delta\varphi(\mathbf{x})} \end{array}\right| \text{constitute the basic local symmetries}\,. \tag{131}$$

(They are not gauge; in fact, the Noether invariants are non-trivial)

Note that they go back to the Evolution Manifold by means of the transformation $(H-J)^{-1}$. "Functions" such as $\int_\Sigma \mathrm{d}^3x\dot{\varphi}(\mathbf{x})$ or $\int_\Sigma \mathrm{d}^3x\varphi(\mathbf{x})$, that is, the integrated basic symmetries, generate rigid symmetries.
For instance, for the <u>massless</u> Klein–Gordon field,

$$\left.\begin{array}{l} f=\int \mathrm{d}^3x\varphi(\mathbf{x}) \;\;\Rightarrow\;\; X_f = x^0\frac{\partial}{\partial\phi}+\frac{\partial}{\partial\dot{\phi}} \\ f=\int \mathrm{d}^3x\dot{\varphi}(\mathbf{x}) \;\;\Rightarrow\;\; X_f=\frac{\partial}{\partial\phi} \end{array}\right| \text{ as vector fields on the bundle } J^1(E)\,. \tag{132}$$

By the way, even in the massive case, $Y_f = f\frac{\partial}{\partial\phi}$ is a symmetry of the Lagrangian if f is a solution of the Klein–Gordon equation:

$$\begin{aligned} \bar{Y}_f &= f\frac{\partial}{\partial\phi}+\partial_\mu f\frac{\partial}{\partial\phi_\mu} \\ \bar{Y}_f\mathcal{L} &= -m^2f\phi+\partial_\mu f\phi^\mu=\partial_\mu(\partial^\mu f\phi) \;\text{ if }\; \partial_\mu\partial^\mu f+m^2f=0\,. \end{aligned}$$

When f is not a solution, symmetry under such a vector field Y_f requires the introduction of compensating *Yang–Mills fields*.

5.6 Current algebra (on the example of the massless Klein–Gordon field)

We write the complete symmetry of the Klein–Gordon field in the form of a semi-direct product group:

$$\text{Poincaré} \otimes_S \text{“Current Group”}.$$

The space–time rigid symmetry provides charges:

$$\left.\begin{array}{l} P_0=\frac{1}{2}\int \mathrm{d}^3x\{\dot{\varphi}(\mathbf{x})\dot{\varphi}(\mathbf{x})+\nabla\varphi(\mathbf{x})\cdot\nabla\varphi(\mathbf{x})\}\equiv\int \mathrm{d}^3x\mathcal{P}_0 \\ P_i=\int \mathrm{d}^3x\dot{\varphi}(\mathbf{x})\varphi_i(\mathbf{x})\equiv\int \mathrm{d}^3x\mathcal{P}_i \end{array}\right| \quad X_{(\mu)}=\frac{\partial}{\partial x^\mu} \tag{133}$$

$$M_{(\mu\nu)}=\int \mathrm{d}^3x(\mathcal{P}_\mu x_\nu-\mathcal{P}_\nu x_\mu)|_{x^0=0} \quad | \;\; X_{(\mu\nu)}=\delta^{\epsilon\sigma}_{\mu\nu}x_\epsilon\frac{\partial}{\partial x^\sigma}\delta^{\epsilon\sigma}_{\mu\nu}x_\epsilon\frac{\partial}{\partial x^\sigma}+\delta^{\epsilon\sigma}_{\mu\nu}\phi_\epsilon\frac{\partial}{\partial\phi^\sigma} \tag{134}$$

The internal symmetries lead to rigid charges:

$$Q_\phi=\int \mathrm{d}^3x\dot{\varphi}(\mathbf{x}) \qquad\qquad X_\phi=\frac{\partial}{\partial\phi} \tag{135}$$

and

$$\left.\begin{array}{l} Q_{\dot{\phi}}=\int \mathrm{d}^3x\varphi(\mathbf{x}) \\ Q_{\phi_i}=\int \mathrm{d}^3xx^i\dot{\varphi}(\mathbf{x}) \end{array}\right| \qquad X_{\phi_\mu}=x^\mu\frac{\partial}{\partial\phi}+\frac{\partial}{\partial\phi_\mu} \tag{136}$$

as well as local ones associated with the Hamiltonian vector fields (131):

$$Q_{\varphi(\mathbf{x})}=\dot{\varphi}(\mathbf{x})\,,\quad Q_{\dot{\varphi}(\mathbf{x})}=\varphi(\mathbf{x}) \quad | \text{ local symmetries.} \tag{137}$$

<u>In other words</u>, given a rigid symmetry, the integrand of the corresponding Noether invariants, that is, the zeroth component of the currents, j^0, are in turn Noether invariants of a current algebra!!.

6 Symmetry and quantum theory

Canonical quantization proved to be inadequate very soon for dealing with nonlinear systems in general, except for certain perturbative conditions. See, for instance the historical paper on "No-Go theorems" [29] as well as, more recently [30,31]. Here, we shall adopt a symmetry-based algorithm more appropriate to formulate basic physical systems irrespective of their (non-)linear character, provided that we are able to parameterize their Solution Manifold by means of Noether charges associated with symmetries [32–36].

6.1 Group Approach to Quantization

The basic idea of GAQ consists in having two mutually commuting copies of the Lie algebra $\tilde{\mathcal{G}}$ of $\tilde{G}$ a central extension by $U(1)$:

$$\mathcal{X}^L(\tilde{G}) \approx \tilde{\mathcal{G}} \approx \mathcal{X}^R(\tilde{G})\,.$$

Then, a copy, let us say $\mathcal{X}^R(\tilde{G})$, constitutes the (pre-)Quantum Operators acting by usual derivations on complex $U(1)$-functions on $\tilde{G}$.

The other copy, now $\mathcal{X}^L(\tilde{G})$, is used to reduce the (pre-)quantum representation in a compatible way $\Rightarrow$ true Quantization

In fact, given a group law, $g'' = g'g$, we have two actions:

$$\begin{aligned} g'' &= g'g = L_{g'}g \qquad \text{left action} \\ g'' &= g'g = R_g g' \qquad \text{right action}\,, \end{aligned}$$

and they do commute: $\left[\tilde{X}^L_a\,,\ \tilde{X}^R_b\right] = 0 \qquad \forall a, b = 1, \ldots \dim G$.

This property also implies:

$$L_{\tilde{X}^R_a}\theta^{Lb} = 0 \qquad \{\theta^{La}\} \ \text{ dual to } \ \{\tilde{X}^L_b\} \quad \text{and}$$

$$L_{\tilde{X}^R_a}(\theta^{Lb} \wedge \theta^{Lc} \wedge \ldots) \equiv L_{\tilde{X}^R_a}\omega = 0 \quad \Rightarrow \quad \omega \ \text{ invariant volume.}$$

The left-invariant form $\theta^{L(U(1))}$ plays the role of generalized Poincaré–Cartan form or quantization form Θ.

The classical Noether invariants are $i_{\tilde{X}^R_a}\Theta$, as they are invariant along the equations of motion, that is, $\tilde{X}^L_a$ in the *characteristic subalgebra* $\mathcal{G}_\Theta$:

$$\begin{aligned} \mathcal{G}_\Theta &= \langle \tilde{X}^L \ / \ i_{\tilde{X}^L}\Theta = 0 = i_{\tilde{X}^L}\mathrm{d}\Theta \rangle \\ \tilde{G}/\mathcal{G}_\Theta \ &\equiv \ \text{Quantum Solution Manifold}\,. \end{aligned}$$

Wave functions ψ are $U(1)$-functions ($\psi(\tilde{g}) = \zeta\Phi(g), \quad \zeta \in U(1)$) invariant under the right action of a polarization subgroup P:

> P is a maximal subgroup of G containing the characteristic subgroup G_Θ and excluding the $U(1)$ central subgroup,

$$\psi(R_g g') = \psi(g') \quad \forall g \in P\,.$$

$\tilde{G}$ acts on ψ from the left, $\hat{g}'\psi(g) = \psi(L_{g'}g)$, providing an irreducible representation of $\tilde{G}$. At the infinitesimal level, the $U(1)$-function condition $\psi(\tilde{g}) = \zeta\Phi(g)$ is written as $\Xi\psi = i\psi$, where Ξ stands for

$$\tilde{X}^L_{(\zeta)} = \tilde{X}^R_{(\zeta)} = i\zeta\frac{\partial}{\partial\zeta} - i\zeta^*\frac{\partial}{\partial\zeta^*}\,,$$

which is the central generator of the group $\tilde{G}$, $\zeta \in U(1)$.
Starting from a complex function $\psi(g, \zeta)$ on $\tilde{G}$, we must impose the Polarization conditions in the form:

$$\tilde{X}_a^L \psi = 0.$$

$\tilde{X}^L \in P$ generate a left-invariant subalgebra $\mathcal{P}$ containing $\mathcal{G}_\Theta$ and excluding the vertical generator Ξ.

If such a polarization subalgebra does not exist, then we may search for a higher-order subalgebra in the left-enveloping algebra substituting a first-order one.
On the space of polarized wave functions, the right-invariant generators $\tilde{X}_a^R$ operate defining the true quantum operators associated with the group variable a, $\hat{a}$. They generate a unitary and irreducible representation of the group $\tilde{G}$, that is, a quantization of the physical system with basic symmetry $\tilde{G}$

6.2 Some examples

6.2.1 Non-relativistic harmonic oscillator (1+1 dimension)

Group law: (central extension by $U(1)$ of the Newton group)

$$\left.\begin{array}{l} t'' = t' + t \\ x'' = x + x'\cos\omega t + \frac{p'}{m\omega}\sin\omega t \\ p'' = p + p'\cos\omega t - m\omega x'\sin\omega t \end{array}\right| \text{ Newton group}$$

$$\zeta'' = \zeta'\zeta e^{\frac{i}{2\hbar}(x'p\cos\omega t - p'x\cos\omega t + (\frac{p'p}{m\omega} + \omega x'x)\sin\omega t)} \tag{138}$$

$$\begin{array}{ll} \tilde{X}_t^L = \frac{\partial}{\partial t} + \frac{p}{m}\frac{\partial}{\partial x} - m\omega^2 x\frac{\partial}{\partial p} & ; \ \tilde{X}_p^L = \frac{\partial}{\partial p} + \frac{x}{2\hbar}\Xi \\ \tilde{X}_x^L = \frac{\partial}{\partial x} - \frac{p}{2\hbar}\Xi & ; \ \tilde{X}_\zeta^L = i\zeta\frac{\partial}{\partial \zeta} - i\zeta^*\frac{\partial}{\partial \zeta^*} \equiv \Xi \end{array} \tag{139}$$

$$\begin{aligned} \tilde{X}_t^R &= \frac{\partial}{\partial t} \\ \tilde{X}_x^R &= \cos\omega t\frac{\partial}{\partial x} - m\omega\sin\omega t\frac{\partial}{\partial p} + \frac{1}{2\hbar}(p\cos\omega t + m\omega x\sin\omega t)\Xi \\ \tilde{X}_p^R &= \frac{1}{m\omega}\sin\frac{\partial}{\partial x} + \cos\omega t\frac{\partial}{\partial p} - \frac{1}{2\hbar}(x\cos\omega t - \frac{p}{m\omega}\sin\omega t)\Xi \\ \tilde{X}_\zeta^R &= i\zeta\frac{\partial}{\partial \zeta} - i\zeta^*\frac{\partial}{\partial \zeta^*} \equiv \Xi \end{aligned} \tag{140}$$

$$\left[\tilde{X}_t^R, \tilde{X}_x^R\right] = -m\omega^2\tilde{X}_p^R \ ; \ \left[\tilde{X}_t^R, \tilde{X}_p^R\right] = \frac{1}{m}\tilde{X}_x^R \ ; \ \left[\tilde{X}_t^R, \tilde{X}_p^R\right] = -\frac{1}{\hbar}\Xi \tag{141}$$

Quantization Form: $\Theta = p\mathrm{d}x - (\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2)\mathrm{d}t + \frac{\mathrm{d}\zeta}{i\zeta}$
Characteristic Module: $\mathcal{C}_\Theta = \langle \tilde{X}^L \rangle$
Polarization (complex): $\mathcal{P} = \langle \tilde{X}_t^L, \tilde{X}_x^L \pm im\omega\tilde{X}_p^L \rangle$
There is no first-order real polarization!

Fock variables: $a \equiv \sqrt{\frac{m\omega}{2\hbar}}x + \frac{i}{\sqrt{2m\hbar\omega}}p$; $a^* \equiv \sqrt{\frac{m\omega}{2\hbar}}x - \frac{i}{\sqrt{2m\hbar\omega}}p$
Group law, vector fields, characteristic module, polarization:

$$\begin{aligned} t'' &= t' + t \\ a'' &= a'e^{-i\omega t} + a \\ a''^* &= a'^* e^{i\omega t} + a^* \\ \zeta'' &= \zeta'\zeta e^{\frac{i}{2}(ia'a^*e^{-i\omega t} - ia^*a'e^{i\omega t})} \end{aligned} \tag{142}$$

$$\left.\begin{aligned} \tilde{X}^L_t &= \frac{\partial}{\partial t} - i\omega a\frac{\partial}{\partial a} + i\omega a^*\frac{\partial}{\partial a^*} \\ \tilde{X}^L_a &= \frac{\partial}{\partial a} - \frac{i}{2}a^*\Xi \\ \tilde{X}^L_{a^*} &= \frac{\partial}{\partial a^*} + \frac{i}{2}a\Xi \end{aligned}\right| \begin{aligned} \tilde{X}^R_t &= \frac{\partial}{\partial t} \\ \tilde{X}^R_a &= e^{-i\omega t}\left(\frac{\partial}{\partial a} + \frac{i}{2}a^*\Xi\right) \\ \tilde{X}^R_{a^*} &= e^{i\omega t}\left(\frac{\partial}{\partial a^*} - \frac{i}{2}a\Xi\right) \end{aligned} \tag{143}$$

$$\mathcal{C}_\Theta = \langle \tilde{X}^L_t \rangle \;\; ; \;\; \mathcal{P} = \langle \tilde{X}^L_t,\, \tilde{X}^L_a \rangle \;\text{ or }\; \mathcal{P}^* = \langle \tilde{X}^L_t,\, \tilde{X}^L_{a^*} \rangle \tag{144}$$

Polarization Equations: $\Psi = \Psi(\zeta, t, a, a^*)$

$$\begin{aligned} (U(1) - \text{function}) \;\; \Xi.\Psi = i\Psi \;\; &\Rightarrow \;\; \Psi = \zeta\Phi(t, a, a^*) \\ \tilde{X}^L_a.\Psi = 0 \;\; &\Rightarrow \;\; \Psi = \zeta e^{-\frac{aa^+}{2}}\phi(t, a) \\ \tilde{X}^L_t.\Psi = 0 \rightarrow \frac{\partial\phi}{\partial t} + i\omega a * \frac{\partial\phi}{\partial a^*} = 0 \;\; &\Rightarrow \;\; \Psi = \zeta e^{-\frac{aa^*}{2}}\sqrt{\frac{\omega}{2\pi}}\Sigma_{n=0}^{\infty}c_n\frac{(a^*e^{-i\omega t})^n}{\sqrt{\pi n!}} \equiv \zeta\Phi_n \end{aligned} \tag{145}$$

Operators: $\hat{E} = i\hbar\tilde{X}^R_t$, $\hat{a} = \tilde{X}^R_{a^*}$, $\hat{a}^\dagger = \tilde{X}^R_a$

$$\left.\begin{aligned} \hat{a}\Phi_n &= \sqrt{n}\Phi_{n-1} \\ \hat{a}^\dagger\Phi_n &= \sqrt{n+1}\Phi_{n+1} \\ \hat{E}\Phi_n &= n\hbar\omega\Phi_n \end{aligned}\right| \quad \langle\Psi'|\Psi\rangle = \int \mathrm{d}(\mathrm{Re}a^*)\mathrm{d}(\mathrm{Im}a)e^{-aa^*}\phi'\phi \tag{146}$$

Configuration space: Higher-order Polarization

$$\mathcal{P}^{HO} = \langle \tilde{X}^L_p,\; \tilde{X}^L_t - \frac{i\hbar}{2m}\tilde{X}^L_x\tilde{X}^L_x \rangle \tag{147}$$

$$\left.\begin{aligned} \Xi.\Psi = i\Psi \;\; &\rightarrow \;\; \Psi = \zeta\Phi(t, x, p) \\ \tilde{X}^L_p\Psi = 0 \;\; &\rightarrow \;\; \Psi = \zeta e^{-i\frac{i}{\hbar}\frac{px}{2}}\phi(t, x) \\ (\tilde{X}^L_t - \frac{i\hbar}{2m}\tilde{X}^L_x\tilde{X}^L_x)\Psi = 0 \;\; &\rightarrow \;\; i\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar}{2m}\frac{\partial^2\phi}{\partial x^2} + \frac{1}{2}m\omega^2x^2\phi \end{aligned}\right| \;\; \Rightarrow \tag{148}$$

$$\phi(t, x) = \Sigma_{n=0}^{\infty}c_n\phi_n(t, x) = \sqrt{\frac{\omega}{2\pi}}\left(\frac{m\omega}{\hbar\pi}\right)^{1/4}\Sigma_{n=0}^{\infty}\frac{c_n}{2^{n/2}\sqrt{n!}}e^{-\frac{m\omega}{2\hbar}x^2}e^{-in\omega t}H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) \tag{149}$$

where H_n are the Hermite polynomials.

6.2.2 Relativistic harmonic oscillator (1+1 dimension)

What is a relativistic harmonic oscillator? A dynamical system characterized by a symmetry that contract to that of the non-relativistic harmonic oscillator in the non-relativistic limit and that contract to the symmetry of the free relativistic particle in the limit of zero frequency [37]. Here is the proposed Lie algebra:

$$\begin{array}{l|l} \left[\hat{E},\hat{K}\right] = -i\frac{\hbar}{m}\hat{P} & c^2 \to \infty \ \text{ N-R oscillator} \\ \left[\hat{E},\hat{P}\right] = im\hbar\omega^2\hat{K} & \omega^2 \to 0 \ \text{ Free Relativistic particle} \\ \left[\hat{K},\hat{P}\right] = i\hbar(\frac{1}{mc^2}\hat{E} + \hat{1}) & c^2 \to \infty\,,\ \omega^2 \to 0 \ \text{ N-R Free particle} \end{array} \tag{150}$$

Group law: (by exponentiating the Lie algebra)

$$\begin{aligned} \sin\omega t'' &= \frac{\omega}{\alpha''}\left(\frac{\alpha}{mc^2\alpha'}p'x'\sin\omega t'\sin\omega t + \frac{\alpha P_0'}{m\omega c\alpha'}\cos\omega t'\sin\omega t\right. \\ &\quad \left. + \frac{\omega}{mc^3\alpha'}xx'P_0'\sin\omega t' + \frac{\alpha'\alpha}{\omega}\cos\omega t\sin\omega t' + \frac{p'x}{mc^2\alpha'}\cos\omega t'\right) \\ x'' &= \frac{p'\alpha}{m\omega}\sin\omega t + \alpha x'\cos\omega t + \frac{xP_0'}{mc} \\ p'' &= \frac{\omega xp}{c^2\alpha}\left(\frac{p'}{m}\sin\omega t + \omega x'\cos\omega t\right) + \frac{P_0}{c\alpha}\left(\frac{p'}{m}\cos\omega t - \omega x'\sin\omega t\right) + \frac{pP_0'}{mc} \\ \zeta'' &= \zeta'\zeta e^{\frac{i}{\hbar}(\delta''-\delta'-\delta)} \end{aligned} \tag{151}$$

where

$$\begin{array}{l|l} P_0 \equiv \sqrt{mc^2 - p^2 + m^2\omega^2x^2} & \delta \equiv \ \text{function generating the} \\ \alpha \equiv \sqrt{1 + \frac{\omega^2x^2}{c^2}} & \text{coboundary such that } \stackrel{c^2\to\infty}{\longrightarrow} \\ \delta \equiv -mc^2t - f & \text{cocycle} \\ f \equiv -\frac{2mc^2}{\omega}\mathrm{Tg}^{-1}\left[\frac{mc^2}{\omega px}(\alpha-1)(\frac{P_0}{mc}-\alpha)\right] & \end{array}$$

Left generators:

$$\begin{array}{l|l} \tilde{X}^L_t = \frac{P_0}{mc\alpha^2}\frac{\partial}{\partial t} + \frac{p}{m}\frac{\partial}{\partial x} - m\omega^2x\frac{\partial}{\partial p} & \left[\tilde{X}^L_t,\ \tilde{X}^L_x\right] = m\omega^2\tilde{X}^L_p \\ \tilde{X}^L_x = \frac{P_0}{mc}\frac{p}{m}\frac{\partial}{\partial x} + \frac{p}{mc^2\alpha^2}\frac{p}{m}\frac{\partial}{\partial t} - \frac{pmc}{P_0+mc}\frac{1}{\hbar}\Xi & \left[\tilde{X}^L_t,\ \tilde{X}^L_p\right] = -\frac{1}{m}\tilde{X}^L_x \\ \tilde{X}^L_p = \frac{P_0}{mc}\frac{\partial}{\partial p} + \frac{mcx}{P_0+mc}\frac{1}{\hbar}\Xi & \left[\tilde{X}^L_x,\ \tilde{X}^L_p\right] = -\frac{1}{mc^2}\tilde{X}^L_t + \frac{1}{\hbar}\Xi \end{array} \tag{152}$$

<u>Configuration space:</u> Higher-order Polarization

$$\mathcal{P}^{HO} = \langle\tilde{X}^L_p,\ \tilde{X}^{LHO}_t \equiv \left(\tilde{X}^L_t\right)^2 - c^2\left(\tilde{X}^L_x\right)^2 + \frac{2imc^2}{\hbar}\tilde{X}^L_t + \frac{imc^2\omega}{\hbar}\Xi\rangle \tag{153}$$

$$\left|\begin{array}{l} \Xi.\Psi = i\Psi \ \Rightarrow\ \Psi = \zeta\Phi(t,x,p) \\ \tilde{X}^L_p.\Psi = 0 \ \Rightarrow\ \Psi = \zeta e^{\frac{i}{\hbar}f}\phi(t,x) \\ \tilde{X}^{LHO}_t.\Psi = 0 \ \Rightarrow\ \frac{1}{\alpha^2}\frac{\partial^2\phi}{\partial t^2} - \frac{2imc^2}{\hbar\alpha^2}\frac{\partial\phi}{\partial t} - 2\omega^2x\frac{\partial\phi}{\partial x} - c^2\alpha^2\frac{\partial^2\phi}{\partial x^2} - \frac{m^2c^4}{\hbar\alpha^2}\phi + \frac{mc^2\omega}{\hbar}\phi = 0\,. \end{array}\right. \tag{154}$$

Restoring the rest energy, that is, $\tilde{X}^R_t \to \tilde{X}^R_t - \frac{mc^2}{\hbar}\Xi\,,\ \ \phi \to \varphi$,
the equation $\tilde{X}^{LHO}_t.\Psi = 0$ becomes

$$\hat{C}\varphi \equiv -\frac{c^2}{\omega^2}\Box\varphi = N(N-1)\varphi \quad \text{(Casimir operator)}\,, \tag{155}$$

where $N \equiv \frac{mc^2}{\hbar\omega}$ and $\Box \equiv \frac{1}{c^2\alpha^2}\frac{\partial^2}{\partial t^2} - \frac{2\omega^2 x}{c^2}\frac{\partial}{\partial x} - \alpha^2\frac{\partial^2}{\partial x^2}$ $\Box$ is the D'Alembert operator in anti-de Sitter space-time. The evolution equation is solved by power series expansion:

$$\varphi_n \equiv e^{-ib_n\omega t}\alpha^{-c_n} H_n^N \quad \Rightarrow \quad \left| \begin{array}{l} b_n = c_n \\ c_n = c_0 + \frac{1}{2} + \frac{\sqrt{1+4N(N-1)}}{2} \equiv c_0 + \frac{1}{2} + \bar{N}\,, \end{array} \right. \tag{156}$$

where H_n^N is a polynomial in the variable $\xi \equiv \sqrt{\frac{m\omega}{\hbar}}x$ satisfying

$$\left(1 + \frac{\xi^2}{N}\right)\frac{d^2}{d\xi^2}H_n^N - \frac{2}{N}(\bar{N} + n - \frac{1}{2})\xi\frac{d}{d\xi}H_n^N + \frac{n}{N}(2\bar{N}+n)H_n^N = 0 \;\Rightarrow \tag{157}$$

$$H_n^N(\xi) = \Sigma_{s=0}^{[n/2]} a_{n,n-2s}^N (2\xi)^{n-2s} \;\; ; \;\; a_{n,n-2s}^N = -\frac{(s+1)(\bar{N}+s+1)}{N(n-2s)(n-2s-1)} a_{n,n-2(s+1)}^N \;\Rightarrow \tag{158}$$

$$a_{n,n-2s}^N = (-1)^s \frac{N^s n!\bar{N}!(2\bar{N}+n)!}{(2N)^n s!(\bar{N}+s)!(2\bar{N})!(n-2s)!}\,, \; s = 0, ..., [\frac{n}{2}]. \tag{159}$$

The polynomials H_n^N are the Relativistic Hermite Polynomials!!

The energy operator provides the value $E_n^N = (\frac{1}{2} + \bar{N} + n)\hbar\omega$.

For $N \equiv \frac{mc^2}{\hbar\omega} \to \infty$, $H_n^N \to H_n$ (Hermite Polynomials). The value $N = \frac{1}{2}$ corresponds to the extreme relativistic regime.

6.2.3 Particle moving on $SU(2)$: PNLσM

The standard classical approach to a particle moving on a Riemann manifold with metric $g_{ij}(x)$ is established by the Lagrangian (see [38,39] and references therein):

$$\mathcal{L} = \frac{1}{2}mg_{ij}(x)\dot{x}^i\dot{x}^j = \frac{1}{2}me_i^{(a)}e_j^{(b)}k_{ab}\,, \tag{160}$$

where $e_i^{(a)}$ are the vierbeins defining the metric $\left| \begin{array}{l} g_{ij} = e_i^{(a)}e_j^{(b)}k_{ab} \\ g^{ij} = e_{(a)}^i e_{(b)}^j k^{ab} \\ e_i^{(a)}e_{(b)}^j = k_b^a \equiv \delta_b^a\,. \end{array} \right.$

Here, $SU(2)$ is parameterized by $\boldsymbol{\epsilon} \in R^3 \;/\; |\boldsymbol{\epsilon}| = 2\sin\frac{\varphi}{2}$

$$e_j^{(i)}(\boldsymbol{\epsilon}) \equiv \theta_j^{(i)}(\boldsymbol{\epsilon}) = \left(\sqrt{1 - \frac{\boldsymbol{\epsilon}^2}{4}}\delta_j^i + \frac{\epsilon^i\epsilon_j}{4\sqrt{1-\frac{\boldsymbol{\epsilon}^2}{4}}} + \frac{1}{2}\eta^i_{.jk}\epsilon^k\right). \tag{161}$$

The form $\theta^{(i)} = \theta_j^{(i)}d\epsilon^j \equiv \theta^{R(i)}$ is the right-invariant canonical 1-form (we could have used the left forms since $\mathcal{L}$ is chiral.

The inverse "vierbeins" are the right-invariant vector fields

$$X_{(i)}^R = \left(\sqrt{1 - \frac{\boldsymbol{\epsilon}^2}{4}}\delta_i^j + \frac{1}{2}\eta^j_{.ik}\epsilon^k\right)\frac{\partial}{\partial\epsilon^j} \quad | \quad X_{(i)}^j\frac{\partial}{\partial\epsilon^j} = e_{(i)}^j\frac{\partial}{\partial\epsilon^j}\,. \tag{162}$$

The momentum, Hamiltonian and Poincaré–Cartan form are

$$\begin{array}{l|l} p_i = \frac{\partial \mathcal{L}}{\partial \dot{\epsilon}^i} = m g_{ij}\dot{\epsilon}^j = m\theta_i^{(k)}\theta_j^{(s)}\dot{\epsilon}^j k_{ks} = m\theta_i^{(k)}\theta^s k_{ks} \equiv m\theta_i^{(k)}\theta_k & \theta^s \equiv \theta_j^{(s)}\dot{\epsilon}^j \\ H = \frac{\partial \mathcal{L}}{\partial \dot{\epsilon}^i}\dot{\epsilon}^i - \mathcal{L} = \frac{1}{2} m g_{ij}\dot{\epsilon}^i\dot{\epsilon}^j = \frac{1}{2m} g^{ij} p_i p_j = \frac{1}{2} m\theta^i\theta_i & \theta_k \equiv \theta^s k_{sk} \end{array} \tag{163}$$

$$\Theta_{PC} = \frac{\partial \mathcal{L}}{\partial \dot{\epsilon}^i}(\mathrm{d}\epsilon^i - \dot{\epsilon}^i \mathrm{d}t) - \mathcal{L}\mathrm{d}t = p_i d\epsilon^i - H\mathrm{d}t = m\theta_i\theta^{(i)} - \frac{1}{2}m\theta_i\theta^i \mathrm{d}t \tag{164}$$

and the solutions to the equations of motion ($\omega \equiv \sqrt{\frac{2}{m}H} = \sqrt{\theta^i\theta_i}$):

$$\begin{array}{l|l} \epsilon^i(t) = \varepsilon^i \cos\omega t + \dot{\varepsilon}^i \frac{\sin\omega t}{\omega} & \text{Hamilton–Jacobi} \\ \dot{\epsilon}^i(t) = \dot{\varepsilon}^i \cos\omega t - \omega\varepsilon^i \sin\omega t & \text{transformation}\,, \end{array} \tag{165}$$

where $\varepsilon^i \equiv \epsilon^i(0)$, $\dot{\varepsilon}^i \equiv \dot{\epsilon}^i(0)$ are constants of motion parameterizing the Solution Manifold. Note that $\theta^i \equiv \vartheta^i$ is also constant of motion.

The symplectic form on the SM turns out to be

$$\Omega = d\Lambda = m d\vartheta_i \wedge \vartheta^{(i)} + \frac{m}{2}\eta^i_{.jk}\vartheta_i\vartheta^{(j)} \wedge \vartheta^{(k)}\,. \tag{166}$$

In local (Darboux) coordinates, we have

$$\Lambda = \pi_i d\varepsilon^i, \quad \Omega = d\pi_i \wedge d\varepsilon^i\ ; \quad \pi_i \equiv m\nu_i = m\vartheta_i^{(k)}\vartheta_k\,. \tag{167}$$

Note that the Hamiltonian, in coordinates $(\varepsilon^i,\ \vartheta_j)$ will be free from normal-order ambiguities as regards quantization.

The basic symmetries are the Hamiltonian vector fields associated with ε^i, ϑ_j and $\rho \equiv \sqrt{1 - \frac{\boldsymbol{\epsilon}^2}{4}}$ when lifted back to the Evolution Manifold by means of the inverse of the H–J transformation.

They lead to the Poisson algebra (beyond Heisenberg–Weyl):

$$\begin{array}{ll} \left\{\varepsilon^i,\ \varepsilon^j\right\} = 0 & \left\{\varepsilon^i,\ \rho\right\} = 0 \\ \left\{\varepsilon^i,\ \vartheta_j\right\} = \frac{1}{2}\eta^i_{.jk}\varepsilon^k + \rho\delta^i_j & \{\vartheta_i,\ \rho\} = \frac{1}{4}k_{ij}\varepsilon^j \\ \left\{\vartheta_i,\ \vartheta_j\right\} = m\eta^k_{.ij}\vartheta_k\,. & \end{array} \tag{168}$$

Remark: The (Hamiltonian) function ϑ_i generate (Killing) symmetries of the Lagrangian, whereas ε^k **only** of the Poincaré–Cartan form, that is, ε^k generate pure contact symmetries. Group Approach to Quantization now proceeds by exponentiating the Poisson algebra above arriving at the $SU(2)$**-sigma group** centrally extended by $U(1)$:

$$\begin{array}{l|l} \boldsymbol{\varepsilon}'' = \rho\boldsymbol{\varepsilon}' + \rho'\boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\varepsilon}' \wedge \boldsymbol{\varepsilon} & \rho' \equiv \rho(\boldsymbol{\varepsilon}') \\ \boldsymbol{\nu}'' = \boldsymbol{\nu}' + X^{L^{SU(2)}}(\boldsymbol{\epsilon}')\boldsymbol{\nu} + \frac{1}{2}\boldsymbol{\varepsilon}' z & \boldsymbol{\nu} \equiv \frac{\boldsymbol{\pi}}{m} \qquad \Rightarrow \\ z'' = z' + \rho' z - \frac{1}{2}\boldsymbol{\varepsilon}' \cdot \boldsymbol{\nu} & \\ \zeta'' = \zeta'\zeta e^{-i\frac{m}{\hbar}(2(\rho'-1)z - \boldsymbol{\varepsilon}'\cdot\boldsymbol{\nu})} & \zeta \in U(1) \end{array} \tag{169}$$

$$\Theta \equiv \theta^{L(\zeta)} = -m\varepsilon_i d\nu^i - 2m(\rho - 1)dz + \frac{\mathrm{d}\zeta}{i\zeta}\,. \tag{170}$$

The characteristic subalgebra and polarization are:

$$\mathcal{G}_\Theta = \langle X^L_{(z)}\rangle\ , \quad \mathcal{P} = \langle X^L_{(\boldsymbol{\nu})},\ X^L_{(z)}\rangle. \tag{171}$$

On the Quantum Solution Manifold, $\tilde{G}/\mathcal{G}_\Theta$, the quantization form is

$$\Theta = -\varepsilon^i \mathrm{d}\pi_i + \frac{\mathrm{d}\zeta}{i\zeta} \quad (\text{or } \ \pi_i \mathrm{d}\varepsilon^i + \frac{\mathrm{d}\zeta}{i\zeta} \ \text{ up to a total differential}) . \tag{172}$$

Wave functions:

$$\Psi(\zeta, \boldsymbol{\varepsilon}, \boldsymbol{\nu}, z) = \zeta e^{-im(\boldsymbol{\varepsilon}\cdot\boldsymbol{\nu}+2(\rho-1)z)}\phi(\boldsymbol{\varepsilon}) \tag{173}$$

Operators:

$$\left| \begin{array}{l} \hat{\varepsilon}^i \phi(\boldsymbol{\varepsilon}) = \varepsilon^i \phi(\boldsymbol{\varepsilon}) \\ \hat{\nu}_j \phi(\boldsymbol{\varepsilon}) = -\frac{i}{m} \overset{SU(2)}{X^{Rk}_{(j)}} \frac{\partial \phi(\boldsymbol{\varepsilon})}{\partial \varepsilon^k} \\ \hat{\rho}\phi(\boldsymbol{\varepsilon}) = (\rho - 1)\phi(\boldsymbol{\varepsilon}) \ \ (\text{to be redefined to remove the } \ 1) \end{array} \right. \tag{174}$$

Hamiltonian:

$$\hat{H}\phi(\boldsymbol{\varepsilon}) = \frac{1}{2} m\delta^{ij} \hat{\nu}_i \hat{\nu}_j \phi(\boldsymbol{\varepsilon}) = -\frac{1}{2m}\Delta_{L-B}\phi(\boldsymbol{\varepsilon}) = E\phi(\boldsymbol{\varepsilon}) \tag{175}$$

Integration measure:

$$\mathrm{d}\mu = \theta^{L(\varepsilon^1)} \wedge \theta^{L(\varepsilon^2)} \wedge \theta^{L(\varepsilon^3)} = \frac{1}{\rho} \mathrm{d}\varepsilon^1 \wedge \mathrm{d}\varepsilon^2 \wedge \mathrm{d}\varepsilon^3 \tag{176}$$

6.2.4 The Klein–Gordon field

Typical infinite-dimensional systems in Physics appear as mappings from a space-time manifold M into a non-(necessarily)Abelian group target G [36]:

$$\phi \ : \ x \in M \ \mapsto \ \phi(x) \in G \, .$$

If g is an element in Diff(M) the following semi-direct group law holds:

$$\left. \begin{array}{l} g'' = g' \circ g \\ \phi''(x) = \phi'(g(x)) * \phi(x) \end{array} \right| \begin{array}{l} \circ \ \text{group law in } \ \mathrm{Diff}(M) \\ * \ \text{group law in } \ G \, . \end{array}$$

Here M is the Minkowski space-time ($x^0 \equiv ct, \ \mathbf{x}$), and Diff$(M)$ is restricted to the Poincaré subgroup or just Translations parameterized by ($a^0 \equiv cb, \ \mathbf{a}$). G is simply the complex (or real) vector space parameterized by ϕ.

A natural parameterization of the Klein–Gordon group is associated with a factorization of M as $\Sigma \times R$ (Cauchy surface times Time): we have parameters $\langle b, \mathbf{a}; \varphi(\mathbf{x}), \dot{\varphi}(\mathbf{x})\rangle$ (the Lorentz subgroup of the Poincaré group can be easily added).

It should be stressed that the action of $\mathbf{a}$ on $\varphi(\mathbf{x})$ just consists in moving the argument by $\mathbf{a}$: $\varphi(\mathbf{x}) \mapsto \varphi(\mathbf{x} - \mathbf{a})$, whereas the action of b requires the knowledge of the equation of motion (although not necessarily their actual solutions).

For K–G fields, $\phi(x)$ satisfies $\ddot{\phi}(x) = (\boldsymbol{\nabla}^2 - m^2)\phi(x) \ | \ \varphi(\mathbf{x}) \sim \phi(0, \mathbf{x})$
Therefore, we write for $\varphi'(b(\mathbf{x}))$:

$$\varphi'(b(\mathbf{x})) \equiv e^{bc\partial_0}\varphi'(\mathbf{x}) = \cos[bc\sqrt{m^2 - \boldsymbol{\nabla}^2}]\varphi'(\mathbf{x}) + \frac{\sin[bc\sqrt{m^2 - \boldsymbol{\nabla}^2}]}{\sqrt{m^2 - \boldsymbol{\nabla}^2}]}\dot{\varphi}'(\mathbf{x}) \tag{177}$$

so that, the Complete Group Law becomes:

$$\begin{array}{ll|l} b'' = b' + b & a''_\mu = a'_\mu + \Lambda'^{\nu}_{\mu} a_\nu & \text{if the Lorentz subgroup} \\ \mathbf{a}'' = \mathbf{a}' + \mathbf{a} & \Lambda'' = \Lambda'\lambda & \text{were included} \end{array}$$

$$\left.\begin{array}{l} \varphi''(\mathbf{x}) = \varphi(\mathbf{x}) + \cos[bc\sqrt{m^2-\nabla^2}]e^{\mathbf{a}\cdot\nabla}\varphi'(\mathbf{x}) + \frac{\sin[bc\sqrt{m^2-\nabla^2}]}{\sqrt{m^2-\nabla^2}}e^{\mathbf{a}\cdot\nabla}\dot\varphi'(\mathbf{x}) \\ \dot\varphi''(\mathbf{x}) = \dot\varphi(\mathbf{x}) + \cos[bc\sqrt{m^2-\nabla^2}]e^{\mathbf{a}\cdot\nabla}\dot\varphi(\mathbf{x}) \\ \qquad -\sqrt{m^2-\nabla^2}\sin[bc\sqrt{m^2-\nabla^2}]e^{\mathbf{a}\cdot\nabla}\varphi'(\mathbf{x}) \\ \zeta'' = \zeta'\zeta\exp\left(\frac{i}{2}\int_\Sigma \mathrm{d}^3x\left[\dot\varphi(\cos[bc\sqrt{m^2-\nabla^2}]e^{\mathbf{a}\cdot\nabla}\varphi' + \frac{\sin[bc\sqrt{m^2-\nabla^2}]}{\sqrt{m^2-\nabla^2}}e^{\mathbf{a}\cdot\nabla}\dot\varphi') - \right.\right. \\ \qquad \left.\left.\varphi(\cos[bc\sqrt{m^2-\nabla^2}]e^{\mathbf{a}\cdot\nabla}\dot\varphi' - \sqrt{m^2-\nabla^2}\sin[bc\sqrt{m^2-\nabla^2}]e^{\mathbf{a}\cdot\nabla}\varphi')\right]\right) \end{array}\right| \begin{array}{l}\text{Central}\\ \\ \text{extension}\\ \text{by } U(1)\\ \\ \end{array} \tag{178}$$

Notice that we can read from the group law the expression of the evolved fields $\phi(x^0,\mathbf{x}),\ \dot\phi(x^0,\mathbf{x})$ in terms of the initial conditions $\varphi(\mathbf{x}),\ \dot\varphi(\mathbf{x})$:

$$\begin{aligned} \phi(x^0,\mathbf{x}) &= \cos[x^0\sqrt{m^2-\nabla^2}]\varphi(\mathbf{x}) + \frac{\sin[x^0\sqrt{m^2-\nabla^2}]}{\sqrt{m^2-\nabla^2}}\dot\varphi(\mathbf{x}) \\ \dot\phi(x^0,\mathbf{x}) &= \cos[x^0\sqrt{m^2-\nabla^2}]\dot\varphi(\mathbf{x}) - \sqrt{m^2-\nabla^2}\sin[x^0\sqrt{m^2-\nabla^2}]\varphi(\mathbf{x})\,. \end{aligned} \tag{179}$$

Left-invariant algebra:

$$\left|\begin{array}{l} \tilde X^L_b = \frac{\partial}{\partial b} + \int_\Sigma \mathrm{d}^3x\dot\varphi(\mathbf{x})\frac{\delta}{\delta\varphi(\mathbf{x})} - \int_\Sigma \mathrm{d}^3x(m^2-\nabla^2)\varphi(\mathbf{x})\frac{\delta}{\delta\dot\varphi(\mathbf{x})} \\ \tilde X^L_{\mathbf{a}} = \frac{\partial}{\partial\mathbf{a}} + \int_\Sigma \mathrm{d}^3x\nabla\varphi(\mathbf{x})\frac{\delta}{\delta\varphi(\mathbf{x})} + \int_\Sigma \mathrm{d}^3x\nabla\dot\varphi(\mathbf{x})\frac{\delta}{\delta\dot\varphi(\mathbf{x})} \\ \tilde X^L_{\varphi(\mathbf{x})} = \frac{\delta}{\delta\varphi(\mathbf{x})} - \frac{1}{2}\dot\varphi(\mathbf{x})\Xi \\ \tilde X^L_{\dot\varphi(\mathbf{x})} = \frac{\delta}{\delta\dot\varphi(\mathbf{x})} + \frac{1}{2}\varphi(\mathbf{x})\Xi \\ \tilde X^L_\zeta = \mathrm{Re}(i\zeta\frac{\partial}{\partial\zeta}) \equiv \Xi \end{array}\right. \tag{180}$$

Quantization form:

$$\begin{aligned} \Theta = &\frac{1}{2}\int_\Sigma \mathrm{d}^3x\ (\dot\varphi(\mathbf{x})\delta\varphi(\mathbf{x}) - \varphi(\mathbf{x})\delta\dot\varphi(\mathbf{x})) \\ &-\frac{1}{2}\int_\Sigma \mathrm{d}^3x\left(\dot\varphi^2(\mathbf{x}) + \nabla\varphi(\mathbf{x})\cdot\nabla\varphi(\mathbf{x}) + m^2\varphi^2(\mathbf{x})\right)db \\ &-\frac{1}{2}\int_\Sigma \mathrm{d}^3x\ (\dot\varphi(\mathbf{x})\nabla\varphi(\mathbf{x}) - \varphi(\mathbf{x})\nabla\dot\varphi(\mathbf{x}))\cdot\mathrm{d}\mathbf{a} + \frac{\mathrm{d}\zeta}{i\zeta}\,. \end{aligned} \tag{181}$$

Commutators:

$$\begin{aligned} \left[\tilde X^L_b,\ \tilde X^L_{\varphi(\mathbf{x})}\right] &= (m^2-\nabla^2)\tilde X^L_{\dot\varphi(\mathbf{x})} \\ \left[\tilde X^L_b,\ \tilde X^L_{\dot\varphi(\mathbf{x})}\right] &= \tilde X^L_{\varphi(\mathbf{x})} \\ \left[\tilde X^L_{\mathbf{a}},\ \tilde X^L_{\varphi(\mathbf{x})}\right] &= -\nabla\tilde X^L_{\varphi(\mathbf{x})} \\ \left[\tilde X^L_{\mathbf{a}},\ \tilde X^L_{\dot\varphi(\mathbf{x})}\right] &= -\nabla\tilde X^L_{\dot\varphi(\mathbf{x})} \\ \left[\tilde X^L_{\varphi(\mathbf{x})},\ \tilde X^L_{\dot\varphi(\mathbf{x})}\right] &= -\delta(\mathbf{x}-\mathbf{x}')\Xi\,. \end{aligned} \tag{182}$$

The Characteristic subalgebra is $\mathcal{G}_\Theta = \langle\tilde X^L_b,\ \tilde X^L_{\mathbf{a}}\rangle$, so that

$$\Theta/\mathcal{G}_\Theta = \frac{1}{2}\int_\Sigma \mathrm{d}^3x(\dot\varphi_0(\mathbf{x})\delta\varphi_0(\mathbf{x}) - \varphi_0(\mathbf{x})\delta\dot\varphi_0(\mathbf{x})) + \frac{\mathrm{d}\zeta_0}{i\zeta_0}\,, \tag{183}$$

where the subscript 0 refers to the initial value in the integration of (generalized) equations of motion corresponding to $\mathcal{G}_\Theta$.

Covariant Formulation. The construction above can be repeated in a form more convenient for the interaction. Now the fields will be defined on the entire Minkowski space-time but supposed to be solutions of the equations of motion

$$\begin{aligned}
a''_\mu &= a'_\mu + \Lambda'^{\nu}_{\mu} a_\nu \\
\Lambda'' &= \Lambda' \Lambda \\
\phi''(x) &= \phi'(\Lambda x + a) + \phi(x) \\
\phi''_\mu(x) &= \phi'_\mu(\Lambda x + a) + \phi_\mu(x) \\
\zeta'' &= \zeta' \zeta \exp \frac{i}{2} \int_\Sigma \mathrm{d}\sigma^\mu \left\{ \phi'_\mu(\Lambda x + a)\phi(x) - \phi_\mu(x)\phi'(\Lambda x + a) \right\} \\
&= \zeta' \zeta \exp \frac{i}{2} \int_\Sigma \mathrm{d}\sigma^\mu_x \int_\Sigma \mathrm{d}\sigma^\nu_y [-\phi'_\mu(\Lambda x + a)\partial^y_\nu \Delta(x-y)\phi(y) \\
&\quad + \phi'(\Lambda x + a)\partial^y_\mu \Delta(x-y)\partial_\nu \phi(y)] \\
&= \zeta' \zeta \exp \frac{-i}{2} \frac{1}{V^2} \int_M \mathrm{d}^4 y \int_M \mathrm{d}^4 z [\phi'(\Lambda y + a)\partial^\mu_y \Delta(y-z)\phi_\mu(z) \\
&\quad - \phi(y)\partial^\mu_y \Delta(y-z)\phi'_\mu(\Lambda z + a)] ,
\end{aligned} \tag{184}$$

where the invariant function Pauli–Jordan $\Delta(x)$ verifies

$$\left.\begin{array}{l}
\Box \Delta + m^2 \Delta = 0 \\
\Delta|_\Sigma = 0 \\
\partial^\mu \Delta|_\Sigma = -\delta^3_\Sigma \\
\Delta(x-y) = -\Delta(y-x)
\end{array}\right| \quad
\begin{array}{l}
\Delta(0, \mathbf{x}) = 0 \\
\partial_0 \Delta(x)_{x^0=0} = -\delta^3(\mathbf{x})
\end{array} \tag{185}$$

$$i\Delta(x-y) = \int \frac{\mathrm{d}^4 k}{(2\pi)^3} \delta(k^2 - m^2)\varepsilon(k^0) e^{-ik\cdot(x-y)} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2\omega(\mathbf{k})} \{e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)}\} , \tag{186}$$

where $\varepsilon(k^0)$ is the sign function, $\omega(\mathbf{k}) \equiv |\mathbf{k}|$ and V is the (infinite) volume of "time."

Left-generators: (formally distinguishing between $\partial_\mu \phi$ and ϕ_μ)

$$\begin{aligned}
\tilde{X}^L_{a_\mu} &= \frac{\partial}{\partial a_\mu} + \int_M \mathrm{d}^4 x \left(\partial_\mu \phi \frac{\delta}{\delta \phi_\mu} + \partial_\mu \phi_\nu \frac{\delta}{\delta \phi_\nu} \right) \\
\tilde{X}^L_{\phi(x)} &= \frac{\delta}{\delta \phi(x)} - \frac{1}{2} \int_M \mathrm{d}^4 z \partial^\mu_z \Delta(x-z)\phi_\mu(z)\Xi \\
\tilde{X}^L_{\phi_\mu(x)} &= \frac{\delta}{\delta \phi_\mu(x)} + \frac{1}{2} \int_M \mathrm{d}^4 z \phi_\mu(z)\partial^\mu_z \Delta(x-z)\Xi ,
\end{aligned} \tag{187}$$

where we have disregarded the infinite volume V.

Commutators:

$$\begin{aligned}
\left[\tilde{X}^L_{a_\mu}, \tilde{X}^L_{\phi(x)}\right] &= -\partial_\mu \frac{\delta}{\delta \phi(x)} - \frac{1}{2} \int_M \mathrm{d}^4 z \partial^\nu \Delta(x-z)\partial_\mu \phi_\nu(z)\Xi = -\partial_\mu \tilde{X}^L_{\phi(x)} \\
\left[\tilde{X}^L_{a_\mu}, \tilde{X}^L_{\phi_\nu(x)}\right] &= -\partial_\mu \frac{\delta}{\delta \phi_\nu(x)} + \frac{1}{2} \int_M \mathrm{d}^4 z \partial^\nu \Delta(x-z)\partial_\mu \phi(z)\Xi = -\partial_\mu \tilde{X}^L_{\phi_\nu(x)} \\
\left[\tilde{X}^L_{\phi(x)}, \tilde{X}^L_{\phi_\mu(y)}\right] &= -\frac{1}{2} \int_M \mathrm{d}^4 u \delta(u-x)\partial^\mu_u \Delta(y-u)\Xi
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_M \mathrm{d}^4 z \partial_z^\nu \Delta(x-z)\delta_\nu^\mu \delta(z-y)\Xi \\
&= \partial_y^\mu \Delta(x-y)\Xi \quad \text{(equal-time commutators)}\,. \qquad (188)
\end{aligned}
$$

Textually, $\left[\tilde{X}^L_{\phi(x)},\ \tilde{X}^L_{\phi(y)}\right]=0$, unless we interpret that $\phi_\mu=\partial_\mu\phi$ (something that happens along the physical trajectories) and in this case we would have the arbitrary-time commutator:

$$
\begin{aligned}
\left[\tilde{X}^L_{\phi(x)},\ \tilde{X}^L_{\phi(y)}\right] &= \left[\frac{\delta}{\delta\phi(x)} - \frac{1}{2m^2}\int_M \mathrm{d}^4 z \partial_z^\mu \Delta(x-z)\partial_\mu\phi(z)\Xi,\ \frac{\delta}{\delta\phi(y)}\right. \\
&\quad \left. - \frac{1}{2m^2}\int_M \mathrm{d}^4 u \partial_z^\nu \Delta(y-u)\partial_\nu\phi(z)\Xi\right] \\
&= -\frac{1}{2m^2}\int_M \mathrm{d}^4 u \partial_u^\nu \Delta(y-u)\partial_\nu\delta(u-x) + \frac{1}{2m^2}\int_M \mathrm{d}^4 z \partial_z^\mu \Delta(x-z)\partial_\mu\delta(z-y) \\
&= \frac{1}{am^2}\Box\Delta(y-x) - \frac{1}{2m^2}\Box\Delta(x-y) = -\frac{1}{2m^2}m^2\Delta(y-x) + \frac{1}{2m^2}\Delta(x-y) \\
&= \Delta(x-y) \qquad (189)
\end{aligned}
$$

where we have "redefined" the fields so as to make explicit the mass m^2.

This computation renders clear the necessity that $\Delta(x-y)$ satisfies the equation of motion.

6.2.5 The Dirac field

(just sketched, Λ of the Lorentz subgroup discarded)

$$
\begin{aligned}
a'' &= a' + a \\
\psi''(x) &= \psi(x) + \psi'(x+a) \\
\bar{\psi}''(x) &= \bar{\psi}(x) + \bar{\psi}'(x+a) \\
\psi''_\mu(x) &= \psi_\mu(x) + \psi'_\mu(x+a) \\
\bar{\psi}''_\mu(x) &= \bar{\psi}_\mu(x) + \bar{\psi}'_\mu(x+a) \\
\zeta'' &= \zeta'\zeta\exp\frac{-1}{2}\int_M \mathrm{d}^4 y \int_M \mathrm{d}^4 z \left\{\bar{\psi}'(y+a)[i\gamma^\mu\partial_\mu^z + m]\Delta(y-z)\psi(z)\right. \\
&\quad \left. - \bar{\psi}(y)[i\gamma^\mu\partial_\mu^z + m]\Delta(y-z)\psi'(z+a)\right\}. \qquad (190)
\end{aligned}
$$

It is customary to use the invariant function $S(x)\equiv(i\gamma^\mu\partial_\mu+m)\Delta(x)$, which satisfies the Dirac equation:

$$
(i\partial\!\!\!/ - m)(i\partial\!\!\!/ + m)\Delta = (\Box + m^2)\Delta = 0\,. \qquad (191)
$$

<u>Remark:</u> If we consider ψ and $\bar{\psi}$ as Fermionic variables, then the relative sign in the cocycle would be $+$

Left-generators:

$$
\begin{aligned}
\tilde{X}^L_{\psi(x)} &= \frac{\delta}{\delta\psi(x)} - \frac{1}{2}\int_M \mathrm{d}^4 z \bar{\psi}(z)[i\gamma^\sigma\partial_\sigma^x + m]\Delta(z-x)\Xi \\
\tilde{X}^L_{\bar{\psi}(x)} &= \frac{\delta}{\delta\bar{\psi}(x)} + \frac{1}{2}\int_M \mathrm{d}^4 z [i\gamma^\nu\partial_\nu^x + m]\Delta(x-z)\psi(z)\Xi \\
\tilde{X}^L_{a^\mu} &= \frac{\partial}{\partial a^\mu} + \int_M \mathrm{d}^4 x \left(\partial_\mu\psi(x)\frac{\delta}{\delta\psi(x)} + \partial_\mu\bar{\psi}(x)\frac{\delta}{\delta\bar{\psi}(x)}\right)\Xi \qquad (192)
\end{aligned}
$$

Right ones:

$$
\begin{aligned}
\tilde{X}^R_{\psi(x)} &= \frac{\delta}{\delta\psi(x-a)} + \frac{1}{2}\int_M \mathrm{d}^4 y \bar{\psi}(y)[i\partial\!\!\!/_x + m]\Delta(y-x+a)\Xi \\
\tilde{X}^R_{\bar{\psi}(x)} &= \frac{\delta}{\delta\bar{\psi}(x-a)} - \frac{1}{2}\int_M \mathrm{d}^4 z[i\partial\!\!\!/_x + m]\Delta(x-z-a)\psi(z)\Xi \\
\tilde{X}^R_{a^\mu} &= \frac{\partial}{\partial a^\mu}
\end{aligned}
\tag{193}
$$

Arbitrary-time commutators:

$$
\begin{aligned}
\left[\tilde{X}^R_{a^\mu},\ \tilde{X}^R_{\psi(x)}\right] &= \frac{\partial}{\partial a^\mu}\int_M \mathrm{d}^4 y\delta(y+a-x)\frac{\delta}{\delta\psi(y)} \\
&\quad - \frac{1}{2}\int_M \mathrm{d}^4 y\bar{\psi}(y)[i\partial\!\!\!/_x + m]\partial^x_\mu\Delta(y-x+a)\Xi = -\partial_\mu\tilde{X}^R_{\psi(x)} \\
\left[\tilde{X}^R_{a^\mu},\ \tilde{X}^R_{\bar{\psi}(x)}\right] &= \ldots = -\partial_\mu\tilde{X}^R_{\bar{\psi}(x)} \\
\left[\tilde{X}^R_{\psi(x)},\ \tilde{X}^R_{\bar{\psi}(y)}\right] &= \ldots = (i\partial\!\!\!/_x + m)\Delta(x-y)\Xi \equiv S(x-y)\Xi.
\end{aligned}
\tag{194}
$$

7 Gauge theory of internal symmetries

Internal symmetries refer to transformations moving only the internal (fiber) components of a matter field [1,28,41,42]. In our language, they are generated by vector fields of the form:

$$
X_{(a)} = X^\alpha_{(a)}\frac{\partial}{\partial\varphi^\alpha} = X^\alpha_{(a)\beta}\varphi^\beta\frac{\partial}{\partial\varphi^\alpha}\,. \tag{195}
$$

Here, $\{\varphi^\alpha\}$ are the coordinates of the fiber of $E \overset{\pi}{\to} M$ on the space-time M, usually the Minkowski space with coordinates $\{x^\mu\}$, $\mu = 0, 1, 2, 3$. The generators above are supposed to close a (finite-dimensional) algebra:

$$
[X_{(a)},\ X_{(b)}] = C^c_{ab}X_{(c)}\,, \tag{196}
$$

to be referred to as the rigid or global symmetry algebra.

The **Minimal Interaction Principle** establishes that a matter Lagrangian $\mathcal{L}_{matt}$ invariant under a rigid group G can be converted into a new one, $\hat{\mathcal{L}}_{matt}$, invariant under the corresponding local (usually called gauge) group $G(M)$, that is, a group generated by $\mathcal{F}(M) \otimes \mathcal{G}$, $\mathcal{F}(M)$ being the algebra of functions on M, $\mathcal{G}$ the Lie algebra of G. The Lie algebra of $G(M)$ satisfies:

$$
[f^{(a)}X_{(a)},\ g^{(b)}X_{(b)}] = f^{(a)}g^{(b)}[X_{(a)},\ X_{(b)}] = f^{(a)}g^{(b)}C^c_{ab}X_{(c)} \tag{197}
$$

where the actual generators of $G(M)$ are locally written as

$$
f^{(a)}X_{(a)},\quad f^{(a)} :\ M \to \mathbf{R}\,.
$$

Note that the ordinary Lie bracket really realizes (197) since $X_{(a)} = X^\alpha_{(a)}\frac{\partial}{\partial\varphi^\alpha}$, with $X^\mu_{(a)} = 0$, for any internal symmetry. Otherwise, the components $X^\mu_{(a)}\frac{\partial}{\partial x^\mu}$ would have derivated the function $g^{(b)}(x)$, giving rise to additional terms in (*) (see later on the space-time gauge symmetries).

The essential consequence of the dependence on x^μ of the local group parameters lies on the different realization of the jet extension of $f^{(a)}X_{(a)}$, which now differs from $f^{(a)}\bar{X}_{(a)}$: $\overline{f^{(a)}X_{(a)}} \neq f^{(a)}\bar{X}_{(a)}$. In fact, computing $\overline{f^{(a)}X_{(a)}}$ according to the standard formulas we get:

$$\overline{f^{(a)}X_{(a)}} = f^{(a)}X^\alpha_{(a)\beta}\varphi^\beta \frac{\partial}{\partial\varphi^\alpha} + (f^{(a)}X^\alpha_{(a)\beta}\varphi^\beta_\mu + X^\alpha_{(a)\beta}\varphi^\beta \frac{\partial f^{(a)}}{\partial x^\mu})\frac{\partial}{\partial\varphi^\alpha_\mu}, \tag{198}$$

so that $\bar{X}_{(a)}\mathcal{L}_{matt} = 0$ **does not** imply $\overline{f^{(a)}(x)X_{(a)}}\mathcal{L}_{matt} = 0$. The extra term $X^\alpha_{(a)\beta}\varphi^\beta \frac{\partial f^{(a)}}{\partial x^\mu}\frac{\partial}{\partial\varphi^\alpha_\mu}$ must be canceled out somehow.

We must introduce extra compensating fields $A^{(a)}_\mu$, the gauge vector bosons, transforming under $G(M)$ as:

$$\delta A^{(a)}_\mu \equiv X_{A^{(a)}_\mu} = f^{(b)}C^a_{bc}A^{(c)}_\mu - \frac{\partial f^{(a)}}{\partial x^\mu}. \tag{199}$$

This way, the complete generators of $G(M)$, acting on φ^α and $A^{(a)}_\mu$, are:

$$f^{(a)}\mathcal{X}_{(a)} = f^{(a)}X_{(a)} + X_{A^{(a)}_\mu}\frac{\partial}{\partial A^{(a)}_\mu}. \tag{200}$$

The transformation properties of $A^{(a)}_\mu$ do correspond to those of a **derivation law** on the sections of E associated with a connection 1-form on the original principal bundle P. The corresponding Christoffel symbols are

$$\Gamma^\alpha_{\mu\beta} \equiv A^{(a)}_\mu (X_{(a)})^\alpha_\beta. \tag{201}$$

However, connections are not the only way of realizing the vector potentials $A^{(a)}_\mu$. We shall construct such fields from the group $G(M)$ **itself** a bit later!!.

Utiyama's Theorem

We establish this theorem in two parts, the first of which refers to the matter field Lagrangian, $\hat{\mathcal{L}}_{matt}$, whereas the second tell us about the Lagrangian, $\mathcal{L}_0$, governing the (free) gauge fields themselves.

Utiyama's Theorem I: The new Lagrangian $\hat{\mathcal{L}}_{matt}$ describing the dynamics of the matter fields along with their interaction with the vector potentials $A^{(a)}_\mu$,

$$\hat{\mathcal{L}}_{matt}(\varphi^\alpha, \varphi^\beta_\mu, A^{(a)}_\nu) \equiv \mathcal{L}_{matt}(\varphi^\alpha, \varphi^\beta_\mu + A^{(a)}_\mu X^\alpha_{(a)\beta}\varphi^\beta) \tag{202}$$

is invariant under the local group $G(M)$, that is,

$$\overline{f^{(a)}(x)\mathcal{X}_{(a)}}\hat{\mathcal{L}}_{matt}(\varphi^\alpha, \varphi^\beta_\mu, A^{(a)}_\nu) = 0. \tag{203}$$

Proof Consider the following change of variables χ:

$$\begin{array}{lcl} \phi^\alpha = \varphi^\alpha & & \varphi^\alpha = \phi^\alpha \\ \phi^\beta_\mu = \varphi^\beta_\mu + A^{(a)}_\mu X^\beta_{(a)\alpha}\varphi^\alpha & \Leftrightarrow & \varphi^\beta_\mu = \phi^\beta_\mu - B^{(a)}_\mu X^\beta_{(a)\alpha}\phi^\alpha \\ B^{(a)}_\nu = A^{(a)}_\nu & & A^{(a)}_\nu = B^{(a)}_\nu \end{array} \tag{204}$$

and the Jacobian:

$$\begin{aligned}
\frac{\partial}{\partial \varphi^\alpha} &= \frac{\partial}{\partial \phi^\alpha} + B_\mu^{(a)} X_{(a)\alpha}^\beta \frac{\partial}{\partial \phi_\mu^\beta} \\
\frac{\partial}{\partial \varphi_\mu^\alpha} &= \frac{\partial}{\partial \phi_\mu^\alpha} \\
\frac{\partial}{\partial A_\mu^{(a)}} &= \frac{\partial}{\partial B_\mu^{(a)}} + X_{(a)\beta}^\alpha \phi^\beta \frac{\partial}{\partial \phi_\mu^\alpha} \, .
\end{aligned} \tag{205}$$

After this change of variables,

$$\begin{aligned}
&\hat{\mathcal{L}}_{matt}(\varphi^\alpha, \varphi_\mu^\beta, A_\nu^{(a)}) \equiv \mathcal{L}_{matt}(\varphi^\alpha, \varphi_\mu^\beta + A_\mu^{(a)} X_{(a)\gamma}^\beta \varphi^\gamma) = \mathcal{L}_{matt}(\phi^\alpha, \phi_\mu^\beta) \\
&= \mathcal{L}_{matt} \circ \chi(\varphi^\alpha, \varphi_\mu^\beta, A_\nu^{(a)}) \, .
\end{aligned} \tag{206}$$

We must now compute $\overline{f^{(a)} \mathcal{X}_{(a)}} \hat{\mathcal{L}}_{matt}$:

$$\begin{aligned}
\overline{f^{(a)} \mathcal{X}_{(a)}} \hat{\mathcal{L}}_{matt} = &\Bigg(f^{(a)} X_{(a)\beta}^\alpha \varphi^\beta \frac{\partial}{\partial \varphi^\alpha} + (f^{(a)} X_{(a)\beta}^\alpha \varphi_\mu^\beta \frac{\partial f^{(a)}}{\partial x^\mu}) \frac{\partial}{\partial \varphi_\mu^\alpha} \\
&+ (f^{(b)} C_{bc}^a A_\mu^{(c)} - \frac{\partial f^{(a)}}{\partial x^\mu}) \frac{\partial}{\partial A_\mu^{(a)}} \Bigg) \hat{\mathcal{L}}_{matt}(\varphi^\gamma, \varphi^\delta, A_\sigma^{(d)}) \\
= &\Bigg(f^{(a)} X_{(a)\beta}^\alpha \phi^\beta (\frac{\partial}{\partial \phi^\alpha} + B_\mu^{(b)} X_{(b)\alpha}^\gamma \frac{\partial}{\partial \phi_\mu^\gamma}) + (f^{(a)} X_{(a)\beta}^\alpha (\phi_\mu^\beta - B_\mu^{(b)} X_{(b)\gamma}^\beta \phi^\gamma) \\
&+ \frac{\partial f^{(a)}}{\partial x^\mu} X_{(a)\beta}^\alpha \phi^\beta) \frac{\partial}{\partial \phi_\mu^\alpha} (f^{(b)} C_{bc}^a B_\mu^{(c)} - \frac{\partial f^{(a)}}{\partial x^\mu})(\frac{\partial}{\partial B_\mu^{(a)}} + X_{(a)\beta}^\alpha \phi^\beta \frac{\partial}{\phi_\mu^\alpha}) \Bigg) \mathcal{L}_{matt}(\phi^\alpha, \phi_\mu^\beta) \\
= &\Bigg(f^{(a)} X_{(a)\beta}^\alpha \phi^\beta \frac{\partial}{\partial \phi^\alpha} + f^{(a)} X_{(a)\beta}^\alpha \phi_\mu^\beta \frac{\partial}{\partial \phi_\mu^\alpha} + (f^{(b)} C_{bc}^a B_\mu^{(c)} - \frac{\partial f^{(a)}}{\partial x^\mu}) \frac{\partial}{\partial B_\mu^{(a)}} \\
&+ (f^{(a)} B_\mu^{(b)} (X_{(b)} X_{(a)})_\beta^\gamma \phi^\beta - f^{(a)} B_\mu^{(b)} (X_{(a)} X_{(b)})_\beta^\gamma \phi^\beta) \frac{\partial}{\partial \phi_\mu^\gamma} \\
&+ f^{(b)} C_{bc}^a B_\mu^{(c)} X_{(a)\beta}^\alpha \phi^\beta \frac{\partial}{\partial \phi_\mu^\alpha} + \frac{\partial f^{(a)}}{\partial x^\mu} X_{(a)\beta}^\alpha \phi^\beta \frac{\partial}{\partial \phi_\mu^\alpha} - \frac{\partial f^{(a)}}{\partial x^\mu} X_{(a)\beta}^\alpha \phi^\beta \frac{\partial}{\partial \phi_\mu^\alpha} \Bigg) \mathcal{L}_{matt}(\phi^\alpha, \phi_\mu^\beta) \\
= &\Bigg(f^{(a)} X_{(a)\beta}^\alpha \phi^\beta \frac{\partial}{\partial \phi^\alpha} + f^{(a)} X_{(a)\beta}^\alpha \phi_\mu^\beta \frac{\partial}{\partial \phi_\mu^\alpha} + (f^{(b)} C_{bc}^a B_\mu^{(c)} - \frac{\partial f^{(a)}}{\partial x^\mu}) \frac{\partial}{\partial B_\mu^{(a)}} \Bigg) \mathcal{L}_{matt}(\phi^\alpha, \phi_\mu^\beta) \\
= &\Bigg(f^{(a)} X_{(a)\beta}^\alpha \phi^\beta \frac{\partial}{\partial \phi^\alpha} + f^{(a)} X_{(a)\beta}^\alpha \phi_\mu^\beta \frac{\partial}{\partial \phi_\mu^\alpha} \Bigg) \mathcal{L}_{matt}(\phi^\alpha, \phi_\mu^\beta) \\
= &f^{(a)} \bar{X}_{(a)} \mathcal{L}_{matt}(\phi^\alpha, \phi_\mu^\beta) = 0 \, .
\end{aligned} \tag{207}$$

Thinking of $A_\mu^{(a)}$ as connections, we may say that under the Minimal Coupling, the covariant "derivative" of φ^α substitutes the ordinary one in $\mathcal{L}_{matt}$:

$$\varphi_\mu^\alpha \;\mapsto\; \varphi_\mu^\alpha + \Gamma_{\mu\beta}^\alpha \varphi^\beta = \varphi_\mu^\alpha + A_\mu^{(a)} X_{(a)\beta}^\alpha \varphi^\beta \equiv \varphi_{\hat{\mu}}^\alpha \text{ (or } \varphi_{;\mu}^\alpha) \, . \tag{208}$$

On jet extensions,

$$\partial_\mu \varphi^\alpha \;\mapsto\; D_\mu \varphi^\alpha = \partial_\mu \varphi^\alpha + \Gamma_{\mu\beta}^\alpha \varphi^\beta \, . \tag{209}$$

Notice that under $G(M)$, $\varphi_{\hat{\mu}}^\alpha$ transforms as a tensor:

$$\delta \varphi_{\hat{\mu}}^\alpha = f^{(a)} X_{(a)\beta}^\alpha \varphi_{\hat{\mu}}^\beta \, . \tag{210}$$

We have introduced new fields $A_\mu^{(a)}$ which must be controlled by a given Lagrangian $\mathcal{L}_0(A_\mu^{(a)}, A_{\nu,\sigma}^{(b)})$ so that the total Lagrangian will be $\hat{\mathcal{L}}_{matt} + \mathcal{L}_0$

$$\mathcal{S}_{tot} = \int \mathrm{d}^4 x (\mathcal{L}_0 + \hat{\mathcal{L}}_{matt}) \,. \tag{211}$$

$\mathcal{L}_0$ should be constrained by the condition of being invariant under $G(M)$ (note that $\hat{\mathcal{L}}_{matt}$ is already invariant). Thus, the second part of the theorem says: □

Utiyama's Theorem II: The necessary condition for $\mathcal{L}_0$ to be invariant under $G(M)$ is that it depends on $A_\mu^{(a)}$ and $A_{\nu,\sigma}^{(a)}$ only through the specific combination

$$F_{\mu,\nu}^{(a)} \equiv A_{\mu,\nu}^{(a)} - A_{\nu,\mu}^{(a)} + \frac{1}{2} C_{bc}^a (A_\mu^{(b)} A_\nu^{(c)} - A_\nu^{(b)} A_\mu^{(c)}) \,, \tag{212}$$

named **curvature** tensor or intensity tensor.

Proof We have to solve the equation

$$\begin{aligned}\overline{f^{(a)}(x)\mathcal{X}_{(a)}}\hat{\mathcal{L}}_0 = X_{A_\mu^{(a)}} \frac{\partial \mathcal{L}_0}{\partial A_\mu^{(a)}} + \bar{X}_{A_{\mu,\nu}^{(a)}} \frac{\partial \mathcal{L}_0}{\partial A_{\mu,\nu}^{(a)}} = (f^{(b)} C_{bc}^a A_\mu^{(c)} - \frac{\partial f^{(a)}}{\partial x^\mu}) \\ +(f^{(b)} C_{bc}^a A_{\mu,\nu}^{(c)} + C_{bc}^a A_\mu^{(c)} \frac{\partial f^{(b)}}{\partial x^\nu} - \frac{\partial^2 f^{(a)}}{\partial x^\nu \partial x^\mu}) \frac{\partial \mathcal{L}_0}{\partial A_{\mu,\nu}^{(a)}} = 0\end{aligned} \tag{213}$$

for arbitrary $f^{(a)}$, which implies that

$$\begin{aligned}&\textbf{a)} \qquad \forall f^{(b)}: C_{bc}^a A_\mu^{(c)} \frac{\partial \mathcal{L}_0}{\partial A_\mu^{(a)}} + C_{bc}^a A_{\mu,\nu}^{(c)} \frac{\partial \mathcal{L}_0}{\partial A_{\mu,\nu}^{(a)}} = 0 \\ &\textbf{b)} \quad \forall \frac{\partial f^{(b)}}{\partial x^\mu}: \frac{\partial \mathcal{L}_0}{\partial A_\mu^{(b)}} - C_{bc}^a A^{(c)} \frac{\partial \mathcal{L}_0}{\partial A_{\nu,\mu}^{(a)}} = 0 \\ &\textbf{c)} \ \forall \frac{\partial^2 f^{(b)}}{\partial x^\nu \partial x^\mu}: \frac{\partial \mathcal{L}_0}{\partial A_{\mu,\nu}^{(a)}} + \frac{\partial \mathcal{L}_0}{\partial A_{\nu,\mu}^{(a)}} = 0 \,.\end{aligned} \tag{214}$$

(a) in turns implies that $\mathcal{L}_0$ is **invariant** under the **rigid group** G.
(c) implies that $\mathcal{L}_0$ depends on $A^{(b)}$ only through the difference $A_{\mu,\nu}^{(b)} - A_{\nu,\mu}^{(b)}$.
(b) then implies the equation

$$\frac{\partial \mathcal{L}_0}{\partial A_\mu^{(b)}} = -C_{bc}^a A_\nu^{(c)} \frac{\partial \mathcal{L}_0}{\partial (A_{\mu,\nu}^{(a)} - A_{\nu,\mu}^{(a)})} \,, \tag{215}$$

which is of the form $\frac{\partial f}{\partial x} = kx \frac{\partial f}{\partial y}$, with general solution $f = f(y + \frac{1}{2}kx^2)$. Therefore,

$$\mathcal{L}_0 = \mathcal{L}_0(F_{\mu\nu}^{(a)}) \,. \tag{216}$$

The additional condition for $\mathcal{L}_0$ of being invariant under the rigid Poincaré group (or any other kinematical space-time rigid symmetry) means a further restriction: $\mathcal{L}_0$ must be a scalar. For internal symmetries the Yang–Mills Lagrangian

$$\mathcal{L}_0^{Y-M} = -\frac{1}{4} F_{\mu\nu}^{(a)} F^{(b)\mu\nu} k_{ab} \,, \tag{217}$$

where k_{ab} is the Killing metric, is usually adopted. □

The Euler–Lagrange equations of the total Lagrangian $\mathcal{L}_{tot} = \mathcal{L}_{matt}(\varphi^\alpha, \varphi^\beta_\mu + A^{(a)}_\mu X^\beta_{(a)\gamma}\varphi^\gamma) + \mathcal{L}_0(F^{(b)}_{\mu\nu})$ corresponding to the independent variables φ^α and $A^{(a)}_\mu$ are:

$$\delta\varphi : \quad \frac{\partial \mathcal{L}_{matt}}{\partial \varphi^\alpha} + A^{(a)}_\mu X^\beta_{(a)\alpha} \frac{\mathcal{L}_{matt}}{\partial \varphi^\beta_\mu} - \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial \mathcal{L}_{matt}}{\partial \varphi^\alpha_\mu}\right) = 0 \tag{218}$$

$$\delta A : \quad -A^{(d)}_\nu C^b_{da} \frac{\partial \mathcal{L}_0}{\partial F^{(b)}_{\mu\nu}} - \frac{\mathrm{d}}{\mathrm{d}x^\nu}\left(\frac{\partial \mathcal{L}_0}{\partial F^{(a)}_{\mu\nu}}\right) = X^\alpha_{(a)\beta}\varphi^\beta \frac{\partial \mathcal{L}_{matt}}{\partial \varphi^\alpha_\mu}\,. \tag{219}$$

In particular, for the Yang–Mills Lagrangian $\mathcal{L}_0$ we have:

$$F^{\mu\nu}_{(a),\nu} + A^{(d)}_\nu C^b_{dc} F^{\mu\nu}_{(b)} = -X^\alpha_{(a)\beta}\varphi^\beta \frac{\partial \mathcal{L}_{matt}}{\partial \varphi^\alpha_\mu}\,,$$

or, using the covariant derivative notation, D_μ,

$$\frac{\partial \mathcal{L}_{matt}}{\partial \varphi^\alpha} - D_\mu\left(\frac{\partial \mathcal{L}_{matt}}{\partial D_\mu \varphi^\alpha}\right) = 0\,; \quad D_\mu F^{\mu\nu}_{(a)} = \hat{J}^\mu_{(a)} \tag{220}$$

where the current $\hat{J}$ is defined as

$$\hat{J}^\mu_{(a)} \equiv -\frac{\partial \mathcal{L}_{matt}}{\partial A^{(a)}_\mu} = -X^\alpha_{(a)\beta}\varphi^\beta \frac{\partial \mathcal{L}_{matt}}{\partial D_\mu \varphi^\alpha} = -X^\alpha_{(a)\beta}\varphi^\beta \frac{\partial \mathcal{L}_{matt}}{\partial \varphi^\alpha_\mu}\,. \tag{221}$$

It is worth noticing that the Euler–Lagrange equations of the Lagrangian $\mathcal{L}_{tot}$, after the change of variables used for proving Utiyama's theorem, would be those of the free fields ϕ and B. In fact, $\mathcal{L}_{tot} = \mathcal{L}_{matt}(\phi, \phi_\mu) + \mathcal{L}_0(B, F)$, **without interacting term!!**. However, this is a consequence of the fact that the mentioned change of variables **does not preserve** the structure 1-forms of the jet-bundle; variational calculus is not invariant under an arbitrary change of variables.

Some remarks on Local vs Gauge symmetries. Let us test explicitly the Gauge symmetry of $\mathcal{L}_{tot}$ under the group $G(M)$ and compute the corresponding Noether invariants, as an exercise.

Generator of $G(M)$ (no sum on (a)): $Y = f^{(a)} X^\alpha_{(a)\beta}\varphi^\beta \frac{\partial}{\partial \varphi^\alpha} + (f^{(b)} C^a_{bc} A^{(c)}_\nu - \frac{\partial f^{(a)}}{\partial x^\nu})\frac{\partial}{\partial A^{(a)}_\nu}$

Noether Current: $J^\mu = Y^\alpha \frac{\partial \mathcal{L}]_{tot}}{\partial \varphi^\alpha_\mu} + Y^{A^{(a)}_\nu} \frac{\partial \mathcal{L}_{tot}}{\partial A^{(a)}_{\nu,\mu}}$

$$= f^{(a)} X^\alpha_{(a)\beta}\varphi^\beta \frac{\partial \mathcal{L}_{matt}}{\partial \varphi^\alpha_\mu} + \left(f^{(b)} C^a_{bc} A^{(c)}_\nu - \frac{\partial f^{(a)}}{\partial x^\nu}\right) \frac{\partial \mathcal{L}_0}{\partial A^{(a)}_{\nu,\mu}}$$

$$= f^{(a)} j^\mu_{(a)matt} + (f^{(b)} C^a_{bc} A^{(c)}_\nu - \frac{\partial f^{(a)}}{\partial x^\nu}) F^{\mu\nu}_{(a)} = \quad \left(\text{if } \mathcal{L}_0 = -\frac{1}{4} F^{(b)}_{\mu\nu} F^{\mu\nu}_{(b)}\right)$$

$$= f^{(a)} j^{\mu\, rigid}_{(a)tot} - \partial_\nu f^{(a)} F^{\mu\nu}_{(a)}$$

Conservation: $\partial_\mu J^\mu|_{sol.}$

$$= \partial_\mu f^{(a)} j^{\mu\, rigid}_{(a)tot} + f^{(a)} \partial_\mu j^{\mu\, rigid}_{(a)tot} - \partial_{\mu\nu} f^{(a)} F^{\mu\nu}_{(a)} - \partial_\nu f^{(a)} \partial_\mu F^{\mu\nu}_{(a)}$$

$$= \partial_\mu f^{(a)} j^{\mu\, rigid}_{(a)tot} + 0 - 0 - \partial_\nu f^{(a)} j^{\nu\, rigid}_{(a)tot} = 0$$

Noether Charge: $Q^{(a)}$

$$= \int_{\Sigma} \mathrm{d}\sigma^{\mu}(f^{(a)} j^{\mu\, rigid}_{(a)tot} - \partial_{\nu} f^{(a)} F^{\mu\nu}_{(a)}) = \int_{R^3} \mathrm{d}^3 x (f^{(a)} j^{0\, rigid}_{(a)tot} - \partial_i f^{(a)} F^{0i}_{(a)})$$
$$= \int_{R^3} \mathrm{d}^3 x (f^{(a)} j^{0\, rigid}_{(a)tot} + f^{(a)} \partial_i F^{0i}_{(a)}) = \int_{R^3} \mathrm{d}^3 x (f^{(a)} j^{0\, rigid}_{(a)tot} - f^{(a)} \partial_i F^{i0}_{(a)}) = \mathbf{0}\,,$$

confirming that the symmetry above is **gauge** indeed.

However, let us also demonstrate that there are local symmetries which **are not** gauge, that is, their associated Noether charges are non-trivial. To this end, consider the massless Klein–Gordon field:

$$\mathcal{L} = \frac{1}{2}\phi_{\mu}\phi^{\mu} \quad \Rightarrow \quad \Box\phi = 0\,.$$

The generator $X \equiv \frac{\partial}{\partial\phi}$ is a symmetry. In fact, $\bar{X} = \frac{\partial}{\partial\phi}$, so that $\bar{X}\mathcal{L} = 0$. But is the local generator $X^f = f(x)\frac{\partial}{\partial\phi}$ a symmetry? We compute the corresponding jet extension and the Lie derivative of the Lagrangian:

$$\overline{X^f}\mathcal{L} = \left(f\frac{\partial}{\partial\phi} + f_{\mu}\frac{\partial}{\partial\phi_{\mu}} \right)\mathcal{L} = f_{\mu}\phi^{\mu} \overset{?}{=} \partial_{\mu}h^{\mu} \tag{222}$$

and we realize that only if f is a solution of the equations of motion, $f_{\mu}\phi^{\mu}$ is a gradient, that is, $h^{\mu} = f^{\mu}\phi$. But in this case, the Noether charge is a non-trivial quantity (see the symmetries parameterizing the solution manifold of the Klein–Gordon field).

7.1 Example of the Dirac field

Free Dirac field $\mathcal{L}^D \equiv \mathcal{L}_{matt} = i\bar{\psi}\gamma^{\mu}\psi_{\mu} - m\bar{\psi}\psi$

The Euler–Lagrange equations of motion become:

$$\frac{\partial\mathcal{L}}{\partial\psi_{\mu}} = i\bar{\psi}\gamma^{\mu} \equiv \pi^{\mu}\,;\ \frac{\partial\mathcal{L}}{\partial\psi} = -m\bar{\psi} \Rightarrow i\partial_{\mu}\bar{\psi}\gamma^{\mu} + m\bar{\psi} = 0 \tag{223}$$
$$\frac{\partial\mathcal{L}}{\partial\bar{\psi}_{\mu}} = 0\,;\ \frac{\partial\mathcal{L}}{\partial\bar{\psi}} = i\gamma^{\mu}\psi_{\mu} - m\psi \Rightarrow i\gamma^{\mu}\psi_{\mu} - m\psi = 0\,. \tag{224}$$

It is assumed (as corresponding to the Ordinary Hamilton Principle) that $\psi_{\mu} = \partial_{\mu}\psi$ but not derived from the equations of motion.

The Poincaré–Cartan form is derived in the standard manner:

$$\begin{aligned}\Theta_{PC} &= \frac{\partial\mathcal{L}}{\partial\psi_{\mu}}(\mathrm{d}\psi - \psi_{\nu}\mathrm{d}x^{\nu})\wedge\theta_{\mu} + \frac{\partial\mathcal{L}}{\partial\bar{\psi}_{\mu}}(d\bar{\psi} - \bar{\psi}_{\nu}\mathrm{d}x^{\nu})\wedge\theta_{\mu} + \mathcal{L}\omega \\ &= i\bar{\psi}\gamma^{\mu}(\mathrm{d}\psi - \psi_{\nu}\mathrm{d}x^{\nu})\wedge\theta_{\mu} + (i\bar{\psi}\gamma^{\mu}\psi_{\mu} - m\bar{\psi}\psi\omega \\ &= i\bar{\psi}\gamma^{\mu}\mathrm{d}\psi\wedge\theta_{\mu} - i\bar{\psi}\gamma^{\mu}\psi_{\mu}\omega + i\bar{\psi}\gamma^{\mu}\psi_{\mu}\omega - m\bar{\psi}\psi\omega \\ &= i\bar{\psi}\gamma^{\mu}\mathrm{d}\psi\wedge\theta_{\mu} - m\bar{\psi}\psi\omega \equiv i\bar{\psi}\gamma^{\mu}\mathrm{d}\psi\wedge\theta_{\mu} - \mathcal{H}\omega\,.\end{aligned} \tag{225}$$

Remark $\mathcal{H}$ is not the ordinary Hamiltonian driving the time evolution. Evolution is driven by the Noether invariant associated with the invariance under time translation $\mathcal{P}_{(0)} \equiv j^0_{(0)} = i\bar{\psi}\boldsymbol{\gamma}\cdot\nabla\psi + m\bar{\psi}\psi$. Even more, if we rewrite the Poincaré–Cartan form in the way

$$\Theta_{PC} \equiv \mathcal{T}^{\mu}_{PC}\wedge\theta_{\mu} = \left\{ \frac{\partial\mathcal{L}}{\partial\psi_{\mu}}(\mathrm{d}\psi - \psi_{\nu}\mathrm{d}x^{\nu}) + \frac{\partial\mathcal{L}}{\partial\bar{\psi}_{\mu}}(d\bar{\psi} - \bar{\psi}_{\nu}\mathrm{d}x^{\nu}) + \mathcal{L}\mathrm{d}x^{\mu} \right\}\wedge\theta_{\mu}\,, \tag{226}$$

as if $\mathcal{T}^{\mu}_{PC}$ where a "conserved current," the "conserved charge," $\int_{\Sigma} \mathrm{d}\sigma_{\mu}\mathcal{T}^{\mu}_{PC}$ plays the role of a Quantum-Mechanics Poincaré–Cartan form:

$$
\begin{aligned}
\Theta_{PC} &= p\mathrm{d}q - H\mathrm{d}t \\
\int_{\Sigma} \mathrm{d}\sigma_{\mu}\mathcal{T}^{\mu}_{PC} &= \int_{\Sigma} \mathrm{d}^3x \left[\frac{\partial \mathcal{L}}{\partial \psi_0}(\mathrm{d}\psi - \psi_0 \mathrm{d}x^0) - \mathcal{L}\mathrm{d}x^0\right] \\
&= \int_{\Sigma} \mathrm{d}^3x[i\bar{\psi}\gamma^0 d\psi - i\bar{\psi}\gamma^0\psi_0\mathrm{d}x^0 + (i\bar{\psi}\gamma^{\mu}\psi_{\mu} - m\bar{\psi}\psi)\mathrm{d}x^0] \\
&= \int_{\Sigma} \mathrm{d}^3x[i\bar{\psi}\gamma^0 d\psi - (i\bar{\psi}\boldsymbol{\gamma}\cdot\nabla\psi + m\bar{\psi}\psi)\mathrm{d}x^0],
\end{aligned} \tag{227}
$$

with $H = i\psi^{\dagger}\boldsymbol{\alpha}\cdot\nabla\psi + m\psi^{\dagger}\beta\psi$.

Coupled Dirac field: $\hat{\mathcal{L}}^D \equiv \hat{\mathcal{L}}_{matt} = i\bar{\psi}\gamma^{\mu}(\psi_{\mu} - iA_{\mu}\psi) - m\bar{\psi}\psi \equiv i\bar{\psi}\gamma^{\mu}\psi_{\mu}D_{\mu}\psi - m\bar{\psi}\psi$

$$
\frac{\partial\hat{\mathcal{L}}}{\partial\bar{\psi}} = i\gamma^{\mu}D_{\mu}\psi - m\psi \Rightarrow\ i\gamma^{\mu}\partial_{\mu}\psi - m\psi = -\gamma^{\mu}\psi A_{\mu} \tag{228}
$$

$$
\frac{\partial\hat{\mathcal{L}}}{\partial\psi} = \bar{\psi}\gamma^{\mu}A_{\mu} - m\bar{\psi}\ ;\ \frac{\partial\hat{\mathcal{L}}}{\partial\psi_{\mu}} = i\bar{\psi}\gamma^{\mu}\ \Rightarrow\ i\partial_{\mu}\bar{\psi}\gamma^{\mu} + m\bar{\psi} = \bar{\psi}\gamma^{\mu}A_{\mu}. \tag{229}
$$

The Poincaré–Cartan form associated with the coupled Lagrangian becomes:

$$
\begin{aligned}
\hat{\Theta}_{PC} = &\ \frac{\partial\hat{\mathcal{L}}}{\partial\bar{\psi}_{\mu}}(d\bar{\psi} - \bar{\psi}_{\nu}\mathrm{d}x^{\nu})\wedge\theta_{\mu} + \frac{\partial\hat{\mathcal{L}}}{\partial\psi_{\mu}}(\mathrm{d}\psi - \psi_{\nu}\mathrm{d}x^{\nu})\wedge\theta_{\mu} + \hat{\mathcal{L}}\omega = i\bar{\psi}\gamma^{\mu}(\mathrm{d}\psi - \psi_{\nu}\mathrm{d}x^{\nu})\wedge\theta_{\mu} \\
&+(i\bar{\psi}\gamma^{\mu}(\psi_{\mu} - iA_{\mu}\psi) - m\bar{\psi}\psi)\omega = i\bar{\psi}\gamma^{\mu}\mathrm{d}\psi\wedge\theta_{\mu} - i\bar{\psi}\gamma^{\mu}\psi_{\mu}\omega + i\bar{\psi}\gamma^{\mu}\psi_{\mu}\omega + i\bar{\psi}\gamma^{\mu}A_{\mu}\psi\omega \\
&-m\bar{\psi}\psi\omega = i\bar{\psi}\gamma^{\mu}\mathrm{d}\psi\wedge\theta_{\mu} - \bar{\psi}(m - \gamma^{\mu}A_{\mu})\psi\omega \equiv i\bar{\psi}\gamma^{\mu}\mathrm{d}\psi\wedge\theta_{\mu} - \hat{\mathcal{H}}_D\omega \\
\equiv &\ \Theta_{PC} + j^{\mu}_{D}A_{\mu}\omega
\end{aligned} \tag{230}
$$

with $\hat{\mathcal{H}}_d = m\bar{\psi}\psi - j^{\mu}_{D}A_{\mu}$, $j^{\mu}_{D} \equiv \bar{\psi}\gamma^{\mu}\psi$.

Integrating again $\hat{\Theta}_{PC}$ over the Cauchy surface, we have:

$$
\begin{aligned}
\int_{\Sigma} \mathrm{d}\sigma_{\mu}\hat{\mathcal{T}}^{\mu}_{PC} &= \int_{\Sigma} \mathrm{d}^3x[\frac{\partial\mathcal{L}}{\partial\psi_0}(\mathrm{d}\psi - \psi_0\mathrm{d}x^0) + \hat{\mathcal{L}}\mathrm{d}x^0] \\
&= \int_{\Sigma} \mathrm{d}^3x[i\bar{\psi}\gamma^0 d\psi - i\bar{\psi}\gamma^0\psi_0\mathrm{d}x^0 + \{i\bar{\psi}\gamma^{\mu}(\psi_{\mu} - iA_{\mu}\psi) - m\bar{\psi}\psi\}\mathrm{d}x^0] \\
&= \int_{\Sigma} \mathrm{d}^3x[i\bar{\psi}\gamma^0 d\psi - (i\bar{\psi}\boldsymbol{\gamma}\cdot\nabla\psi + m\bar{\psi}\psi - j^{\mu}_{D}A_{\mu})\mathrm{d}x^0]
\end{aligned} \tag{231}
$$

with $\hat{H}_D = i\bar{\psi}\boldsymbol{\gamma}\cdot\nabla\psi + m\bar{\psi}\psi - j^{\mu}_{D}A_{\mu}$.

7.2 Brief report on the Group Quantization of Electrodynamics

One way of proceeding in facing the quantization of a system whose full symmetry (to be precise, the basic symmetry evolved in time) is unknown consists in quantizing the basic symmetry that characterizes the Solution Manifold and then realizes the right-enveloping algebra, which preserves the representation space (Hilbert space) of the basic algebra of quantum operators. In other words, the exponential of the **complete** Hamiltonian will act perturbatively on the wave functions defined on the Solution Manifold (The complete Hamiltonian is a constant of motion in any isolated system and, thus, it is well defined on the SM).

This procedure, proposed here, is related to the approach followed in "Landau's series" text books when dealing with formal perturbation theory in that which concerns with exact propagators and exact vertices in the Heisenberg picture (see [44]).

Another way would be that of closing perturbatively the classical Poisson algebra, exponentiating the approximate algebra at each order and applying GAQ at the corresponding order. This more precise method will not be considered here.

Quantum Basic Symmetry: General case (Space-time symmetry excluded; internal indices of the matter fields are not explicit)

Since we aim at representing just the basic symmetry on SM and then realize the quantum evolution perturbatively, we ignore the semi-direct action of the Poincaré group and think of the arguments of the fields, **x**, on the Cauchy surface, only as (infinitely many) indices. In the same way, spatial derivatives do act as infinitesimal translations on those indices, whereas time derivative of the fields correspond to different field coordinates with initial values on SM. Roughly spiking, $\partial_i\phi$ is not independent of ϕ, although $\partial_0\phi$ indeed is. Nevertheless, we intend to take the Lorentz covariance as far as possible in the proposed group law:

$$
\begin{array}{l|l}
U'' = U'U & U = e^{-i\varphi^a T_{(a)}} \; ; \; A_\mu \equiv T_{(a)} A_\mu^{(a)} \\
A''_\mu = U' A_\mu U'^{-1} + A'_\mu & A_\mu^{(a)} \stackrel{U'}{\to} A_\mu'^{(a)} = A_\mu^{(a)} + C^a_{bc} A_\mu^{(b)} \varphi'^c + \frac{1}{g}\partial_\mu \varphi'^b + ... \\
F''_{\mu\nu} = U' F_{\mu\nu} U'^{-1} + F'_{\mu\nu} & D_\mu\phi = \partial_\mu\phi + igA_\mu^{(a)} T_{(a)}\phi \\
\phi'' = U'\phi + \phi' & \phi \stackrel{U'}{\to} \phi' = e^{i\varphi'^a T_{(a)}}\phi \\
\phi''^* = U'^{-1}\phi^* + \phi'^* & A_\mu = iU_\mu U^{-1} \equiv A_\mu^R \\
\phi''_\mu = \phi'_\mu + U'\phi_\mu - gA_\mu\phi' & U_\mu = iA_\mu U \; ; \; U_\mu^{-1} = -iU^{-1}A_\mu \\
\phi_\mu^{''*} = \phi_\mu^{'*} + U'^{-1}\phi^*_\mu - gA_\mu\phi^{'*} & \phi_\mu^{-1} = -U^{-1}(\phi_\mu - gA_\mu T\phi) \\
\zeta'' = \zeta'\zeta e^{i\xi_0(g',g)} e^{i\hat{\xi}_{matt}(g',g)} & \xi_0 = \frac{1}{4}\int_\Sigma d\sigma_\nu (A'_\mu F^{[\mu\nu]} - F'^{[\mu\nu]} A_\mu)
\end{array}
\tag{232}
$$

$$
\hat{\xi}_{matt} \equiv \frac{1}{2}\int_\Sigma d\sigma^\mu \{\phi'(U'^{-1}\phi^*_\mu + gA_\mu\phi^*) - \phi U'\phi'^*_\mu + \phi'^*(U'\phi_\mu + gA_\mu\phi) - \phi^* U'^{-1}\phi'_\mu\} \,. \tag{233}
$$

It must be stressed that the co-cycle $\hat{\xi}_{matt}$ can be written as if it where the sum of the co-cycle for the free matter ξ_{matt} plus an interaction term proportional to the coupling constant, that is:

$$
\hat{\xi}_{matt} = \xi_{matt} + \frac{1}{2} g A_\mu (\phi^*\phi' + \phi'^*\phi) \,, \tag{234}
$$

but the "interaction" term, itself, is not a co-cycle. The reason for this fact is that the unextended group, for which $\hat{\xi}_{matt}$ is a co-cycle, is a deformation of the direct product of the unextended groups corresponding to the free matter and free gauge fields. We leave as an exercise the verification of the co-cycle condition (55) for $\hat{\xi}_{matt}$ and ξ_0, that is,

$$
\xi(g', g) + \xi(g' * g, g'') - \xi(g', g * g'') - \xi(g, g'') = 0.
$$

7.2.1 Scalar Electrodynamics

For Scalar Electrodynamics, we have:

$$\left.\begin{array}{l}
\varphi'' = \varphi' + \varphi \\
A''_\mu = A'\mu + A_\mu \\
F''_{\mu\nu} = F'_{\mu\nu} + F_{\mu\nu} \\
\phi'' = e^{-i\varphi'}\phi + \phi \\
\phi''^* = e^{i\varphi'}\phi^* + \phi'^* \\
\phi''^*_\mu = \phi'^*_\mu + e^{i\varphi'}\phi^*_\mu - eA_\mu\phi'^* \\
\zeta'' = \zeta'\zeta e^{i\xi_0(g',g)}e^{i\hat{\xi}_{matt}(g',g)}
\end{array}\right|
\begin{array}{l}
\xi_0 = \frac{1}{4}\int_\Sigma \mathrm{d}\sigma_\nu (A'_\mu F'^{[\mu\nu]} - F'^{[\mu\nu]}A_\mu) \\
\hat{\xi}_{matt} = \frac{1}{2}\int_\Sigma \mathrm{d}\sigma^\mu\{\phi'(e^{i\varphi'}\phi^*_\mu + eA_\mu\phi^*) - \phi e^{-i\varphi'}\phi'^*_\mu \\
\qquad\qquad + \phi'^*(e^{-i\varphi'}\phi_\mu + eA_\mu\phi) - \phi^* e^{i\varphi'}\phi'_\mu\}
\end{array} \tag{235}$$

Left generators and Θ form:

$$\left.\begin{array}{l}
\tilde{X}^L_\varphi = \frac{\partial}{\partial\varphi} \\
\tilde{X}^L_{A_\mu} = \frac{\partial}{\partial A_\mu} - e\left(\phi\frac{\partial}{\partial\phi_\mu} + \phi^*\frac{\partial}{\partial\phi^*_\mu}\right) - \frac{1}{4}\hat{n}_\sigma F^{[\sigma\mu]}\Xi \\
\tilde{X}^L_{F_{\mu\nu}} = \frac{\partial}{\partial F_{\mu\nu}} + \frac{1}{4}(\hat{n}^\mu A^\nu - \hat{n}^\nu A^\mu)\Xi \\
\tilde{X}_\phi = e^{-i\varphi}\frac{\partial}{\partial\phi} - \frac{1}{2}e^{-i\varphi}\phi^*_\mu\hat{n}^\mu\Xi \\
\tilde{X}_{\phi^*} = e^{i\varphi}\frac{\partial}{\phi^*} - \frac{1}{2}e^{i\varphi}\phi_\mu\hat{n}^\mu\Xi \\
\tilde{X}_{\phi_\mu} = e^{-i\varphi}\frac{\partial}{\partial\phi_\mu} + \frac{1}{2}e^{-i\varphi}\phi^*\hat{n}^\mu\Xi \\
\tilde{X}_{\phi^*_\mu} = e^{i\varphi}\frac{\partial}{\partial\phi^*_\mu} + \frac{1}{2}e^{i\varphi}\phi\hat{n}^\mu\Xi
\end{array}\right|
\begin{array}{l}
[\tilde{X}^L_{A_\mu},\ \tilde{X}_\phi] = e\delta^\mu_\nu\tilde{X}^L_{\phi_\nu},\ [\tilde{X}^L_{A_\mu},\ \tilde{X}^L_{\phi^*}] = e\delta^\mu_\nu\tilde{X}^L_{\phi^*_\nu} \\
\\
\Theta = \frac{1}{2}\int_\Sigma \mathrm{d}\sigma^\mu\{(\phi^*_\mu + eA_\mu\phi^*)\delta\phi \\
\quad + (\phi_\mu + eA_\mu\phi)\delta\phi^* \\
\quad - \phi^*\delta(\phi_\mu + eA_\mu\phi) - \phi\delta(\phi^*_\mu + eA_\mu\phi^*)\} \\
\quad - \frac{1}{4}\int_\Sigma \mathrm{d}\sigma_\rho\{A_\mu\delta F^{[\rho\nu]} - F^{[\rho\mu]}\delta A_\mu\} + \frac{\mathrm{d}\zeta}{i\zeta}
\end{array} \tag{236}$$

Right generators:

$$\begin{aligned}
\tilde{X}^R_\varphi &= \frac{\partial}{\partial\varphi} - i\phi\frac{\partial}{\partial\phi} + i\phi^*\frac{\partial}{\partial\phi^*} - i\phi_\mu\frac{\partial}{\partial\phi_\mu} + i\phi^*_\mu\frac{\partial}{\partial\phi^*_\mu} \\
\tilde{X}^R_{A_\mu} &= \frac{\partial}{\partial A_\mu} + \frac{1}{4}\hat{n}_\nu F^{[\nu\mu]}\Xi \\
\tilde{X}^R_{F_{\mu\nu}} &= \frac{\partial}{\partial F_{\mu\nu}} - \frac{1}{4}(\hat{n}^\mu A^\nu - \hat{n}^\nu A^\mu)\Xi \\
\tilde{X}^R_\phi &= \frac{\partial}{\partial\phi} - eA_\mu\frac{\partial}{\partial\phi_\mu} + \frac{1}{2}(\phi^*_\mu + eA_\mu\phi^*)\hat{n}^\mu\Xi \\
\tilde{X}^R_{\phi^*} &= \frac{\partial}{\partial\phi^*} - eA_\mu\frac{\partial}{\partial\phi^*_\mu} + \frac{1}{2}(\phi_\mu + eA_\mu\phi)\hat{n}^\mu\Xi \\
\tilde{X}_{\phi_\mu} &= \frac{\partial}{\partial\phi_\mu} - \frac{1}{2}\phi^*\hat{n}^\mu\Xi \\
\tilde{X}_{\phi^*_\mu} &= \frac{\partial}{\partial\phi^*_\mu} - \frac{1}{2}\phi\hat{n}^\mu\Xi
\end{aligned} \tag{237}$$

with structure constants which are the opposite to the left ones.

Noether invariants:

$$\begin{aligned}
i_{\tilde{X}^R_\varphi}\Theta = \frac{1}{2}\big(&-i\phi(\phi^*_\mu + eA_\mu\phi^*) + i\phi^*(\phi_\mu + eA_\mu\phi) \\
&+ i\phi^*\phi_\mu - i\phi^*\mu\phi + ie\phi^*\phi A_\mu - ie\phi^*\phi A_\mu\big)\hat{n}^\mu
\end{aligned}$$

$$
\begin{aligned}
&= \frac{i}{2}(\phi^*\phi_\mu - \phi^*_\mu\phi)\hat{n}^\mu \\
i_{\tilde{X}^R_{A_\mu}}\Theta &= \frac{1}{4}F^{[\nu\mu]}\hat{n}_\nu + \frac{1}{4}F^{[\nu\mu]}\hat{n}_\nu - e\phi^*\phi\hat{n}^\mu = \frac{1}{2}F^{[\nu\mu]}\hat{n}_\nu - e\phi^*\phi\hat{n}^\mu \\
i_{\tilde{X}^R_{F_{\mu\nu}}}\Theta &= -\frac{1}{2}(A_\mu\hat{n}_\nu - A_\nu\hat{n}_\mu) - \frac{1}{2}(A_\mu\hat{n}_\nu - A_\nu\hat{n}_\mu) = A_\nu\hat{n}_\mu - A_\mu\hat{n}_\nu \\
i_{\tilde{X}^R_\phi}\Theta &= \left(\frac{1}{2}(\phi^*_\mu + eA_\mu\phi^*) + \frac{e}{2}A_\mu\phi^*\right)\hat{n}^\mu + \frac{1}{2}(\phi^*_\mu + eA_\mu\phi^*)\hat{n}^\mu - \frac{1}{2}e\phi^*A_\mu\hat{n}^\mu \\
&= (\phi^*_\mu + eA_\mu\phi^*)\hat{n}^\mu \\
i_{\tilde{X}^R_{\phi^*}}\Theta &= \ldots = (\phi_\mu + eA_\mu\phi)\hat{n}^\mu \\
i_{\tilde{X}^R_{\phi_\mu}}\Theta &= (-\frac{1}{2}\phi^* - \frac{1}{2}\phi^*)\hat{n}^\mu = -\phi^*\hat{n}^\mu \\
i_{\tilde{X}^R_{\phi^*_\mu}}\Theta &= -\phi\hat{n}^\mu .
\end{aligned}
\tag{238}
$$

Note that the commutators $[\tilde{X}^R_{A_\mu},\ \tilde{X}^R_\phi] = -e\tilde{X}^R_{\phi_\mu}$ will only imply the quantum commutators:

$$
[\hat{A}_0,\ \hat{\dot{\phi}}] = -e\hat{\phi}\ ,\quad [\hat{A}_0,\ \hat{\dot{\phi}}^*] = -e\hat{\phi}^*\ , \tag{239}
$$

where the association of right generators with quantum operators is:

$$
\begin{aligned}
&\hat{A}_0 \sim \tilde{X}^R_{A_0},\quad \hat{A}_i \sim \tilde{X}^R_{F_{0i}},\quad \hat{F}_{0i} \sim \tilde{X}^R_{A_i},\quad \hat{\phi} \sim \tilde{X}^R_{\dot{\phi}^*}, \\
&\hat{\phi}^* \sim \tilde{X}^R_{\dot{\phi}},\quad \hat{\dot{\phi}} \sim \tilde{X}^R_{\phi^*},\quad \hat{\dot{\phi}}^* \sim \tilde{X}^R_{\phi},\quad \hat{\varphi} \sim \tilde{X}^R_{\varphi}\, .
\end{aligned}
\tag{240}
$$

7.2.2 Time Evolution from the Solution Manifold

The methodology to be here sketched is quite general and can be applied to any physical system whose basic operators do not close algebra in "finite" dimension with the Hamiltonian.

In that which follows we shall consider the time evolution of either a classical function $f(q, p)$ on the classical Solution Manifold or a function $f(\hat{q}, \hat{p})$ of quantum operators $\hat{q}, \hat{p}$ represented on (polarized) wave functions Ψ of classical variables (q or p, or some combination). In the same way, a bracket [,] will mean Poisson bracket as regarding classical evolution, or quantum commutators in the case of the quantum evolution.
Schematically: (Time evolution by **Magnus Series** [45])
With a given function on the solution manifold we associate the following "evolutive" version:

$$
f(q, p) \text{ on } SM \quad \overset{\text{evolutive version}}{\longrightarrow} \quad F(t, x, p) = U(t)F_0 \equiv e^{\Omega(t)}F_0 \tag{241}
$$

where

$$
\begin{aligned}
F_0 &\equiv F(0, q, p) \equiv f(q, p) \\
\Omega(t) &= \lim_{n\to\infty}\Omega^{[n]}(t) \\
\Omega^{[n]}(t) &= \Sigma^\infty_{k=0}\frac{B_k}{k!}\int_0^t \mathrm{d}t_1 \mathrm{ad}^k_{\Omega^{[n-1]}(t_1)}(-H(t_1))
\end{aligned}
\tag{242}
$$

where B_k are Bernoulli numbers and the "powers" of ad_f means

$$
\mathrm{ad}^0_f(g) \equiv g,\quad \mathrm{ad}^1_f(g) \equiv [f, g],\quad \mathrm{ad}^k_f(g) \equiv [\mathrm{ad}^{k-1}_f(g), g]\, . \tag{243}
$$

Magnus series (versus Dyson-like series) offers “unitarity” even at finite orders (at the classical level we would say “symplecticity”).

For t-independent Hamiltonians, as corresponds to objects on SM, we arrive at a rather simpler formula:

$$F(t) = \Sigma_{k=0}^{\infty} \frac{1}{k!}(-t)^k a\mathrm{d}_H^k(F_0)\,, \tag{244}$$

which constitutes the Inverse Hamilton–Jacobi transformation by H.

In particular, we can compute the “arbitrary-time” commutator of two (field) operators

$$[\hat{A}_\mu^{(a)}(\mathbf{x},t),\ \hat{A}_\nu^{(b)}(\mathbf{x}',t')] = [U(t)\hat{A}_\mu^{(a)}(\mathbf{x})U^\dagger(t),\ U(t')\hat{A}_\nu^{(b)}(\mathbf{x}')U^\dagger(t')]\,, \tag{245}$$

or the exact propagator of the field $\hat{A}$ as

$$\mathcal{D}(x,x') = \langle 0|T\hat{A}(x)\hat{A}(x')|0\rangle, \tag{246}$$

where T stands for “time-order” in the traditional way, to be further developed in terms of the free propagator $D(x,x')$.

8 Massive Gauge Theory

Weak Interactions were originally described by a “current-current” term in the Lagrangian to account for the property of being very local. To turn them into a gauge theory would require a very massive intermediate particle, a fact which makes quite difficult the corresponding renormalizability beyond the Abelian case [46]. To avoid this difficulty, a mechanism [47,48], imported from solid-state physics, was introduced in Particle theory [49]. For a review, we recommend Ref. [50].

8.1 Giving dynamical content to the gauge parameters: Massive Gauge Theory and the Generalized Non-Abelian Stueckelberg formalism

The group $G^1(M)$ of the 1-jets of $G(M)$: We start from the local or gauge group $G(M)$ as the group of mappings from the space-time M to the rigid symmetry of a supposed matter Lagrangian $\mathcal{L}_{matt}$.

As in the case of the formulation of the variational calculus, where we construct the bundle of 1-jets of the sections of E, $\Gamma(E)$, we proceed in much the same way with $G(M)$. We think of $G(M)$ as if it was a space of (some sort of matter) scalar fields on M, though valued on a non-flat internal space. Then, we construct

$$J^1(G(M)) \equiv \frac{G(M)\times M}{\overset{1}{\sim}}\,,$$

where $\overset{1}{\sim}$ is the equivalence relation (quite analogous to (89))

$$(g,m) \quad \overset{1}{\sim} \quad (g',m') \quad \Longleftrightarrow \quad \left|\begin{array}{c} m = m' \\ g(m) = g'(m) \\ \partial_\mu g(m) = \partial_\mu g'(m)\,. \end{array}\right. \tag{247}$$

$J^1(G(M))$ has dimension $\dim M + \dim G + \dim M \times \dim G$ and can be locally parameterized by $\{x^\mu, g^a, g^b_\nu\}$.

We now consider the group of sections of the bundle $J^1(G(M)) \to M$, $G^1(M)$, parameterized by $\{g^a(x),\ g^b_\mu(x)\}$.

Group Law:

$$
\begin{aligned}
g''(x) &= g'(x)g(x) \\
g''_\mu(x) &= g'_\mu(x)g(x) + g'(x)g_\mu(x)\,,
\end{aligned}
\tag{248}
$$

where $g_\mu(x)$ above is not, necessarily, $\partial_\mu g(x)$.

Equivalently, we may define new coordinates:

$$
g \ \mapsto\ g\,,\ \ g_\mu \ \mapsto\ g^{-1}g_\mu \equiv A_\mu\,. \tag{249}
$$

Notice that, now, A_μ is not, necessarily, $g^{-1}\partial_\mu g \equiv \theta^L_\mu$ $(\theta^L(x) = \theta^L_\mu(x)\mathrm{d}x^\mu)$. Explicitly, the left-invariant canonical 1-form on the group $G^1(M)$ is written as

$$
\theta^{L(a)}(x) = \theta^{L(a)}_b(x)dg^b(x) = \theta^{L(a)}_b(x)\partial_\mu g^b(x)\mathrm{d}x^\mu \equiv \theta^{L(a)}_\mu(x)\mathrm{d}x^\mu\,, \tag{250}
$$

with $\theta^{L(a)}_\mu = \theta^{L(a)}_b\partial_\mu g^b$. In terms of the coordinates $\{g^a,\ A^{(a)}_\mu(x)\}$, the group law reads:

$$
\begin{aligned}
g''(x) &= g'(x)g(x) \\
A''_\mu(x) &= g'(x)A_\mu(x)g'^{-1}(x) + A'_\mu(x)\,.
\end{aligned}
\tag{251}
$$

Note also that the group $G(M)$ is naturally contained in $G^1(M)$ by means of the jet extension: $j^1(G(M)) \in G^1(M)$. In fact, if the element $A'_\mu(x)$ in the group law corresponds to a jet extension

$$
A''_\mu(x) = g'(x)A_\mu(x)g'^{-1}(x) + g'^{-1}(x)\partial_\mu g'(x)\,. \tag{252}
$$

Then, if we think of A_μ in the group law as an ordinary Yang–Mills physical field, of g' as an ordinary gauge transformation, to be call g, and of A''_μ as the transformed of A_μ, A'_μ, we can read:

$$
A'_\mu(x) = g(x)A_\mu(x)g^{-1}(x) + g^{-1}(x)\partial_\mu g(x)\,, \tag{253}
$$

just as corresponds to the transformation law of a physical Yang–Mills field.

Ordinary connections can be derived from $G^1(M)$ by simply taking the quotient by $G(M)$ (that is to say, by $j^1(G(M)) \in G^1(M)$).

However, we should not take the mentioned quotient but, rather, A_μ and θ_μ will live together and they will combine in the proper way in due time.

Massive Gauge Theory

We may repeat Utiyama's theory on the grounds of some exotic matter $g^a(x)$. The action of G on the scalar fields $g^a(x)$ proceeds as the own right action with generators $X^L_{(b)}$. This way, the generators of $G(M)$ on $(g^a,\ A^{(b)}_\mu)$ take the expression

$$
f^{(a)}\mathcal{X}_{(a)} = f^{(a)}X^L_{(a)} + (f^{(b)}C^a_{bc}A^{(c)}_\mu - \frac{\partial f^{(a)}}{\partial x^\mu})\frac{\partial}{\partial A^{(a)}_\mu}\,, \tag{254}
$$

and the minimal coupling is realized as

$$
g^a_\mu \ \mapsto\ g^a_\mu + A^{(b)}_\mu X^{La}_{(b)}\,, \tag{255}
$$

to be compared with the standard expression for ordinary fields

$$
\varphi^\alpha_\mu \ \mapsto\ \varphi^\alpha_\mu + A^{(b)}_\mu X^\alpha_{(b)} \ \ (= \varphi^\alpha_\mu + A^{(b)}_\mu X^\alpha_{(b)\beta}\varphi^\beta)\,. \tag{256}
$$

It should also be compared the expressions of the group generators acting on g and φ:

$$X^L_{(a)} = X^{Lb}_{(a)}(g)\frac{\partial}{\partial g^b} \quad \text{vs} \quad X_{(a)} = X^\alpha_{(a)}(\varphi)\frac{\partial}{\partial \varphi^\alpha} \overset{\text{linearity}}{=} X^\alpha_{(a)\beta}\varphi^\beta\frac{\partial}{\partial \varphi^\alpha},$$

the main difference being that now $X^{Lb}_{(a)}(g)$ is an invertible function (though nonlinear, in general) of g. In fact, the inverse matrix is $[X^{Lb}_{(a)}]^{-1} = \theta^{L(a)}_b$!!. This means that the minimal coupling stated in terms of $\theta^{L(a)}_\mu$, instead of g^a_μ, becomes:

$$\theta^{L(a)}_\mu \quad \mapsto \quad \theta^{L(a)}_\mu + A^{(a)}_\mu, \tag{257}$$

which is an affine coupling (it is not linear in g as $\varphi_\mu + A_\mu\varphi$ was in φ).

So then, giving dynamics to the "exotic matter" g^a through a kinetic term in the Lagrangian, $\mathcal{L}_{\text{“}matt\text{”}}$, of the form

$$\mathcal{L}_{\text{“}matt\text{”}} = \frac{1}{2}\mu^2\theta^{L(a)}_\mu\theta^{L(b)}_\nu\eta^{\mu\nu}k_{ab} \equiv \frac{1}{2}\mu^2\theta^{L(a)}_\mu\theta^{L\mu}_{(a)} \equiv \frac{1}{2}\mu^2 T^G_{race}[\theta^{L\mu}\theta^L_\mu], \tag{258}$$

the Minimal Coupling Principle provides mass to the fields A_μ **without damaging gauge invariance !!**. (In the expression above, $\eta_{\mu\nu}$ stand for the metric in the space-time manifold M and k_{ab} for the Killing metric in G).

In fact, $\mathcal{L}_{\text{“}matt\text{”}}$ becomes $\hat{\mathcal{L}}_{\text{“}matt\text{”}}$:

$$\hat{\mathcal{L}}_{\text{“}matt\text{”}} = \frac{1}{2}\mu^2(\theta^{L(a)}_\mu - A^{(a)}_\mu)(\theta^{L\mu}_{(a)} - A^\mu_{(a)}), \tag{259}$$

which contains the mass term $\frac{1}{2}\mu^2 A^{(a)}_\mu A^\mu_{(a)}$. It is a Minimal coupling with affine character.

This Lagrangian $\mathcal{L}_{\text{“}matt\text{”}}$ addresses part of the Non-Abelian Stueckelberg Lagrangian in massive gauge theory:

$$\mathcal{L}_{MYM} = \hat{\mathcal{L}}_{\text{“}matt\text{”}} + \mathcal{L}_{YM} = \frac{1}{2}\mu^2(\theta^{(a)}_\mu - A^{(a)}_\mu)(\theta^\mu_{(a)} - A^\mu_{(a)}) - \frac{1}{4}F^{(a)}_{\mu\nu}F^{\mu\nu}_{(a)}. \tag{260}$$

After the change of variables $\tilde{A}_\mu = U^\dagger(A_\mu - \theta_\mu)U$, that is, the "unitary gauge", this Lagrangian is written

$$\mathcal{L}_{MYM} = -\frac{1}{4}F^{(a)}_{\mu\nu}(\tilde{A})F^{\mu\nu}_{(a)}(\tilde{A}) + \frac{1}{2}\mu^2\tilde{A}^{(a)}_\mu\tilde{A}^\mu_{(a)}, \tag{261}$$

as corresponding to a Non-Abelian Proca Field.

8.1.1 Standard attempt to the quantization of massive gauge theory: Nonlinear Sigma Model (N-LSM)

The Lagrangian $\mathcal{L}_\sigma = \frac{1}{2}\theta^{L(a)}_\mu\theta^{L\mu}_{(a)}$ $(= \frac{1}{2}\theta^{R(a)}_\mu\theta^{R\mu}_{(a)}$, then chiral) is usually referred to as σ−Lagrangian, the origin of the name being traced back to the low-energy models for strong interactions, where a set of field $(\sigma, \boldsymbol{\pi})$, $SU(2)$-valued obeyed a Lagrangian of this kind.

The Euler–Lagrange equations for $\mathcal{L}_\sigma = \frac{1}{2}g_{ab}\eta^{\mu\nu}\partial_\mu\varphi^a\partial_\nu\varphi^b$, where $g_{ab} \equiv \theta^{(c)}_a\theta^{(d)}_b k_{cd}$, $k_{ab} \equiv$ Killing metric in G, become

$$\Box\varphi^a = \mathcal{L}_\sigma\varphi^a \quad \text{or} \quad \partial^\mu\theta^{(a)}_\mu = 0. \tag{262}$$

A similar scheme, but with external scalar fields ϕ^a behaving as our g^a, had been considered in the Literature in an attempt to make the massive gauge theory renormalizable. This scheme is called non-Abelian Stueckelberg formalism as it generalizes the Abelian case,

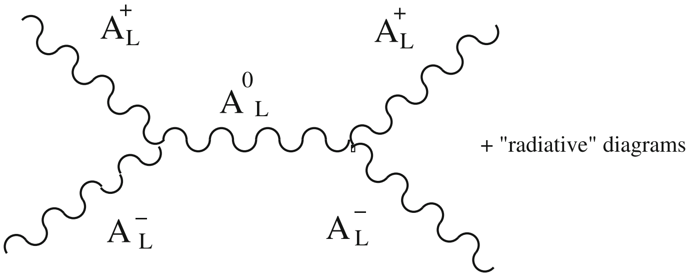


Fig. 10 Feynman diagrams involving longitudinal components of massive vector potentials

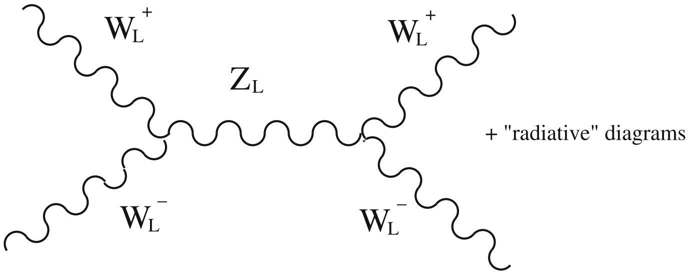


Fig. 11 The case of the Standard Model

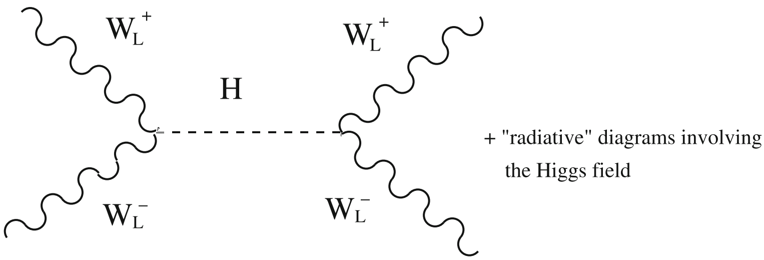


Fig. 12 The role of the Higgs field

the Massive Electrodynamics, introduced by this physicist. The main difference is that the Abelian case is renormalizable under Canonical Quantization whereas the non-Abelian one is not [51,52].

Canonical Quantization renders divergent the amplitude for processes of the form (L stands for the **longitudinal** components of A_μ)
where $(A^+_\mu, A^-_\mu, A^0_\mu)$ are the gauge fields associated with a "root" of the semi-simple group G (Fig. 10).

In the specific case of the Standard Model, it would read:
and the infinite contribution has to be substracted by means of processes involving the Higgs field (Figs. 11, 12):

8.1.2 Brief note on the Higgs–Kibble Mechanism

A conventional field has a self-interacting potential $V(\phi) = m^2|\phi|^2$ (like the spring potential $V(x) = kx^2$) (Fig. 13):

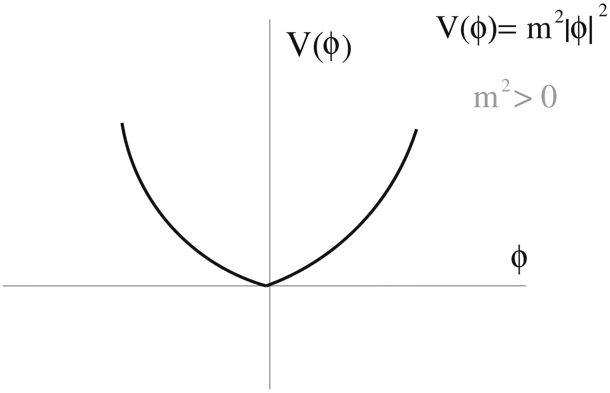


Fig. 13 Standard self-interacting potential

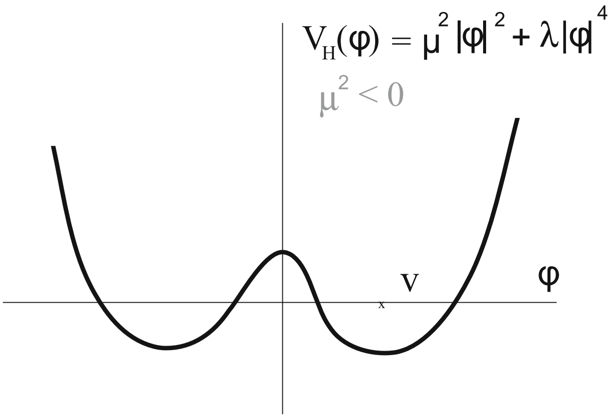


Fig. 14 Potential with degenerated vacuum

but that of the Higgs field is a bit different, $V_H(\phi) = \mu^2|\phi|^2 + \lambda|\phi|^4$, corresponding to an imaginary mass (Fig. 14). The minimum of the potential is degenerated, at a distance $v^2 \equiv -\frac{\mu^2}{\lambda}$ of the origin, which implies that we have to decide which one should be the best !!.

"We break down" the symmetry by moving the origin to one of the local minima:

$$\phi \equiv v + \eta \quad \text{(v constant)} \quad \rightarrow \quad V_H(\phi) = -2\mu^2\eta^2 + ... \quad \Rightarrow \quad m_\eta^2 = -2\mu^2 > 0\,.$$

When ϕ couples to a field W_μ according to the Minimal Interaction Principle, the interaction term $|\phi|^2$ turns to $v^2W^2 + ...$ giving mass to W_μ.

In the same way, coupling ϕ to a fermion ψ à la Yukawa, that is, $\kappa\phi\bar{\psi}\psi$, the displacement of ϕ leads to the mass term $\kappa v\bar{\psi}\psi$, where κ is a constant, providing the mass κv to the fermion.

General case: G semi-simple group of dimension r

$H \in G$ of dimension s, preserving the vacuum

ϕ representing G in dimension n

η (Higgses) $n-(r-s)$ massive real fields

ξ (Goldstone bosons) $r-s$ massless real fields, which will be gauged away

A_μ (massless vector bosons s

$\tilde{A}_\mu$ (massive vector bosons) $r-s$

When $n=r$ we shall have as many η's as A_μ's (n-(n-s) = s), provided that $n \geq r$, of course.

8.2 Group Quantization of Non-Abelian Stueckelberg Field Model for Massive Gauge Theory: Thinking of $SU(2)$

The original non-Abelian Stueckelberg model was addressed by the Lagrangian given above (260)

$$\mathcal{L}_{MYM} = \hat{\mathcal{L}}_{\text{“matt”}} + \mathcal{L}_{YM} = \frac{1}{2}\mu^2(\theta_\mu^{(a)} - A_\mu^{(a)})(\theta_{(a)}^\mu - A_{(a)}^\mu) - \frac{1}{4}F_{\mu\nu}^{(a)}F_{(a)}^{\mu\nu}$$

but $\theta_\mu^{(a)}$ were made of <u>external</u> scalar fields $\phi^{(a)}(x)$ behaving under the group G just like the own group parameters $\varphi^a(x)$ do.

Here we just turn ϕ^a into group parameters $\varphi^a \equiv g^a$ and find the complete group law bearing the corresponding <u>Solution Manifold</u> as a co-adjoint orbit. Then, we apply GAQ instead of CQ [53].

Inspired on the symmetry of the particle S^3-sigma model we directly guess the proper symmetry for the Massive Yang–Mills field theory associated with $SU(2)$ gauge group (generalizations for other semi-simple groups are also possible).

The σ-sector is the more relevant one. The $\tilde{\Sigma}SU(2)_{local}$ group law (a central extension by $U(1)$ of a group $\Sigma SU(2)_{local}$) for elements of the form $\check{U} \equiv (U,\, U_\mu U^{-1},\, z_\nu) \sim (\varphi^a,\, \theta_\mu^{(b)},\, z_\nu)$, $U \in SU(2)_{local}$ can be written in the form:

$$\begin{aligned}
\varphi''^a(\mathbf{x}) &= \rho(\mathbf{x})\varphi'^a(\mathbf{x}) + \rho'(\mathbf{x})\varphi^a(\mathbf{x}) + \frac{1}{2}\eta^a_{.bc}\varphi'^b(\mathbf{x})\varphi^c(\mathbf{x}) \\
\theta_\mu''^{(b)}(\mathbf{x}) &= \theta_\mu'^{(b)}(\mathbf{x}) + \rho'(\mathbf{x})\theta_\mu^{(b)}(\mathbf{x}) + \frac{1}{2}\eta^b_{.ca}\varphi'^c(\mathbf{x})\theta_\mu^{(a)}(\mathbf{x}) + \frac{1}{4}z_\mu(\mathbf{x})\varphi'^b(\mathbf{x}) \\
z_\mu''(\mathbf{x}) &= z_\mu'(\mathbf{x}) + \sqrt{1-\frac{\boldsymbol{\varphi}'^2(\mathbf{x})}{4}}\,z_\mu(\mathbf{x}) - \varphi'^a(\mathbf{x})\theta_\mu^{(b)}(\mathbf{x})\delta_{ab} \qquad \rho(\mathbf{x}) \equiv \sqrt{1-\frac{\boldsymbol{\varphi}'^2(\mathbf{x})}{4}} \\
\zeta'' &= \zeta'\zeta e^{i\mu\int_\Sigma d\sigma^\mu[(\rho'(\mathbf{x})-1)z_\mu(\mathbf{x}) - \varphi'^a(\mathbf{x})\theta_\mu^{(b)}(\mathbf{x})\delta_{ab}]} \qquad \mathbf{x} \in \Sigma \equiv \text{Cauchy Surface}\,.
\end{aligned} \tag{263}$$

<u>Note that</u> (φ, θ, z) is a <u>non-central extension</u> of (φ, θ) by z.

*** Remark: The unextended local group $\Sigma SU(2)_{local}$ can be formally rewritten as:

$$\begin{aligned}
\varphi''^a &= \rho\varphi'^a + \rho'\varphi^a + \frac{1}{2}\eta^a_{.bc}\varphi'^b\varphi^c \\
\theta_\mu''^{(a)} &= \theta_\mu'^{(a)} + \{(1-\lambda)R(\varphi)^a_b + \lambda X^L(\varphi')^a_b\} + \frac{\lambda}{4}\varphi'^a z_\mu \\
z_\mu'' &= z_\mu' + (1+\lambda(\rho'-1))z_\mu - \lambda\varphi'^a\theta_\mu^{(b)}\delta_{ab}\,,
\end{aligned} \tag{264}$$

where $R(\varphi)$ is the adjoint rotation in $SU(2)$. For $\lambda = 1$, we obtain the (generalized) gauge symmetry of Massive Yang–Mills fields, whereas for $\lambda = 0$ we recover the ordinary gauge symmetry of the Massless ones.***

Complete Group Law: (Including the Yang–Mills fields)

By $\check{U}$, we shall understand $\check{U} \equiv (U,\ U_\mu U^{-1},\ z_\nu) \equiv (\varphi^a,\ \theta^{(b)}_\mu,\ z_\nu)$

$$\begin{aligned}
\check{U}''(\mathbf{x}) &= \check{U}'(\mathbf{x}) * \check{U}(\mathbf{x}) \ \ (\Rightarrow \theta''_\mu = U'\theta_\mu U'^\dagger + \theta'_\mu + 0(\lambda);\ z''_\mu = z'_\mu + z_\mu + 0(\lambda))\\
A''_\mu(\mathbf{x}) &= U'(\mathbf{x})A_\mu(\mathbf{x})U'^\dagger(\mathbf{x}) + A'_\mu(\mathbf{x})\\
F''_{\mu\nu}(\mathbf{x}) &= U'(\mathbf{x})F_{\mu\nu}(\mathbf{x})U'\dagger(\mathbf{x}) + F'_{\mu\nu}(\mathbf{x})\\
\zeta'' &= \zeta'\zeta e^{i\int_\Sigma \mathrm{d}\sigma^\mu J_\mu(\tilde{U}',A',F';\tilde{U},A,F)}\\
J_\mu &= J^{YM}_\mu + J^\sigma_\mu = \frac{1}{2}[(A'^\nu - \theta'^\nu)U'F_{\mu\nu}U'^\dagger - F'_{\mu\nu}U'(A^\nu - \theta^\nu)U'^\dagger]\\
&\quad + U'(A_\mu - \theta_\mu)U'^\dagger - (\rho' - 1)z_\mu
\end{aligned} \tag{265}$$

Lie algebra commutators: Sigma Sector

$$\begin{aligned}
\left[X_{\varphi^a(\mathbf{x})},\ X_{\varphi^b(\mathbf{y})}\right] &= \eta^c_{.ab}X_{\varphi^c(\mathbf{x})}\delta(\mathbf{x}-\mathbf{y})\\
\left[X_{\varphi^a(\mathbf{x})},\ X_{\theta^{(b)}_\mu(\mathbf{y})}\right] &= \frac{1}{2}\eta^c_{.ab}X_{\theta^{(c)}_\mu(\mathbf{x})}\delta(\mathbf{x}-\mathbf{y}) + \delta_{ab}X_{z_\mu(\mathbf{x})}\delta(\mathbf{x}-\mathbf{y}) + \delta_{ab}\delta_{\mu 0}X_\zeta\delta(\mathbf{x}-\mathbf{y})\\
\left[X_{\theta^{(a)}_0(\mathbf{x})},\ X_{\theta^{(b)}_0(\mathbf{y})}\right] &= 0\\
\left[X_{\varphi^a(\mathbf{x})},\ X_{z_\mu(\mathbf{y})}\right] &= \frac{1}{4}X_{\theta^{(a)}_\mu(\mathbf{x})}\delta(\mathbf{x}-\mathbf{y})\\
\left[X_{\theta^{(a)}_0(\mathbf{x})},\ X_{z_\mu(\mathbf{y})}\right] &= 0\,.
\end{aligned} \tag{266}$$

It should be remarked that $X_{\theta^{(a)}_{1,2,3}}$ are non-basic generators; they are derived, as operators, from X_{φ^a}. Note also that the parameters $z_\mu(\mathbf{x})$ do not contribute to the SM.

Adding **Vector Bosons**: (only nonzero commutators)

$$\begin{aligned}
\left[X_{\varphi^a(\mathbf{x})},\ X_{A^b_\mu(\mathbf{y})}\right] &= \eta^{\ c}_{ab.}X_{A^c_\mu(\mathbf{x})}\delta(\mathbf{x}-\mathbf{y})\\
\left[X_{\varphi^a(\mathbf{x})},\ X_{E^b_j(\mathbf{y})}\right] &= \eta^{\ c}_{ab.}X_{E^c_j(\mathbf{x})} + \delta_{ab}\partial^{\mathbf{x}}_j\delta(\mathbf{x}-\mathbf{y})X_\zeta \qquad E^a_i \equiv F^a_{0i}\\
\left[X_{A^a_j(\mathbf{x})},\ X_{E^b_k(\mathbf{y})}\right] &= \delta_{ij}\delta_{ab}\delta(\mathbf{x}-\mathbf{y})X_\zeta\ .
\end{aligned} \tag{267}$$

Massive Yang–Mills fields interacting with Fermionic Matter

As commented above, the relevant modification concerning the quantization of massive Yang–Mills interaction lies on the vector boson sector. Let us justify this fact by looking at the group action of this sector on the (Fermionic) matter fields.

The group law of the massive gauge symmetry must be completed with the Fermionic sector in the way: (U acting on ψ is assumed to be the standard linear action. The arguments of the fields are omitted whereas no confusion could arise)

$$\begin{aligned}
\psi'' &= U'\psi + \psi'\\
\psi''_\mu &= U'_\mu\psi + U'\psi_\mu + \psi'_\mu\,.
\end{aligned} \tag{268}$$

Then, the full expression of the generators of the symmetry group (omitting the local indices for the sake of simplicity) is as follows:

$$X_{(a)} \equiv X_{\varphi^a} = X^b_{(a)}\frac{\partial}{\partial\varphi^b} + \frac{\partial X^b_{(a)}}{\partial\varphi^c}\varphi^c_\mu\frac{\partial}{\partial\varphi^b_\mu} + X^\alpha_{(a)\beta}\psi^\beta\frac{\partial}{\partial\psi^\alpha} + \frac{\partial X^\alpha_{(a)\beta}}{\partial\varphi^b}\varphi^b_\mu\psi^\beta\frac{\partial}{\partial\psi^\alpha_\mu}X^\alpha_{(a)\beta}\psi^\beta_\mu\frac{\partial}{\partial\psi^\alpha_\mu}$$

$$Y^{\mu}_{(b)} \equiv X_{\theta^{(b)}_{\mu}} = X^{c}_{(b)} X^{d}_{(c)} \frac{\partial}{\partial \varphi^{d}_{\mu}} \qquad | \quad X^{\alpha}_{(a)\beta} = \frac{i}{2}\sigma^{\alpha}_{(a)\beta} \quad \text{(Pauli matrices)}$$
$$Z^{\nu} \equiv X_{z_{\nu}} = \frac{1}{4}\rho\varphi^{c} \frac{\partial}{\partial \varphi^{c}_{\mu}} \qquad | \quad \frac{\partial X^{\alpha}_{(a)\beta}}{\partial \varphi^{b}} = 0\,, \tag{269}$$

reproducing the $\Sigma SU(2)_{local}$ Lie algebra (266). In fact:

$$\begin{aligned}
[X_{(a)},\, Y^{\mu}_{(b)}] &= X^{m}_{(a)} \frac{\partial}{\partial \varphi^{m}} (X^{c}_{(b)} X^{d}_{(c)}) \frac{\partial}{\partial \varphi^{d}_{\mu}} - X^{c}_{(b)} X^{d}_{(c)} \frac{\partial}{\partial \varphi^{d}_{\mu}} \left(\frac{\partial}{\partial \varphi^{m}} X^{n}_{(a)} \varphi^{m}_{\nu} \right) \frac{\partial}{\partial \varphi^{n}_{\nu}} \\
&\quad - X^{c}_{(b)} X^{d}_{(c)} \frac{\partial}{\partial \varphi^{d}_{\mu}} \left(\frac{\partial}{\partial \varphi^{m}} X^{\alpha}_{(a)\beta} \varphi^{m}_{\nu}) \psi^{\beta} \frac{\partial}{\partial \psi^{\alpha}_{\nu}} = X^{m}_{(a)} \frac{\partial}{\partial \varphi^{m}} (X^{c}_{(b)} X^{d}_{(c)} \right) \frac{\partial}{\partial \varphi^{d}_{\mu}} \\
&\quad - X^{c}_{(b)} X^{d}_{(c)} \frac{\partial}{\partial \varphi^{d}} \frac{\partial}{\partial \varphi^{d}} X^{n}_{(a)} \frac{\partial}{\partial \varphi^{n}_{\mu}} - X^{c}_{(b)} X^{d}_{(c)} \frac{\partial X^{\alpha}_{(a)\beta}}{\partial \varphi^{d}} \psi^{\beta} \frac{\partial}{\partial \psi^{\alpha}_{\mu}} \\
&= -\delta_{ab} \left(\frac{1}{4}\rho\varphi^{c} \frac{\partial}{\varphi^{c}_{\mu}} \right) - \frac{1}{2}\eta_{ab.}^{\;\;\;c} Y^{\mu}_{(c)} - X^{c}_{(b)} X^{d}_{(c)} \frac{\partial X^{\alpha}_{(a)\beta}}{\partial \varphi^{d}} \psi^{\beta} \frac{\partial}{\partial \psi^{\alpha}_{\mu}} \\
&= -\delta_{ab} Z^{\mu} - \frac{1}{2}\eta_{ab.}^{\;\;\;c} Y^{\mu}_{(c)}
\end{aligned} \tag{270}$$

as expected!!

Then, neither $X_{\theta^{(a)}_{\mu}} \equiv Y^{\mu}_{(a)}$, nor the extra generator Z^{μ}, act on the matter fields; they only affect the Goldstone sector.

8.2.1 Electroweak Interactions: Some remarkable new facts

Electroweak interactions are mediated by a vector potential B_{μ}, associated with the invariance under a local $U(1)$ group, as well as three massive Yang–Mills fields $W^{(\pm)}_{\mu},\ W^{(0)}_{\mu}$, associated with a local $SU(2)$ group. However, the rigid symmetry is not properly the group $SU(2) \otimes U(1)$, but a particular mixture where the $U(1)$ subgroup of $SU(2)$ and the external $U(1)$ group combine in a way intended to provide a final electromagnetic vector potential, and a new $W^{(0)}_{\mu}$-like in the form:

$$\begin{aligned}
A_{\mu} &\equiv \sin(\vartheta_W) W^{(0)}_{\mu} + \cos(\vartheta_W) B_{\mu} \\
Z_{\mu} &\equiv \cos(\vartheta_W) W^{(0)} - \sin(\vartheta_W) B_{\mu} \\
W^{(\pm)} &\equiv W^{(\pm)}\,,
\end{aligned} \tag{271}$$

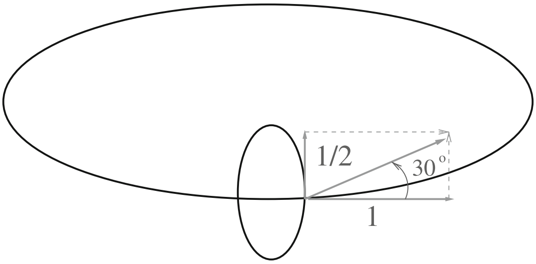
with a certain angle ϑ_W named Weinberg angle.

The mass of Z_{μ} and $W^{(\pm)}_{\mu}$ will be provided through the generalized Stueckelberg mechanism associated with the new $SU(2)$ subgroup.

Denoting the new (after the Weinberg rotation) rigid group $SU(2) \,\tilde{\otimes}\, U(1)$, we shall call $\Sigma SU(2)_{local} \,\tilde{\otimes}\, U(1)_{local}$ the relevant (unextended) symmetry addressing the electroweak interaction.

** There is, nevertheless, an obscure handling in the Standard Model in that which refers to the Weinberg rotation, mainly if we pretend to describe the quantum theory by means of a Group Approach: The Weinberg rotation could not be performed without the associated proper rotation in the Lie algebra and, accordingly, in the $SU(2) \otimes U(1)$ group, thus leading to what we have called $SU(2) \,\tilde{\otimes}\, U(1)$.

Fig. 15 Quantization of the Weinberg angle



In fact, a proper geometric analysis of the possible mixture of the involved $U(1)$ subgroups, the Cartan subgroups, concludes that ϑ_W should be quantized with a non-trivial ground value of 30^o (Fig. 15).

Graphically, this can be easily depicted by looking at the possible closed geodesic curves on the Cartan Torus taking into account that the "velocity" in one direction is twice than in the other

** Another remarkable fact related to the group approach to quantization of the electroweak interactions is that the mass generation in the Stueckelberg-like treatment involves the vector potentials but not, a priori, the Fermionic matter. Then, we have to be able to provide some group-theoretical algorithm to give mass to fermions.

In fact, as will be widely developed in the last chapter, devoted to possible generalizations of the gauge formulation of Gravitation, we resort to another mixing of the rigid symmetry, that time involving the Electromagnetic $U(1)$ group and the Translation subgroup of the Poincaré group:

$$T^4 \tilde{\otimes} U(1).$$

This mixing leads to a momentum operator $P'_0 = P_0 + \kappa Q$, combining the old energy and electric charge, so that the new mass operator for a charged fermion ψ is:

$$M'^2\psi = (m_0^2 + 2m_0\kappa Q + \kappa^2 Q^2)\psi.$$

Then, for "originally" massless particles ($m_0 = 0$) we get

$$M'^2\psi = \kappa^2 Q^2 \psi \,. \tag{272}$$

This mass-generation mechanism might be further developed involving more "sophisticate" mixings.

9 Gauge theory of space-time symmetries

As reports concerning a gauge approach to Gravity, more recent that the pioneer papers by Utiyama [2] and Kibble [3], we would recommend Refs. [54–58].

9.1 Generalization of the Gauge Invariance Principle

General global symmetries are generated by vector fields of the form

$$X_{(a)} = X^{\mu}_{(a)}(x)\frac{\partial}{\partial x^{\mu}} + X^{\alpha}_{(a)\beta}\varphi^{\beta}\frac{\partial}{\partial \varphi^{\beta}} \quad \text{(linear action on the fiber)}\,, \tag{273}$$

where $X^{\mu}_{(a)}$ are function only of x^{μ}. This fact causes essentially two new phenomena:

$$
\begin{aligned}
&a) \quad L_{X_{(a)}}\omega = \partial_\mu X^{\mu}_{(a)}\omega \neq 0 \quad \text{(in general)}\\
&b) \quad \bar{X}_{(a)} \quad \text{contains new terms in } \ \bar{X}^{\alpha}_{(a)\mu}\,,
\end{aligned}
$$

so that $\overline{f X_{(a)}}(\mathcal{L}\omega)$ is now <u>much more different</u> from $f\bar{X}_{(a)}(\mathcal{L}\omega)$.

The Lie algebra of the local group $G(M)$ also departs from the tensor product $\mathcal{F}(M)\otimes\mathcal{G}$. In fact, given two generators $X_{(a)}$, $X_{(b)}$ in $\mathcal{G}$, the commutator of the corresponding local generators is:

$$
\begin{aligned}
[f^{(a)}X_{(a)}, g^{(b)}X_{(b)}] &= f^{(a)}g^{(b)}[X_{(a)}, X_{(b)}] + f^{(a)}X^{\mu}_{(a)}\frac{\partial g^{(b)}}{\partial x^{\mu}}X_{(b)} - g^{(a)}X^{\mu}_{(a)}\frac{\partial f^{(b)}}{\partial x^{\mu}}X_{(b)}\\
&= \left(f^{(a)}g^{(b)}\left(X^{\nu}_{(a)}\frac{\partial X^{\mu}_{(b)}}{\partial x^{\nu}} - X^{\nu}_{(b)}\frac{\partial X^{\mu}_{(a)}}{\partial x^{\nu}}\right) + \left(f^{(a)}\frac{\partial g^{(b)}}{\partial x^{\nu}} - g^{(a)}\frac{\partial f^{(b)}}{\partial x^{\nu}}\right)X^{\nu}_{(a)}X^{\mu}_{(b)}\right)\frac{\partial}{\partial x^{\mu}}\\
&\quad + \left(f^{(a)}g^{(b)}C_a{}^c{}_b + \left(f^{(a)}\frac{\partial g^{(c)}}{\partial x^{\nu}} - g^{(a)}\frac{\partial f^{(c)}}{\partial x^{\nu}}\right)X^{\nu}_{(a)}\right)X^{\alpha}_{(c)}\frac{\partial}{\partial\varphi^{\alpha}}\,. \qquad (274)
\end{aligned}
$$

Note that since $X^{\mu}_{(a)}$ is a function of only x^{μ}, the action of $f^{(a)}X^{\mu}_{(a)}$ on space-time is of the form:

$$
f^{(a)}(x)X^{\mu}_{(a)}\frac{\partial}{\partial x^{\mu}} \equiv f^{\mu}(x)\frac{\partial}{\partial x^{\mu}}\,. \qquad (275)
$$

This means that the space-time action corresponds to a subgroup of Diff(M). The vertical action, however, remains as in the internal case, that is,

$$
\mathcal{G}(M)^{vertical} \approx \mathcal{F}(M)\otimes\mathcal{G}^{vertical}\,.
$$

Therefore, the general local symmetry algebras are contained in diff$(M)\otimes_S\mathcal{G}(M)^{vertical}$ with commutation relations (semi-direct action):

$$
\begin{aligned}
\left[f^{\mu}(x)\frac{\partial}{\partial x^{\mu}}, g^{\nu}(x)\frac{\partial}{\partial x^{\nu}}\right] &= \left(f^{\nu}\frac{\partial g^{\mu}}{\partial x^{\nu}} - g^{\nu}\frac{\partial f^{\mu}}{\partial x^{\nu}}\right)\frac{\partial}{\partial x^{\mu}}\\
\left[f^{(a)}(x)X^{\alpha}_{(a)}\frac{\partial}{\partial\varphi^{\alpha}}, g^{(b)}(x)X^{\beta}_{(b)}\frac{\partial}{\partial\varphi^{\beta}}\right] &= f^{(a)}g^{(b)}C_a{}^c{}_bX^{\alpha}_{(a)}\frac{\partial}{\partial\varphi^{\alpha}}\\
\left[f^{\mu}(x)\frac{\partial}{\partial x^{\mu}}, f^{(a)}(x)X^{\alpha}_{(a)}\frac{\partial}{\partial\varphi^{\alpha}}\right] &= f^{\mu}\frac{\partial f^{(a)}}{\partial x^{\mu}}X^{\alpha}_{(a)}\frac{\partial}{\partial\varphi^{\alpha}}\,. \qquad (276)
\end{aligned}
$$

As a consequence of the differences above with respect to the internal case, we have to introduce compensating fields $A^{(a)}_{\mu}$ with **modified** transformation properties, and also **new** compensating fields, noted $h^{(a)\nu}_{\mu\rho}$, both sets associated with each generator of the global group, as before.

The transformation properties of $\{A^{(a)}_{\mu},\ h^{(a)\nu}_{\mu\rho}\}$ must be:

$$
X_{A^{(a)}_{\mu}} \equiv \delta A^{(a)}_{\mu} = f^{(b)}C_b{}^a{}_cA^{(c)}_{\mu} - \frac{\partial f^{(a)}}{\partial x^{\mu}} - A^{(a)}_{\nu}\partial_\mu(f^{(b)}X^{\nu}_{(b)})\,, \qquad (277)
$$

$$
X_{h^{(a)\nu}_{\mu\rho}} \equiv \delta h^{(a)\nu}_{\mu\rho} = \frac{\partial f^{(a)}}{\partial x^{\mu}}\delta^{\nu}_{\rho} + h^{(a)\sigma}_{\mu\rho}\partial_\sigma(f^{(b)}X^{\nu}_{(b)}) - f^{(b)}\frac{\partial X^{\sigma}_{(b)}}{\partial x^{\mu}}h^{(a)\nu}_{\sigma\rho}\,. \qquad (278)
$$

The expression for $\delta h^{(a)\nu}_{\mu\rho}$ can be taken to the form:

$$
X_{h^{(a)\nu}_{\mu\rho}} \equiv \delta h^{(a)\nu}_{\mu\rho} = \frac{\partial f^{(a)}}{\partial x^{\mu}}\delta^{\nu}_{\rho} + M^{\nu}_{\sigma}h^{(a)\sigma}_{\mu\rho} - M^{\sigma}_{\mu}h^{(a)\nu}_{\sigma\rho} + \frac{\partial f^{(b)}}{\partial x^{\mu}}X^{\sigma}_{(b)}h^{(a)\nu}_{\sigma\rho}\,,
$$

where

$$M^{\nu}_{\sigma} \equiv \frac{\partial f^{(b)}}{\partial x^{\sigma}} X^{\nu}_{(b)} + f^{(b)} \frac{\partial X^{\nu}_{(b)}}{\partial x^{\sigma}}$$

is the expected transformation matrix for a tensorial index. This means that only the ν index above is tensorial!!. The final expression for the local generators is:

$$\begin{aligned}
f^{(a)} \mathcal{X}_{(a)} &= f^{(a)} X_{(a)} + X_{A^{(a)}_{\mu}} \frac{\partial}{\partial A^{(a)}_{\mu}} + X_{h^{(a)\nu}_{\mu\rho}} \frac{\partial}{\partial h^{(a)\nu}_{\mu\rho}} \\
&= f^{(a)} X^{\mu}_{(a)} \frac{\partial}{\partial x^{\mu}} + f^{(a)} X^{\alpha}_{(a)\beta} \varphi^{\beta} \frac{\partial}{\partial \varphi^{\alpha}} \\
&\quad + \left(f^{(b)} C_{b\ c}^{\ a} A^{(c)}_{\mu} - \frac{\partial f^{(a)}}{\partial x^{\mu}} - A^{(a)}_{\nu} X^{\nu}_{(b)} \frac{\partial f^{(b)}}{\partial x^{\mu}} - f^{(b)} A^{(a)}_{\nu} \frac{\partial X^{\nu}_{(b)}}{\partial x^{\mu}} \right) \frac{\partial}{\partial A^{(a)}_{\mu}} \\
&\quad + \left(\frac{\partial f^{(a)}}{\partial x^{\mu}} \delta^{\nu}_{\rho} + \frac{\partial f^{(b)}}{\partial x^{\sigma}} h^{(a)\sigma}_{\mu\rho} X^{\nu}_{(b)} + f^{(b)} \Big(\frac{\partial X^{\nu}_{(b)}}{\partial x^{\sigma}} h^{(a)\sigma}_{\mu\rho} - \frac{\partial X^{\sigma}_{(b)}}{\partial x^{\mu}} h^{(a)\nu}_{\sigma\rho} \Big) \right) \frac{\partial}{\partial h^{(a)\nu}_{\mu\rho}} .
\end{aligned} \tag{279}$$

Utiyama's Theorem

As in the case of internal symmetries, the theorem of Utiyama will be established in two parts, one for the matter Lagrangian, the other for the Lagrangian driving the compensating fields.

Utiyama's Theorem I: *Given a matter Lagrangian $\mathcal{L}_{matt}$ depending on $(\varphi^{\alpha}, \varphi^{\beta}_{\mu})$, the new Lagrangian $\widehat{L}_{\mathrm{mat}}(\varphi^{\alpha}, \varphi^{\alpha}_{\mu}, A^{(a)}_{\mu}, h^{(a)\nu}_{\mu\rho})$, invariant under the local algebra $\mathcal{G}(M)$, describing the dynamics of the matter fields, as well as their interaction with the compensating fields $\{A^{(a)}_{\mu}, h^{(a)\nu}_{\mu\rho}\}$, takes the following structure*:

$$\widehat{L}_{\mathrm{mat}}(\varphi^{\alpha}, \varphi^{\alpha}_{\mu}, A^{(a)}_{\mu}, h^{(a)\nu}_{\mu\rho}) \equiv \Lambda \widehat{\mathcal{L}}_{\mathrm{mat}}(\varphi^{\alpha}, \varphi^{\alpha}_{\mu}, A^{(a)}_{\nu}, h^{(a)\nu}_{\mu\rho}) \ , \tag{280}$$

where

$$\begin{aligned}
\widehat{\mathcal{L}}_{\mathrm{matt}}(\varphi^{\alpha}, \varphi^{\alpha}_{\mu}, A^{(a)}_{\mu}, h^{(a)\nu}_{\mu\rho}) &\equiv \mathcal{L}_{\mathrm{matt}}(\varphi^{\alpha}, k^{\nu}_{\mu}(\varphi^{\alpha}_{\nu} + A^{(a)}_{\nu} X^{\alpha}_{(a)\beta} \varphi^{\beta})), \\
k^{\nu}_{\mu} &\equiv \delta^{\nu}_{\mu} + h^{(a)\nu}_{\mu\sigma} X^{\sigma}_{(a)} \ , \\
\Lambda &\equiv det(q^{\mu}_{\nu}) \ ,
\end{aligned} \tag{281}$$

the objects q^{μ}_{ν} being the inverse of k^{ν}_{μ}, i.e.,

$$\begin{aligned}
k^{\nu}_{\mu} q^{\mu}_{\sigma} &= \delta^{\nu}_{\sigma}, \\
k^{\nu}_{\mu} q^{\sigma}_{\nu} &= \delta^{\sigma}_{\mu} \ .
\end{aligned}$$

Proof Since space-time transformations can modify the integration volume, the invariance condition on the action now means:

$$\overline{f^{(a)} \mathcal{X}_{(a)}} (\Lambda \widehat{\mathcal{L}}_{\mathrm{mat}}) + \Lambda \widehat{\mathcal{L}}_{\mathrm{mat}} \partial_{\mu} (f^{(a)} X^{\mu}_{(a)}) = 0 \, . \tag{282}$$

Otherwise, the proof follows the same scheme. Let us consider the change of variables χ:

$$\begin{aligned}
\Phi^{\alpha} &= \varphi^{\alpha} \\
\Phi^{\alpha}_{\mu} &= k^{\nu}_{\mu}(\varphi^{\alpha}_{\nu} + A^{(a)}_{\nu} X^{\alpha}_{(a)\beta}\varphi^{\beta}) = (\delta^{\nu}_{\mu} + h^{(b)\nu}_{\mu\sigma} X^{\sigma}_{(b)})(\varphi^{\alpha}_{\nu} + A^{(a)}_{\nu} X^{\alpha}_{(a)\beta}\varphi^{\beta}) \\
B^{(a)}_{\mu} &= A^{(a)}_{\mu} \\
H^{(a)\sigma}_{\mu\nu} &= h^{(a)\sigma}_{\mu\nu} ,
\end{aligned} \tag{283}$$

and the corresponding change of the partial derivatives:

$$\begin{aligned}
\frac{\partial}{\partial\varphi^{\alpha}} &= \frac{\partial}{\partial\Phi^{\alpha}} + k^{\nu}_{\mu} B^{(a)}_{\nu} X^{\beta}_{(a)\alpha} \frac{\partial}{\partial\Phi^{\beta}_{\mu}} \\
\frac{\partial}{\partial\varphi^{\alpha}_{\mu}} &= k^{\mu}_{\nu} \frac{\partial}{\partial\Phi^{\alpha}_{\nu}} \\
\frac{\partial}{\partial A^{(a)}_{\mu}} &= \frac{\partial}{\partial B^{(a)}_{\mu}} + k^{\mu}_{\nu} X^{\alpha}_{(a)\beta}\Phi^{\beta} \frac{\partial}{\partial\Phi^{\alpha}_{\mu}} \\
\frac{\partial}{\partial h^{(a)\sigma}_{\mu\nu}} &= \frac{\partial}{\partial H^{(a)\sigma}_{\mu\nu}} + X^{\nu}_{(a)} q^{\rho}_{\sigma} \Phi^{\alpha}_{\rho} \frac{\partial}{\partial\Phi^{\alpha}_{\mu}} .
\end{aligned} \tag{284}$$

Using this change of variables, we can write:

$$\begin{aligned}
\widehat{\mathcal{L}}_{\text{mat}}(\varphi^{\alpha}, \varphi^{\alpha}_{\mu}, A^{(a)}_{\nu}, h^{(a)\nu}_{\mu\rho}) &\equiv \mathcal{L}_{\text{mat}}(\varphi^{\alpha}, k^{\nu}_{\mu}(\varphi^{\alpha}_{\nu} + A^{(a)}_{\mu} X^{\alpha}_{(a)\beta}\varphi^{\beta})) = \mathcal{L}_{\text{mat}}(\Phi^{\alpha}, \Phi^{\alpha}_{\mu}) \\
&= \mathcal{L}_{\text{mat}} \circ \chi(\varphi^{\alpha}, \varphi^{\alpha}_{\mu}, A^{(a)}_{\nu}, h^{(a)\nu}_{\mu\rho}) .
\end{aligned} \tag{285}$$

Let us compute $\overline{f^{(a)}\mathcal{X}_{(a)}}\widehat{\mathcal{L}}_{\text{mat}}$:

$$\begin{aligned}
\overline{f^{(a)}\mathcal{X}_{(a)}}\widehat{\mathcal{L}}_{\text{mat}} = &\Bigg(f^{(a)} X^{\mu}_{(a)} \frac{\partial}{\partial x^{\mu}} + f^{(a)} X^{\alpha}_{(a)\beta}\Phi^{\beta}\Big(\frac{\partial}{\partial\Phi^{\alpha}} + k^{\nu}_{\mu} B^{(b)}_{\nu} X^{\gamma}_{(b)\alpha} \frac{\partial}{\partial\Phi^{\gamma}_{\mu}}\Big) \\
&+\Big(f^{(a)} X^{\alpha}_{(a)\beta}(q^{\nu}_{\mu}\Phi^{\beta}_{\nu} - B^{(b)}_{\mu} X^{\beta}_{(b)\gamma}\Phi^{\gamma}) - f^{(a)} \frac{\partial X^{\nu}_{(a)}}{\partial x^{\mu}} (q^{\sigma}_{\nu}\Phi^{\alpha}_{\sigma} \\
&- B^{(b)}_{\nu} X^{\alpha}_{(b)\gamma}\Phi^{\gamma}) - \frac{\partial f^{(a)}}{\partial x^{\mu}} X^{\nu}_{(a)} (q^{\sigma}_{\nu}\Phi^{\alpha}_{\sigma} - B^{(b)}_{\nu} X^{\alpha}_{(b)\gamma}\Phi^{\gamma}) \\
&+ \frac{\partial f^{(a)}}{\partial x^{\mu}} X^{\alpha}_{(a)\beta}\Phi^{\beta}\Big) k^{\mu}_{\rho} \frac{\partial}{\partial\Phi^{\alpha}_{\rho}} + (f^{(b)} C_{b}{}^{a}{}_{c} B^{(c)}_{\mu} - \frac{\partial f^{(a)}}{\partial x^{\mu}} - \frac{\partial f^{(b)}}{\partial x^{\mu}} X^{\nu}_{(b)} B^{(a)}_{\nu} \\
&- f^{(b)} \frac{\partial X^{\nu}_{(b)}}{\partial x^{\mu}} B^{(a)}_{\nu}) \frac{\partial}{\partial B^{(a)}_{\mu}} + (f^{(b)} C_{b}{}^{a}{}_{c} B^{(c)}_{\mu} - \frac{\partial f^{(a)}}{\partial x^{\mu}} - \frac{\partial f^{(b)}}{\partial x^{\mu}} X^{\nu}_{(b)} B^{(a)}_{\nu} \\
&- f^{(b)} \frac{\partial X^{\nu}_{(b)}}{\partial x^{\mu}} B^{(a)}_{\nu}) k^{\mu}_{\rho} X^{\alpha}_{(a)\beta}\Phi^{\beta} \frac{\partial}{\partial\Phi^{\alpha}_{\rho}} + (\frac{\partial f^{(b)}}{\partial x^{\mu}}\delta^{\nu}_{\rho} + \frac{\partial f^{(a)}}{\partial x^{\sigma}} H^{(b)\sigma}_{\mu\rho} X^{\nu}_{(a)} \\
&+ f^{(a)} \frac{\partial X^{\nu}_{(a)}}{\partial x^{\sigma}} H^{(b)\sigma}_{\mu\rho} - f^{(a)} \frac{\partial X^{\sigma}_{(a)}}{\partial x^{\mu}} H^{(b)\nu}_{\sigma\rho}) \frac{\partial}{\partial H^{(b)\nu}_{\mu\rho}} + (\frac{\partial f^{(b)}}{\partial x^{\mu}}\delta^{\nu}_{\rho} + \frac{\partial f^{(a)}}{\partial x^{\sigma}} H^{(b)\sigma}_{\mu\rho} X^{\nu}_{(a)} \\
&+ f^{(a)} \frac{\partial X^{\nu}_{(a)}}{\partial x^{\sigma}} H^{(b)\sigma}_{\mu\rho} - f^{(a)} \frac{\partial X^{\sigma}_{(a)}}{\partial x^{\mu}} H^{(b)\nu}_{\sigma\rho}) X^{\rho}_{(b)} q^{\kappa}_{\nu}\Phi^{\alpha}_{\kappa} \frac{\partial}{\partial\Phi^{\gamma}_{\mu}}\Bigg) \mathcal{L}_{\text{mat}}(\Phi^{\alpha}, \Phi^{\alpha}_{\mu}) \\
= &\Bigg(f^{(a)} X^{\mu}_{(a)} \frac{\partial}{\partial x^{\mu}} + f^{(a)} X^{\alpha}_{(a)\beta}\Phi^{\beta} \frac{\partial}{\partial\Phi^{\alpha}} + (f^{(a)} X^{\alpha}_{(a)\beta}\Phi^{\beta}_{\mu} \\
&- f^{(a)} \frac{\partial X^{\sigma}_{(a)}}{\partial x^{\mu}} \Phi^{\alpha}_{\sigma}) \frac{\partial}{\partial\Phi^{\alpha}_{\mu}}\Bigg) \mathcal{L}_{\text{mat}}(\Phi^{\alpha}, \Phi^{\alpha}_{\mu}) = \overline{X}_{(a)} \mathcal{L}_{\text{mat}}(\Phi^{\alpha}, \Phi^{\alpha}_{\mu}) .
\end{aligned} \tag{286}$$

Then, the invariance condition (given that for an arbitrary field Y, $Y(\Lambda\hat{\mathcal{L}}\omega) = Y(\Lambda\hat{\mathcal{L}})\omega + (\text{div}Y)\omega\Lambda\hat{\mathcal{L}}$)

$$\overline{f^{(a)}\mathcal{X}_{(a)}}(\Lambda\hat{\mathcal{L}}_{matt}) + \Lambda\hat{\mathcal{L}}_{matt}\partial_\mu(f^{(a)}X^\mu_{(a)}) = 0 \tag{287}$$

becomes:

$$\Lambda\overline{f^{(a)}\mathcal{X}_{(a)}}\hat{\mathcal{L}}_{matt} + \hat{\mathcal{L}}_{matt}\overline{f^{(a)}\mathcal{X}_{(a)}}\Lambda \;+\; \Lambda\hat{\mathcal{L}}_{matt}X^\mu_{(a)}\partial_\mu f^{(a)} + \Lambda\hat{\mathcal{L}}_{matt}f^{(a)}\partial_\mu X^\mu_{(a)} = 0$$
or
$$\Lambda f^{(a)}\big(\bar{X}_{(a)}\mathcal{L}_{matt} + \mathcal{L}_{matt}\partial_\mu X^\mu_{(a)}\big) \;+\; \hat{\mathcal{L}}_{matt}\big(\overline{f^{(a)}\mathcal{X}_{(a)}}\Lambda + \Lambda X^\mu_{(a)}\partial_\mu f^{(a)}\big) = 0$$

where the first term is **zero**, due to the **rigid invariance**, so that

$$\overline{f^{(a)}\mathcal{X}_{(a)}}\Lambda + \Lambda\partial_\mu f^{(a)}X^\mu_{(a)} = 0\,. \tag{288}$$

For the sake of simplicity, we shall assume that Λ depends only on the fields $h^{(a)\nu}_{\mu\rho}$ but not on their derivatives. This way, the expression (288) reduces to:

$$\left(\frac{\partial f^{(a)}}{\partial x^\mu}\delta^\nu_\rho + \frac{\partial f^{(b)}}{\partial x^\sigma}h^{(a)\sigma}_{\mu\rho}X^\nu_{(b)} + f^{(b)}\Big(\frac{\partial X^\nu_{(b)}}{\partial x^\sigma}h^{(a)\sigma}_{\mu\rho} - \frac{\partial X^\sigma_{(b)}}{\partial x^\mu}h^{(a)\nu}_{\sigma\rho}\Big)\right)\frac{\partial\Lambda}{\partial h^{(a)\nu}_{\mu\rho}}$$
$$+\Lambda\frac{\partial f^{(a)}}{\partial x^\mu}X^\mu_{(a)} = 0\,.$$

Since $f^{(a)}$ are arbitrary, we can factorize the functions and their derivatives:

$$\textbf{a)}\quad f^{(b)}:\ \Big(\frac{\partial X^\nu_{(b)}}{\partial x^\sigma}h^{(a)\sigma}_{\mu\rho} - \frac{\partial X^\sigma_{(b)}}{\partial x^\mu}h^{(a)\nu}_{\sigma\rho}\Big)\frac{\partial\Lambda}{\partial h^{(a)\nu}_{\mu\rho}} = 0 \tag{289}$$

$$\textbf{b)}\quad \frac{\partial f^{(b)}}{\partial x^\sigma}:\ (\delta^\sigma_\mu\delta^a_b\delta^\nu_\rho + h^{(a)\sigma}_{\mu\rho}X^\nu_{(b)})\frac{\partial\Lambda}{\partial h^{(a)\nu}_{\mu\rho}} + \Lambda X^\sigma_{(b)} = 0\,. \tag{290}$$

Taking into account that $\frac{\partial\Lambda}{\partial h^{(a)\nu}_{\mu\rho}} = X^\rho_{(a)}\frac{\partial\Lambda}{\partial k^\nu_\mu}$, Eqs. (289) and (290) become:

$$\textbf{a)}\quad \Big(k^\sigma_\mu\frac{\partial X^\nu_{(b)}}{\partial x^\sigma} - k^\nu_\sigma\frac{\partial X^\sigma_{(b)}}{\partial x^\mu}\Big)\frac{\partial\Lambda}{\partial k^\nu_\mu} = 0 \tag{291}$$

$$\textbf{b)}\quad X^\nu_{(b)}k^\sigma_\mu\frac{\partial\Lambda}{\partial k^\nu_\mu} + \Lambda X^\sigma_{(b)} = 0\,, \tag{292}$$

whose solution is (save for a constant):

$$\Lambda = \det(q^\nu_\mu)\,. \tag{293}$$

□

The generalized Minimal Coupling Principle claims for the replacement of $\partial_\mu\varphi^\alpha$ with the **generalized "covariant" derivative**:

$$\begin{aligned}\mathcal{D}_\mu\varphi^\alpha &\equiv \varphi^\alpha_\mu + A^{(a)}_\mu X^\alpha_{(a)\beta}\varphi^\beta + h^{(a)\nu}_{\mu\sigma}X^\sigma_{(a)}\varphi^\alpha_\nu + h^{(a)\nu}_{\mu\sigma}X^\sigma_{(a)}A^{(b)}_\nu X^\alpha_{(b)\beta}\varphi^\beta\\ &\equiv \varphi^\alpha_\mu + \mathcal{A}^{(a)}_\mu X^\alpha_{(a)\beta}\varphi^\beta + h^{(a)\nu}_{\mu\sigma}X^\sigma_{(a)}\varphi^\alpha_\nu\\ &\equiv k^\nu_\mu(\varphi^\alpha_\nu + A^{(a)}_\nu X^\alpha_{(a)\beta}\varphi^\beta) \equiv k^\nu_\mu D_\nu\varphi^\alpha\,,\end{aligned} \tag{294}$$

where we have introduced the notation $\mathcal{A}^{(a)}_{\mu} \equiv k^{\nu}_{\mu} A^{(a)}_{\nu}$.

Big Remark: The compensating fields $h^{(a)\nu}_{\mu\sigma}$ are not of common usage. We have introduced them in order to generalize more properly the theory of internal gauge symmetry: this way, the pair $(A^{(a)}_{\mu},\, h^{(b)\sigma}_{\nu\rho})$ is associated with the generator $X_{(a)}$ of the rigid group. However, the fields $h^{(a)\nu}_{\mu\sigma}$ always appear in the theory in a sum over the index (a), so that the association of the field $h^{(a)\nu}_{\mu\sigma}$ with $X_{(a)}$ loses consistence. In fact, it is possible to sum up all the $h^{(a)}$'s in a simpler quantity, precisely $k^{\nu}_{\mu} = \delta^{\nu}_{\mu} + h^{(a)\nu}_{\mu\sigma} X^{\sigma}_{(a)}$.

The objects k^{ν}_{μ} will recover an algebraic role as associated with the symmetry group under a slightly different viewpoint (see below) and, for the time being, they simplify in general the transformation properties. In fact, the variation of k^{ν}_{μ}, δk^{ν}_{μ}, restrict to:

$$
\begin{aligned}
X_{k^{\nu}_{\mu}} \equiv \delta k^{\nu}_{\mu} &= X^{\nu}_{(a)} k^{\sigma}_{\mu} \frac{\partial f^{(a)}}{\partial x^{\sigma}} + f^{(a)} \left(k^{\sigma}_{\mu} \frac{\partial X^{\nu}_{(a)}}{\partial x^{\sigma}} - k^{\nu}_{\sigma} \frac{\partial X^{\sigma}_{(a)}}{\partial x^{\mu}} \right) \\
&= k^{\sigma}_{\mu} \partial_{\sigma} (f^{(a)} X^{\nu}_{(a)}) - k^{\nu}_{\sigma} f^{(a)} \frac{\partial X^{\sigma}_{(a)}}{\partial x^{\mu}} \, .
\end{aligned}
\tag{295}
$$

Let us repeat Utiyama's Theorem I, very briefly, in terms of k^{ν}_{μ}:
Given $\mathcal{L}_{matt}(\varphi^{\alpha}, \varphi^{\beta}_{\mu})$ invariant under G, the minimally coupled Lagrangian

$$
\hat{L}_{matt}(\varphi^{\alpha}, \varphi^{\alpha}_{\mu}, A^{(a)}_{\mu}, k^{\nu}_{\mu}) \equiv \Lambda \mathcal{L}_{matt}(\varphi^{\alpha}, k^{\nu}_{\mu}(\varphi^{\alpha}_{\nu} + A^{(a)}_{\nu} X^{\alpha}_{(a)\beta} \varphi^{\beta})) \, , \; \Lambda \equiv \det(q),
$$

leads to an invariant action $\hat{S}_{matt} = \int \omega \Lambda \hat{\mathcal{L}}_{matt}$.
In fact, the change of variables

$$
\begin{aligned}
\Phi^{\alpha} &= \varphi^{\alpha} \\
\Phi^{\alpha}_{\mu} &= k^{\nu}_{\mu} (\varphi^{\alpha}_{\nu} + A^{(a)}_{\nu} X^{\alpha}_{(a)\beta} \varphi^{\beta}) \\
B^{(a)}_{\mu} &= A^{(a)}_{\mu} \\
K^{\nu}_{\mu} &= k^{\nu}_{\mu}
\end{aligned}
\tag{296}
$$

accomplishes the same task as before, that is,

$$
\overline{f^{(a)} \mathcal{X}_{(a)}} \mathcal{L}_{matt}(\varphi, \varphi_{\nu}, A, k) = f^{(a)} \bar{X}_{(a)} \mathcal{L}_{matt}(\phi, \phi_{\nu}) \, .
$$

We must find now the structure of the Lagrangian driving the dynamics of the fields $(A^{(a)}_{\mu}, k^{\nu}_{\mu})$ themselves.

Utiyama's Theorem II: *The Lagrangian $L_0(A^{(a)}_{\mu}, A^{(a)}_{\mu,\nu}, k^{\nu}_{\mu}, k^{\nu}_{\mu,\sigma})$ of the free compensating fields invariant under the local group $G(M)$ must be, except for a factor Λ, an arbitrary function $\mathcal{L}_0(\mathcal{T}^{\sigma}{}_{\mu\nu}, \mathcal{F}^{(a)}_{\nu\mu})$, where*

$$
\begin{aligned}
\mathcal{T}^{\sigma}{}_{\mu\nu} &\equiv T^{\sigma}{}_{\nu\mu} - A^{(a)}_{\rho} (k^{\rho}_{\mu} \partial_{\nu} X^{\sigma}_{(a)} - k^{\rho}_{\nu} \partial_{\mu} X^{\sigma}_{(a)}) \\
T^{\sigma}{}_{\nu\mu} &\equiv q^{\sigma}_{\rho} (k^{\rho}_{\nu,\tau} k^{\tau}_{\mu} - k^{\rho}_{\mu,\tau} k^{\tau}_{\nu}) \\
\mathcal{F}^{(a)}_{\nu\mu} &\equiv k^{\rho}_{\nu} k^{\sigma}_{\mu} F^{(a)}_{\rho\sigma} \, ,
\end{aligned}
\tag{297}
$$

and F is the already known object

$$
F^{(a)}_{\mu\nu} \equiv A^{(a)}_{\mu,\nu} - A^{(a)}_{\nu,\mu} - \frac{1}{2} C_{b}{}^{a}{}_{c} (A^{(b)}_{\mu} A^{(c)}_{\nu} - A^{(b)}_{\nu} A^{(c)}_{\mu}) \, .
$$

Proof We require that

$$\overline{f^{(a)}\mathcal{Y}_{(a)}}(\Lambda\mathcal{L}_0)+\Lambda\mathcal{L}_0\partial_\mu(f^{(a)}X^\mu_{(a)})=0$$

or

$$\Lambda\overline{f^{(a)}\mathcal{X}_{(a)}}\mathcal{L}_0+\Lambda\mathcal{L}_0 f^{(a)}\partial_\mu X^\mu_{(a)}+\mathcal{L}_0\overline{f^{(a)}\mathcal{X}_{(a)}}\Lambda+\Lambda\mathcal{L}_0 X^\mu_{(a)}\partial_\mu f^{(a)}=0\,,$$

from which, the first two terms fixe $\mathcal{L}_0$, whereas the other two fixe Λ, as before. Then, let us solve

$$\overline{f^{(a)}\mathcal{Y}_{(a)}}\mathcal{L}_0+\mathcal{L}_0 f^{(a)}\frac{\partial X^\mu_{(a)}}{\partial x^\mu}=0\,.$$

Using the standard expressions for jet extensions:

$$\begin{aligned}\overline{X}_{k^\nu_{\mu,\sigma}}&=\frac{\partial X_{k^\nu_\mu}}{\partial x^\sigma}+\frac{\partial X_{k^\nu_\mu}}{\partial k^\rho_\xi}k^\rho_{\xi,\sigma}-\frac{\partial(f^{(a)}X^\rho_{(a)})}{\partial x^\sigma}k^\nu_{\mu,\rho}\\ \overline{X}_{A^{(a)}_{\mu,\nu}}&=\frac{\partial X_{A^{(a)}_\mu}}{\partial x^\nu}+\frac{\partial X_{A^{(a)}_\mu}}{\partial A^{(b)}_\rho}A^{(b)}_{\rho,\nu}-\frac{\partial(f^{(b)}X^\rho_{(b)})}{\partial x^\nu}A^{(a)}_{\mu,\rho}\,,\end{aligned}$$

we have:

$$\begin{aligned}&X_{A^{(a)}_\mu}\frac{\partial\mathcal{L}_0}{\partial A^{(a)}_\mu}+X_{k^\nu_\mu}\frac{\partial\mathcal{L}_0}{\partial k^\nu_\mu}+\overline{X}_{A^{(a)}_{\mu,\nu}}\frac{\partial\mathcal{L}_0}{\partial A^{(a)}_{\mu,\nu}}+\overline{X}_{k^\nu_{\mu,\sigma}}\frac{\partial\mathcal{L}_0}{\partial k^\nu_{\mu,\sigma}}+\mathcal{L}_0 f^{(a)}\frac{\partial X^\mu_{(a)}}{\partial x^\mu}\\
&=\left(f^{(b)}C_b{}^a{}_c A^{(c)}_\mu-\frac{\partial f^{(a)}}{\partial x^\mu}-A^{(a)}_\nu X^\nu_{(b)}\frac{\partial f^{(b)}}{\partial x^\mu}-f^{(b)}A^{(a)}_\nu\frac{\partial X^\nu_{(b)}}{\partial x^\mu}\right)\frac{\partial\mathcal{L}_0}{\partial A^{(a)}_\mu}\\
&+\left(X^\nu_{(a)}k^\sigma_\mu\frac{\partial f^{(a)}}{\partial x^\sigma}+f^{(a)}\left(k^\sigma_\mu\frac{\partial X^\nu_{(a)}}{\partial x^\sigma}-k^\nu_\sigma\frac{\partial X^\sigma_{(a)}}{\partial x^\mu}\right)\right)\frac{\partial\mathcal{L}_0}{\partial k^\nu_\mu}\\
&+\left(\frac{\partial f^{(b)}}{\partial x^\nu}C_b{}^a{}_c A^{(c)}_\mu-\frac{\partial^2 f^{(a)}}{\partial x^\mu\partial x^\nu}-A^{(a)}_\theta\frac{\partial X^\theta_{(b)}}{\partial x^\nu}\frac{\partial f^{(b)}}{\partial x^\mu}-A^{(a)}_\theta X^\theta_{(b)}\frac{\partial^2 f^{(b)}}{\partial x^\mu\partial x^\nu}\right.\\
&-\frac{\partial f^{(b)}}{\partial x^\nu}A^{(a)}_\theta\frac{\partial X^\theta_{(b)}}{\partial x^\mu}-f^{(b)}A^{(a)}_\theta\frac{\partial^2 X^\theta_{(b)}}{\partial x^\mu\partial x^\nu}+f^{(b)}\left(C_b{}^a{}_c A^{(c)}_{\mu,\nu}-\frac{\partial X^\theta_{(b)}}{\partial x^\mu}A^{(a)}_{\theta,\nu}\right)\\
&\left.-\frac{\partial f^{(b)}}{\partial x^\mu}X^\theta_{(b)}A^{(a)}_{\theta,\nu}-\frac{\partial f^{(b)}}{\partial x^\nu}X^\rho_{(b)}A^{(a)}_{\mu,\rho}-f^{(b)}\frac{\partial X^\rho_{(b)}}{\partial x^\nu}A^{(a)}_{\mu,\rho}\right)\frac{\partial\mathcal{L}_0}{\partial A^{(a)}_{\mu,\nu}}\\
&+\left(\frac{\partial X^\nu_{(a)}}{\partial x^\sigma}k^\theta_\mu\frac{\partial f^{(a)}}{\partial x^\theta}+X^\nu_{(a)}k^\theta_\mu\frac{\partial^2 f^{(a)}}{\partial x^\theta\partial x^\sigma}+\frac{\partial f^{(a)}}{\partial x^\sigma}\left(k^\theta_\mu\frac{\partial X^\nu_{(a)}}{\partial x^\theta}-k^\nu_\theta\frac{\partial X^\theta_{(a)}}{\partial x^\mu}\right)\right.\\
&+f^{(a)}\left(k^\theta_\mu\frac{\partial^2 X^\nu_{(a)}}{\partial x^\theta\partial x^\sigma}-k^\nu_\theta\frac{\partial^2 X^\theta_{(a)}}{\partial x^\mu\partial x^\sigma}\right)+k^\theta_{\mu,\sigma}X^\nu_{(a)}\frac{\partial f^{(a)}}{\partial x^\theta}+f^{(a)}\left(k^\theta_{\mu,\sigma}\frac{\partial X^\nu_{(a)}}{\partial x^\theta}-k^\nu_{\theta,\sigma}\frac{\partial X^\theta_{(a)}}{\partial x^\mu}\right)\\
&\left.-\frac{\partial f^{(a)}}{\partial x^\sigma}X^\rho_{(a)}k^\nu_{\mu,\rho}f^{(a)}\frac{\partial X^\rho_{(a)}}{\partial x^\sigma}k^\nu_{\mu,\rho}\right)\frac{\partial\mathcal{L}_0}{\partial k^\nu_{\mu,\sigma}}+f^{(a)}\frac{\partial X^\mu_{(a)}}{\partial x^\mu}\mathcal{L}_0=0\,.\end{aligned}\tag{298}$$

Since the functions $f^{(a)}$ are arbitrary, we arrive at the following system of differential equations:

$$\textbf{a)}\ \ f^{(b)}:\ \ \left(C_b{}^a{}_c A^{(c)}_\mu-A^{(a)}_\theta\frac{\partial X^\theta_{(b)}}{\partial x^\mu}\right)\frac{\partial\mathcal{L}_0}{\partial A^{(a)}_\mu}+\left(k^\theta_\mu\frac{\partial X^\nu_{(b)}}{\partial x^\theta}-k^\nu_\theta\frac{\partial X^\theta_{(b)}}{\partial x^\mu}\right)\frac{\partial\mathcal{L}_0}{\partial k^\nu_\mu}$$

$$+\left(C_b{}^a{}_c A^{(c)}_{\mu,\nu} - A^{(a)}_{\theta,\nu}\frac{\partial X^\theta_{(b)}}{\partial x^\mu} - A^{(a)}_{\mu,\rho}\frac{\partial X^\rho_{(b)}}{\partial x^\nu} - A^{(a)}_\theta\frac{\partial^2 X^\theta_{(b)}}{\partial x^\mu \partial x^\nu}\right)\frac{\partial \mathcal{L}_0}{\partial A^{(a)}_{\mu,\nu}}$$
$$+\left(k^\theta_\mu\frac{\partial^2 X^\nu_{(b)}}{\partial x^\theta \partial x^\sigma} - k^\nu_\theta\frac{\partial^2 X^\theta_{(b)}}{\partial x^\mu \partial x^\sigma} + k^\theta_{\mu,\sigma}\frac{\partial X^\nu_{(b)}}{\partial x^\theta} - k^\nu_{\theta,\sigma}\frac{\partial X^\theta_{(b)}}{\partial x^\mu}\right.$$
$$\left.-k^\nu_{\mu,\rho}\frac{\partial X^\rho_{(b)}}{\partial x^\sigma}\right)\frac{\partial \mathcal{L}_0}{\partial k^\nu_{\mu,\sigma}} + \mathcal{L}_0\frac{\partial X^\mu_{(a)}}{\partial x^\mu} = 0 \tag{299}$$

b) $\dfrac{\partial f^{(b)}}{\partial x^\theta}$: $(\delta^a_b - A^{(a)}_\nu X^\nu_{(b)})\dfrac{\partial \mathcal{L}_0}{\partial A^{(a)}_\theta} + k^\theta_\mu X^\nu_{(b)}\dfrac{\partial \mathcal{L}_0}{\partial k^\nu_\mu}$

$$+\left(\delta^\theta_\nu C_b{}^a{}_c A^{(c)}_\mu - \delta^\theta_\mu A^{(a)}_\rho\frac{\partial X^\rho_{(b)}}{\partial x^\nu} - \delta^\theta_\nu A^{(a)}_\rho\frac{\partial X^\rho_{(b)}}{\partial x^\mu} - \delta^\theta_\mu A^{(a)}_{\rho,\nu}X^\rho_{(b)}\right.$$
$$\left.-\delta^\theta_\nu A^{(a)}_{\mu,\rho}X^\rho_{(b)}\right)\frac{\partial \mathcal{L}_0}{\partial A^{(a)}_{\mu,\nu}} + \left(k^\theta_\mu\frac{\partial X^\nu_{(b)}}{\partial x^\sigma} + \delta^\theta_\sigma\left(k^\rho_\mu\frac{\partial X^\nu_{(b)}}{\partial x^\rho} - k^\nu_\rho\frac{\partial X^\rho_{(b)}}{\partial x^\mu}\right)\right.$$
$$\left.+k^\theta_{\mu,\sigma}X^\nu_{(b)} - \delta^\theta_\sigma k^\nu_{\mu,\rho}X^\rho_{(b)}\right)\frac{\partial \mathcal{L}_0}{\partial k^\nu_{\mu,\sigma}} = 0 \tag{300}$$

c) $\dfrac{\partial^2 f^{(b)}}{\partial x^\theta}\partial x^\sigma$: $(\delta^a_b + A^{(a)}_\rho X^\rho_{(b)})\left(\dfrac{\partial \mathcal{L}_0}{\partial A^{(a)}_{\theta,\sigma}} + \dfrac{\partial \mathcal{L}_0}{\partial A^{(a)}_{\sigma,\theta}}\right) - k^\theta_\mu X^\nu_{(b)}\dfrac{\partial \mathcal{L}_0}{\partial k^\nu_{\mu,\sigma}}$

$$-k^\sigma_\mu X^\nu_{(b)}\frac{\partial \mathcal{L}_0}{\partial k^\nu_{\mu,\theta}} = 0\,. \tag{301}$$

Equation (299) establishes the invariance of $\mathcal{L}_0$ under the rigid group G, and using (301) and then (300), it is proven that $\mathcal{L}_0 = \mathcal{L}_0(\mathcal{T}^\sigma{}_{\mu\nu}, \mathcal{F}^{(a)}_{\mu\nu})$, so that the action for the compensating fields becomes:

$$\mathcal{S}_0 = \int L_0\,\omega \equiv \int \Lambda\mathcal{L}_0(\mathcal{T}^\sigma{}_{\mu\nu}, \mathcal{F}^{(a)}_{\mu\nu})\,\omega\,. \tag{302}$$

Note that the tensorial objects $\mathcal{T}^\sigma{}_{\mu\nu}, \mathcal{F}^{(a)}_{\mu\nu}$ naturally appear in the commutator of covariant derivatives:

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\varphi^\alpha = \mathcal{T}^\sigma{}_{\mu\nu}\mathcal{D}_\sigma\varphi^\alpha + \mathcal{F}^{(a)}_{\nu\mu}X^\alpha_{(a)\beta}\varphi^\beta\,,$$

and their tensorial character is manifest when they transform under the local group:

$$\delta\mathcal{T}^\sigma{}_{\mu\nu} = f^{(a)}(x)\left(\frac{\partial X^\sigma_{(a)}}{\partial x^\rho}\mathcal{T}^\rho{}_{\mu\nu} - \frac{\partial X^\rho_{(a)}}{\partial x^\mu}\mathcal{T}^\sigma{}_{\rho\nu} - \frac{\partial X^\rho_{(a)}}{\partial x^\nu}\mathcal{T}^\sigma{}_{\mu\rho}\right) \tag{303}$$
$$\delta\mathcal{F}^{(a)}_{\mu\nu} = f^{(b)}(x)\left(C_b{}^a{}_c\mathcal{F}^{(c)}_{\mu\nu} - \frac{\partial X^\rho_{(b)}}{\partial x^\mu}\mathcal{F}^{(a)}_{\rho\nu} - \frac{\partial X^\rho_{(b)}}{\partial x^\nu}\mathcal{F}^{(a)}_{\mu\rho}\right). \tag{304}$$

In terms of the fields $\{\mathcal{A}^{(a)}_\mu, k^\nu_\mu\}$, these objects are written in the form:

$$\mathcal{T}^\sigma{}_{\mu\nu} = T^\sigma{}_{\nu\mu} - \mathcal{A}^{(a)}_\mu\partial_\nu X^\sigma_{(a)} + \mathcal{A}^{(a)}_\nu\partial_\mu X^\sigma_{(a)}$$
$$\mathcal{F}^{(a)}_{\mu\nu} = \mathcal{A}^{(a)}_{\mu,\sigma}k^\sigma_\nu - \mathcal{A}^{(a)}_{\nu,\sigma}k^\sigma_\mu - \frac{1}{2}C_b{}^a{}_c(\mathcal{A}^{(b)}_\mu\mathcal{A}^{(c)}_\nu - \mathcal{A}^{(b)}_\nu\mathcal{A}^{(c)}_\mu) - \mathcal{A}^{(a)}_\sigma T^\sigma{}_{\mu\nu}\,. \tag{305}$$

□

9.1.1 Geometric interpretation

The objects q^{ν}_{μ} and the inverse, k^{μ}_{σ}, can be given the role of **tetrads**, so that we may define a **metric tensor** g in the form:

$$g_{\mu\nu} \equiv q^{\sigma}_{\mu} q^{\rho}_{\nu} \eta_{\sigma\rho} \ , \quad g^{\mu\nu} \equiv k^{\mu}_{\sigma} k^{\nu}_{\rho} \eta^{\sigma\rho} \ ,$$

where η is the Minkowski metric tensor.

We also may define a **connection** Γ **compatible** with the metric as:

$$\Gamma^{\sigma}{}_{\mu\nu} \equiv q^{\rho}_{\mu}(A^{(a)}_{\nu} \partial_{\rho} X^{\theta}_{(a)} k^{\sigma}_{\theta} - k^{\sigma}_{\rho,\nu}) \ . \tag{306}$$

The compatibility relies on the **metricity** condition $g_{\mu\nu;\sigma} = 0$, where the covariant derivative $;\mu$ is defined as an extension of D_{μ} such that

$$\varphi_{\rho;\nu} = D_{\nu} k^{\mu}_{\rho} + \Gamma^{\mu}_{\sigma\nu} k^{\sigma}_{\rho} \quad (D_{\nu} k^{\mu}_{\rho} = k^{\mu}_{\rho,\nu} - A^{(a)}_{\nu} \partial_{\rho} X^{\lambda}_{(a)} k^{\mu}_{\lambda}) \ . \tag{307}$$

This constitutes a **metric-affine** theory equipped with **curvature** and **torsion**:

$$R^{\rho}{}_{\sigma\mu\nu} \equiv \Gamma^{\rho}{}_{\sigma\mu,\nu} - \Gamma^{\rho}{}_{\sigma\nu,\mu} - \Gamma^{\rho}{}_{\lambda\mu} \Gamma^{\lambda}{}_{\sigma\nu} + \Gamma^{\rho}{}_{\lambda\nu} \Gamma^{\lambda}{}_{\sigma\mu} \tag{308}$$

$$\theta^{\sigma}{}_{\mu\nu} \equiv \Gamma^{\sigma}{}_{\mu\nu} - \Gamma^{\sigma}{}_{\nu\mu} \ , \tag{309}$$

which can be written in terms of the arguments of $\mathcal{L}_0$:

$$R^{\rho}{}_{\sigma\mu\nu} = k^{\rho}_{\theta} q^{\lambda}_{\sigma} q^{\omega}_{\mu} q^{\xi}_{\nu} \mathcal{F}^{(a)}_{\omega\xi} \partial_{\lambda} X^{\theta}_{(a)} \tag{310}$$

$$\theta^{\sigma}{}_{\mu\nu} = k^{\sigma}_{\theta} q^{\rho}_{\mu} q^{\lambda}_{\nu} \mathcal{T}^{\theta}{}_{\rho\lambda} \ . \tag{311}$$

(Γ is not the Levi-Civita connection associated with g, nor R is its curvature)

Equations of motion of k^{μ}_{ν} and $A^{(a)}_{\mu}$:

We start from the total Lagrangian $L_{tot} = \hat{L}_{matt} + L_0$, where

$$\hat{L}_{\text{matt}} = \Lambda \mathcal{L}_{\text{matt}}(\varphi^{\alpha}, k^{\nu}_{\mu}(\varphi^{\alpha}_{\nu} + A^{(a)}_{\mu} X^{\alpha}_{(a)\beta} \varphi^{\beta})) \tag{312}$$

$$L_0 = \Lambda \mathcal{L}_0(\mathcal{T}^{\sigma}{}_{\mu\nu}, \mathcal{F}^{(a)}_{\mu\nu}) \tag{313}$$

$$\Lambda \equiv det(q^{\nu}_{\mu}) \quad \left(\frac{\partial \Lambda}{\partial k^{\mu}_{\nu}} = -\Lambda q^{\nu}_{\mu}\right) . \tag{314}$$

The Euler–Lagrange equations of motion for $k,\ A$ are:

$$\begin{aligned} k^{\mu}_{\nu} \ : \ & 2\Lambda \frac{\partial \mathcal{L}_0}{\partial \mathcal{F}^{(a)}_{\nu\sigma}} k^{\lambda}_{\sigma} F^{(a)}_{\mu\lambda} - \Lambda \frac{\partial \mathcal{L}_0}{\partial \mathcal{T}^{\sigma}{}_{\rho\lambda}} q^{\sigma}_{\mu} T^{\nu}{}_{\lambda\rho} + 2\Lambda \frac{\partial \mathcal{L}_0}{\partial \mathcal{T}^{\sigma}{}_{\rho\nu}} q^{\sigma}_{\theta} k^{\lambda}_{\rho} \Gamma^{\theta}{}_{\lambda\mu} \\ & - \frac{d}{dx^{\sigma}} \left(2\Lambda \frac{\partial \mathcal{L}_0}{\partial \mathcal{T}^{\lambda}{}_{\rho\nu}} q^{\lambda}_{\mu} k^{\sigma}_{\rho} \right) - q^{\nu}_{\mu} \Lambda \mathcal{L}_0 = -\mathcal{T}^{\nu}_{\mu} \end{aligned} \tag{315}$$

$$A^{(a)}_{\mu} \ : \ 2\Lambda \frac{\partial \mathcal{L}_0}{\partial \mathcal{F}^{(e)}_{\sigma\rho}} k^{\mu}_{\rho} k^{\lambda}_{\sigma} C_{a}{}^{e}{}_{b} A^{(b)}_{\lambda} + 2\Lambda \frac{\partial \mathcal{L}_0}{\partial \mathcal{T}^{\theta}{}_{\sigma\rho}} k^{\mu}_{\rho} \partial_{\sigma} X^{\theta}_{(a)} - \frac{d}{dx^{\nu}} \left(2\Lambda \frac{\partial \mathcal{L}_0}{\partial \mathcal{F}^{(a)}_{\mu\nu}} \right) = \mathcal{S}^{\mu}_{(a)} \ , \tag{316}$$

where the matter currents are given by:

$$\mathcal{T}^{\nu}_{\mu} \equiv \frac{\partial \hat{L}_{matt}}{\partial k^{\mu}_{\nu}} = \Lambda q^{\sigma}_{\mu} \left(\frac{\partial \hat{\mathcal{L}}_{matt}}{\partial \mathcal{D}_{\nu} \varphi^{\alpha}} \mathcal{D}_{\sigma} \varphi^{\alpha} - \delta^{\nu}_{\sigma} \hat{\mathcal{L}}_{matt} \right) \tag{317}$$

$$\mathcal{S}^{\mu}_{(a)} \equiv -\frac{\partial \hat{L}_{matt}}{\partial A^{(a)}_{\mu}} = -\Lambda k^{\mu}_{\sigma} \frac{\partial \hat{\mathcal{L}}_{matt}}{\partial \mathcal{D}_{\sigma}\varphi^{\alpha}} X^{\alpha}_{(a)\beta}\varphi^{\beta} \,. \tag{318}$$

For the **special case** $L_0 = \Lambda \mathcal{L}_0(\mathcal{F}^{(a)}_{\mu\nu})$, the equation for k soundly simplifies and **generalizes General Relativity** equations for nonlinear Lagrangians:

$$\frac{\partial L_0}{\partial \mathcal{F}^{(a)}_{\nu\sigma}} k^{\lambda}_{\sigma} F^{(a)}_{\mu\lambda} - \frac{1}{2} q^{\nu}_{\mu} L_0 = -\frac{1}{2} \mathcal{T}^{\nu}_{\mu} \,. \tag{319}$$

Note that for the linear case $L_0 = \Lambda \mathcal{F}^{(\mu\nu)}_{\mu\nu}$, it becomes:

$$\mathcal{F}^{(\mu\sigma)}_{\nu\sigma} - \frac{1}{2}\delta^{\mu}_{\nu}\mathcal{F}^{(\rho\sigma)}_{\rho\sigma} = -\Lambda^{-1}\mathcal{T}^{\mu}_{\sigma} k^{\sigma}_{\nu} \,, \tag{320}$$

which looks very much like $R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu} R = T^{\mu}_{\nu}$!! ($T^{\mu}_{\nu} \equiv \mathcal{T}^{\mu}_{\nu}$).

Conservation laws: Given the matter Lagrangian $\mathcal{L}_{matt}(\varphi^{\alpha}, \varphi^{\beta}_{\mu})$, invariant under the rigid group G, we derive the general expression for conserved currents:

$$\mathcal{J}^{\mu(\mathrm{rigid})}_{(a)} \equiv -\frac{\partial \mathcal{L}_{\mathrm{matt}}}{\partial \varphi^{\alpha}_{\mu}} X^{\alpha}_{(a)\beta}\varphi^{\beta} + X^{\nu}_{(a)}\varphi^{\alpha}_{\nu}\frac{\partial \mathcal{L}_{\mathrm{matt}}}{\partial \varphi^{\alpha}_{\mu}} - X^{\mu}_{(a)}\mathcal{L}_{\mathrm{matt}} \,. \tag{321}$$

The local symmetry permits the construction of extra conservation laws for (or identities among) the currents above $\mathcal{T}^{\nu}_{\mu}$, $\mathcal{S}^{\mu}_{(a)}$:

$$\widetilde{\mathcal{T}}^{\mu}_{\nu;\mu} - \theta_{\mu}\widetilde{\mathcal{T}}^{\mu}_{\nu} + \theta^{\sigma}{}_{\nu\mu}\widetilde{\mathcal{T}}^{\mu}_{\sigma} = F^{(a)}_{\nu\sigma}\mathcal{S}^{\sigma}_{(a)} \tag{322}$$

$$\mathcal{S}^{\mu}_{(a);\mu} - \theta_{\mu}\mathcal{S}^{\mu}_{(a)} = \widetilde{\mathcal{T}}^{\mu}_{\sigma} q^{\nu}_{\mu}\partial_{\nu} X^{\rho}_{(a)} k^{\sigma}_{\rho} \,, \tag{323}$$

where $\widetilde{\mathcal{T}}^{\mu}_{\nu} \equiv k^{\mu}_{\rho}\mathcal{T}^{\rho}_{\nu}$, $\theta_{\mu} \equiv \theta^{\sigma}{}_{\mu\sigma}$.

9.2 Gauge Theory of Gravitation

In that which follows, we shall restrict ourselves to rigid groups G acting only on space-time (except for some attempt to gravitational mixing, to be briefly considered later).

There are many possibilities for the kinematical group G related to possible asymptotic symmetries of space-time: G may be Poincaré, de Sitter, Anti-de Sitter, Weyl (Poincaré + Dilatations) or even Conformal (SO(4,2)) group, apart from any **invariant** subgroup of them.

Note that we shall have to address more “gravitational fields” than those strictly required, so that many constraints among them must be handled.

9.2.1 Translations: (Teleparallelism)

We start with (and pay special attention to) the simplest case of the translation subgroup, $G = T^4$, of the Poincaré group. The group index (a) now reads (μ), the unbracketed indices $\mu, \nu, \sigma, \ldots$, representing coordinate ones.

The generators of the rigid translations have components $X^{\nu}_{(\mu)} = \delta^{\nu}_{\mu}$, $X^{\alpha}_{(\mu)\beta} = 0$, corresponding to

$$X_{(\mu)} = \frac{\partial}{\partial x^{\mu}} \,. \tag{324}$$

The local algebra then becomes (Non-Abelian):

$$[f^{(\mu)}X_{(\mu)},\ g^{(\nu)}X_{(\nu)}] = (f^{(\mu)}\partial_{\mu}g^{(\nu)} - g^{(\mu)}\partial_{\mu}f^{(\nu)})X_{(\nu)} \,. \tag{325}$$

We shall consider non-trivial compensating potentials $A_\mu^{(a)}$ even though the covariant derivative of the matter fields coincides with the ordinary one, that is, $D_\mu \varphi^\alpha = \varphi^\alpha_{,\mu}$, so that the generalized compensating derivative just becomes

$$\mathcal{D}_\mu \varphi^\alpha \equiv k^\nu_\mu \varphi^\alpha_{,\nu} \, . \tag{326}$$

Keeping $A_\nu^{(\mu)}$ will prove relevant, in particular, in mixing gravity and internal interactions (see later), although it should not represent an increased number of degrees of freedom. We expect the need for some (natural) constraints.

According to the general scheme, $\mathcal{L}_0$ must be an (scalar under the rigid group) arbitrary function of $\mathcal{T}^\sigma{}_{\mu\nu}$, $\mathcal{F}^{(\sigma)}_{\mu\nu}$, which now acquire the expression:

$$\mathcal{T}^\sigma{}_{\mu\nu} = T^\sigma{}_{\nu\mu} \ \ ; \ \ \mathcal{F}^{(\sigma)}_{\mu\nu} = k^\lambda_\mu k^\rho_\nu (A^{(\sigma)}_{\lambda,\rho} - A^{(\sigma)}_{\rho,\lambda}) \, . \tag{327}$$

If we assume the constraint (compatible with the equations of motion)

$$A^{(\sigma)}_\mu = \delta^\sigma_\mu + q^\sigma_\mu \, , \tag{328}$$

we obtain

$$\mathcal{T}^\sigma{}_{\mu\nu} = \mathcal{F}^{(\sigma)}_{\mu\nu} = T^\sigma{}_{\nu\mu} \, , \tag{329}$$

and, therefore,

$$L_0 = \Lambda \mathcal{L}_0 (T^\sigma{}_{\nu\mu}) \, . \tag{330}$$

Resulting Geometry (Weitzenbock space-time): The connection, curvature and torsion, all indexed by the superscript T^4, are:

$$\Gamma^{\sigma(T(4))}_{\mu\nu} \equiv -q^\rho_\mu k^\sigma_{\rho,\nu} = k^\sigma_\rho q^\rho_{\mu,\nu} \tag{331}$$

$$R^{\sigma(T(4))}_{\mu\rho\nu} = 0 \qquad \text{(null curvature)} \tag{332}$$

$$\theta^{\sigma(T(4))}_{\mu\nu} = k^\sigma_\rho q^\lambda_\mu q^\xi_\nu T^\rho{}_{\lambda\xi} \quad \text{(pure torsion !!)} \, . \tag{333}$$

On the other hand, we have at our disposal the metric tensor $g_{\mu\nu}$ in terms of which a Levi-Civita connection $\Gamma^{(L-C)}$ can be constructed, as well as the corresponding curvature $R^{(L-C)}$, that is:

$$g_{\mu\nu} \equiv q^\rho_\mu q^\sigma_\nu \eta_{\rho\sigma} \, , \Gamma^{\sigma(L-C)}_{\mu\nu} \equiv \frac{1}{2} g^{\sigma\rho} (g_{\rho\nu,\mu} + g_{\rho\mu,\nu} - g_{\mu\nu,\rho}) \, , \tag{334}$$

which is symmetric in μ and ν and, therefore, provides null torsion, although non-trivial curvature tensor $R^{\sigma(\Gamma^{(L-C)})}_{\mu\rho\nu}$. The relationship between $\Gamma^{\sigma(T(4))}_{\mu\nu}$ and $\Gamma^{\sigma(L-C)}_{\mu\nu}$ is:

$$\Gamma^{\sigma(T(4))}_{\mu\nu} = \Gamma^{\sigma(L-C)}_{\mu\nu} + \mathcal{K}^\sigma{}_{\mu\nu} \, , \tag{335}$$

where

$$\mathcal{K}^\sigma{}_{\mu\nu} \equiv \frac{1}{2} (\theta^{\sigma(T(4))}_{\mu\nu} - \theta_\mu{}^{\sigma(T(4))}{}_\nu - \theta_\nu{}^{\sigma(T(4))}{}_\mu) \, , \tag{336}$$

with $\theta_\mu{}^{\sigma(T(4))}{}_\nu \equiv \theta^{\lambda(T(4))}_{\tau\nu} g_{\lambda\mu} g^{\tau\sigma}$.

Teleparallelism: The "gauge" symmetry, through Utiyama's Theorem, is only able to fix the argument of $\mathcal{L}_0$ as the Cartan Torsion, but the actual functional expression still remains to be determined. Among all possible Lagrangians there is one that reproduces the Hilbert–Einstein Lagrangian (except for a total derivative). This Lagrangian is called Teleparallelism Lagrangian, and is given by:

$$L_0^{(Tel)} \equiv \Lambda \mathcal{L}_0^{(Tel)} \equiv \Lambda T^{\mu}_{\ \nu\sigma} T^{\rho}_{\ \lambda\theta} \left(\frac{1}{4} \eta^{\lambda\nu} \eta^{\sigma\theta} \eta_{\mu\rho} + \frac{1}{2} \delta^{\theta}_{\mu} \eta^{\nu\lambda} \delta^{\sigma}_{\rho} - 1 \delta^{\sigma}_{\mu} \delta^{\theta}_{\rho} \eta^{\nu\lambda} \right), \quad (337)$$

where the numerical coefficients have been determined by hand in order to achieve our purpose, that is:

$$L_0^{(Tel)} = \sqrt{-g} R^{(L-C)} + \partial_\mu (2\Lambda \theta_\sigma^{\ \mu\nu}) \ \ , \ \theta_\sigma^{\ \mu\nu} = g^{\mu\nu} \theta^{\sigma}_{\ \lambda\sigma} \, . \quad (338)$$

It must be noticed that the equations of motion of a particle, derived from the Gauge Theory are:

$$\frac{\mathrm{d}u_\mu}{\mathrm{d}\tau} = \Gamma^{\sigma(T(4))}_{\nu\mu} u_\sigma u^\nu \, , \quad (339)$$

and it turns out to be equivalent to those of geodesic motion in the pseudo-Riemannian geometry addressed by $\Gamma^{(L-C)}$:

$$\frac{\mathrm{d}u_\mu}{\mathrm{d}\tau} = \Gamma^{\sigma(L-C)}_{\nu\mu} u_\sigma u^\nu \, , \quad (340)$$

although the formers do not correspond to a geodesic motion.

9.2.2 The Poincaré Group

In the case G is the Poincaré group, the index (a) splits in (μ) for the translation subgroup, and $(\mu\nu)$ for the Lorentz one. The generators of the rigid group are:

$$X_{(\mu)} = X^{\nu}_{(\mu)} \frac{\partial}{\partial x^\nu} \quad (341)$$

$$X_{(\mu\nu)} = X^{\sigma}_{(\mu\nu)} \frac{\partial}{\partial x^\sigma} + X^{\alpha}_{(\mu\nu)} \frac{\partial}{\partial \varphi^\alpha} \quad (342)$$

with

$$\begin{aligned} X^{\nu}_{(\mu)} &= \delta^{\nu}_{\mu} \\ X^{\sigma}_{(\mu\nu)} &= \delta^{\sigma}_{(\mu\nu),\rho} x^\rho \equiv (\delta^{\sigma}_{\mu} \eta_{\nu\rho} - \delta^{\sigma}_{\nu} \eta_{\mu\rho}) x^\rho \\ X^{\alpha}_{(\mu\nu)} &= X^{\alpha}_{(\mu\nu)\beta} \varphi^\beta \quad (\sim [\gamma_\mu, \gamma_\nu]^{\alpha}_{\beta} \varphi^\beta) \, . \end{aligned} \quad (343)$$

According to the general theory, $\mathcal{L}_0 = \mathcal{L}_0(\mathcal{T}^{\sigma}_{\mu\nu}, \mathcal{F}^{(\sigma)}_{\mu\nu}, \mathcal{F}^{(\sigma\rho)}_{\mu\nu})$. The simplest possibility corresponds to the particular choice $\mathcal{L}_0 = \mathcal{L}_0(\mathcal{F}^{(\sigma\rho)}_{\mu\nu})$,

$$\mathcal{L}_0 = \frac{1}{2} \Lambda \mathcal{F}^{(\mu\nu)}_{\mu\nu} \quad (344)$$

$$\begin{aligned} \mathcal{F}^{(\sigma\rho)}_{\lambda\theta} &\equiv k^{\mu}_{\lambda} k^{\nu}_{\theta} F^{(\sigma\rho)}_{\mu\nu} \\ &\equiv k^{\mu}_{\lambda} k^{\nu}_{\theta} \left(A^{(\sigma\rho)}_{\mu,\nu} - A^{(\sigma\rho)}_{\nu,\mu} - (A^{(\sigma\kappa)}_{\mu} A^{(\xi\rho)}_{\nu} - A^{(\sigma\kappa)}_{\nu} A^{(\xi\rho)}_{\mu}) \eta_{\kappa\xi} \right) . \end{aligned} \quad (345)$$

The equations of motion become:

$$k) \quad \Lambda (\mathcal{F}^{(\mu\sigma)}_{\nu\sigma} - \frac{1}{2} \delta^{\mu}_{\nu} \mathcal{F}^{(\rho\sigma)}_{\rho\sigma}) = -\mathcal{T}^{\mu}_{\sigma} k^{\sigma}_{\nu} \quad (346)$$

$$
\begin{aligned}
A)\quad & \Lambda(k^{\mu}_{\theta}T^{\theta}{}_{\rho\sigma} - k^{\mu}_{\rho}T^{\theta}{}_{\theta\sigma} + k^{\mu}_{\sigma}T^{\theta}{}_{\theta\rho} + (k^{\mu}_{\theta}k^{\nu}_{\rho} - k^{\mu}_{\rho}k^{\nu}_{\theta})A^{(\theta}{}_{\sigma)\nu} \\
& -(k^{\mu}_{\sigma}k^{\nu}_{\theta} + k^{\mu}_{\theta}k^{\nu}_{\sigma})A^{(\theta}{}_{\rho)\nu}) = 2\mathcal{S}^{\mu}_{(\sigma\rho)}\,, \qquad (347)
\end{aligned}
$$

where

$$
\mathcal{T}^{\mu}_{\sigma} \equiv \frac{\partial}{\partial k^{\sigma}_{\mu}}(\Lambda\mathcal{L}_{\mathrm{mat}}(\varphi^{\alpha}, \mathcal{D}_{\nu}\varphi^{\alpha})) \qquad (348)
$$

$$
\mathcal{S}^{\mu}_{(\sigma\rho)} \equiv -\frac{\partial}{\partial A^{(\sigma\rho)}_{\mu}}(\Lambda\mathcal{L}_{\mathrm{mat}}(\varphi^{\alpha}, \mathcal{D}_{\nu}\varphi^{\alpha})) \qquad (349)
$$

are the matter currents already defined.

In terms of $\mathcal{T}^{\theta}{}_{\sigma\rho}$, the equation associated with A is written as:

$$
\Lambda(k^{\mu}_{\theta}\mathcal{T}^{\theta}{}_{\sigma\rho} - k^{\mu}_{\rho}\mathcal{T}^{\theta}{}_{\sigma\theta} - k^{\mu}_{\sigma}\mathcal{T}^{\theta}{}_{\theta\rho}) = 2\mathcal{S}^{\mu}_{(\sigma\rho)}\,, \qquad (350)
$$

from which it follows the conservation law:

$$
\frac{d}{\mathrm{d}x^{\mu}}(\mathcal{S}^{\mu}_{(\sigma\rho)} + \int^{\mu}_{(\sigma\rho)}) = 0\,, \quad \int^{\mu}_{(\sigma\rho)} \equiv -\frac{\partial\mathcal{L}_0}{\partial A^{(\sigma\rho)}_{\mu}}\,. \qquad (351)
$$

This means the conservation of the total spin density, $\mathcal{S}^{\mu}_{(\sigma\rho)}$ corresponding to the matter, and $\int^{\mu}_{(\sigma\rho)}$ corresponding to the gravitational fields itself.

Riemann–Cartan Geometry: We have again two geometric objects at our disposal. On the one hand, the metric tensor $g_{\mu\nu} \equiv q^{\rho}_{\mu}q^{\sigma}_{\nu}\eta_{\rho\sigma}$ and, on the other, the connection associated with the Poincaré gauge group:

$$
\begin{aligned}
\Gamma^{\sigma(P)}_{\mu\nu} &\equiv q^{\rho}_{\mu}\left(\frac{1}{2}A^{(\lambda\omega)}_{\nu}\partial_{\rho}X^{\theta}_{(\lambda\omega)}k^{\sigma}_{\theta} - k^{\sigma}_{\rho,\nu}\right) = -q^{\rho}_{\mu}k^{\sigma}_{\rho,\nu} + q^{\rho}_{\mu}A^{(\theta\lambda)}_{\nu}\eta_{\lambda\rho}k^{\sigma}_{\theta} \\
&= \Gamma^{\sigma(T(4))}_{\mu\nu} + q^{\rho}_{\mu}A^{(\theta\lambda)}_{\nu}\eta_{\lambda\rho}k^{\sigma}_{\theta}\,, \qquad (352)
\end{aligned}
$$

and the corresponding curvature and torsion:

$$
R^{\rho(P)}{}_{\sigma\mu\nu} = k^{\rho}_{\theta}q^{\lambda}_{\sigma}q^{\omega}_{\mu}q^{\xi}_{\nu}\frac{1}{2}\mathcal{F}^{(\kappa\zeta)}_{\omega\xi}\partial_{\lambda}X^{\theta}_{(\kappa\zeta)} = k^{\rho}_{\theta}q^{\lambda}_{\sigma}q^{\omega}_{\mu}q^{\xi}_{\nu}\mathcal{F}^{(\theta\zeta)}_{\omega\xi}\eta_{\zeta\lambda} \qquad (353)
$$

$$
\begin{aligned}
\theta^{\sigma(P)}{}_{\mu\nu} &= k^{\sigma}_{\theta}q^{\rho}_{\mu}q^{\lambda}_{\nu}\mathcal{T}^{\theta(P)}{}_{\rho\lambda} \\
&= k^{\sigma}_{\theta}q^{\rho}_{\mu}q^{\lambda}_{\nu}(T^{\theta}{}_{\lambda\rho} + A^{(\theta\zeta)}_{\kappa}(k^{\kappa}_{\lambda}\eta_{\zeta\rho} - k^{\kappa}_{\rho}\eta_{\zeta\lambda}))\,. \qquad (354)
\end{aligned}
$$

Omitting the (P) superscript and using the following derived currents:

$$
\widetilde{T}_{\mu\nu} \equiv g_{\sigma\mu}\widetilde{T}^{\sigma}_{\nu} \equiv g_{\sigma\mu}k^{\sigma}_{\rho}T^{\rho}_{\nu} = q^{\sigma}_{\mu}\eta_{\rho\sigma}\mathcal{T}^{\rho}_{\nu} \qquad (355)
$$

$$
\widetilde{\mathcal{S}}^{\lambda}_{\mu\nu} \equiv q^{\sigma}_{\mu}q^{\rho}_{\nu}\mathcal{S}^{\lambda}_{(\sigma\rho)}\,, \qquad (356)
$$

the equations of motion acquire the form:

$$
\Lambda\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = -\widetilde{T}_{\mu\nu}\,, \qquad (357)
$$

$$
\Lambda\theta^{\lambda}{}_{\mu\nu} = 2\widetilde{\mathcal{S}}^{\lambda}_{\mu\nu} - \delta^{\lambda}_{\mu}\widetilde{\mathcal{S}}^{\rho}_{\rho\nu} - \delta^{\lambda}_{\nu}\widetilde{\mathcal{S}}^{\rho}_{\mu\rho}\,, \qquad (358)
$$

from which we conclude that the source for the torsion is the spin of the matter.

Comparison with the standard theory: We shall limit ourselves to the case of absence of matter. In the vacuum case, the equation of motion for the field $A_\mu^{(\sigma\rho)}$, becomes:

$$k_\theta^\mu \mathcal{T}^\theta_{\ \sigma\rho} - k_\rho^\mu \mathcal{T}^\theta_{\ \sigma\theta} - k_\sigma^\mu \mathcal{T}^\theta_{\ \theta\rho} = 0 \tag{359}$$

and can be solved explicitly in terms of the Cartan torsion $T^\sigma_{\ \mu\nu}$:

$$A^{\text{vacuum}}_{(\sigma\rho)\mu} = \frac{1}{2} q_\mu^\lambda (T_{\sigma\rho\lambda} + T_{\rho\lambda\sigma} - T_{\lambda\sigma\rho})\,, \tag{360}$$

with $A^{\text{vacuum}}_{(\sigma\rho)\mu} \equiv A_\mu^{(\lambda\theta)\text{vacuum}} \eta_{\lambda\sigma}\eta_{\theta\rho}$, $T_{\sigma\rho\lambda} \equiv T^\mu_{\ \rho\lambda}\eta_{\mu\sigma}$, that is to say, $A^{\text{vacuum}}_{(\sigma\rho)\mu}$ are the so-called Ricci rotation coefficients in the standard theory.

Then, in the vacuum, we arrive at

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{361}$$

$$\theta^\lambda_{\ \mu\nu} = 0\,. \tag{362}$$

The second equation implies that $\Gamma^\sigma_{\ \mu\nu}$ is symmetric and, therefore, it coincides with the Levi-Civita connection associated with $g_{\mu\nu}$. Likewise, $R_{\mu\nu}$ coincides with the Ricci tensor providing the ordinary Einstein equations.

Remark on the "gauge theory" of the Lorentz group: The Lorentz group is not an invariant subgroup of the Poincaré group and if we desire to keep the rigid invariance under the whole Poincaré group, making local the Lorentz subgroup entails necessarily the local character of the Translation subgroup and, then, of the total Poincaré group.

9.3 Beyond the Poincaré group as rigid symmetry

Naively, the more natural generalization of the Poincaré group as the starting rigid symmetry is the group $GL(4, R)$, which had been considered in Literature long ago. It leads to Edington Geometry. The simplest and best motivated generalization is that addressed by the Weyl group, made of Poincaré and Dilatations transformations.

Even more interesting proves to be the generalization of GR combining the Weyl group with the mass-generating scheme, discussed above, giving dynamics to only the field associated with the dilatation parameter [56,57]. This constitutes some sort of

"Stueckelberg" model for the Weyl group (Brief comments): We consider the Weyl group as G and start from a very special "matter" Lagrangian constituted by the partial-trace σ-Lagrangian associated with the dilatation subgroup of W. That is to say:

$$\mathcal{L}_{\text{“}matt''} = \mathrm{Tr}_{W/P}(\theta_\mu \theta^\mu) \equiv \theta_\mu^{(dil)} \theta^{(dil)\mu}\,. \tag{363}$$

The minimal coupling principle entails the minimal substitution:

$$\hat{\mathcal{L}}_{\text{“}matt''} = (\theta_\mu^{(dil)} - A_\mu^{(dil)})(\theta_\nu^{(dil)} - A_\nu^{(dil)})\eta^{\mu\nu}\,, \tag{364}$$

where $\theta_\mu^{(dil)}$ is just $\partial_\mu \varphi^{dil}$.

As far as the Lagrangian $\mathcal{L}_0$ is concerned, we resort to the simplest, yet new possibility:

$$\begin{aligned}\mathcal{L}_0 &= \mathcal{L}_0(\mathcal{F}_{\mu\nu}^{(\sigma\rho)}, \mathcal{F}_{\mu\nu}^{(dil)}) \\ &= \mathcal{F}_{\mu\nu}^{(\mu\nu)} + \mathcal{F}_{\mu\nu}^{(dil)}\mathcal{F}_{\sigma\rho}^{(dil)}\eta^{\mu\sigma}\eta^{\nu\rho} = k_\mu^\sigma k_\nu^\rho F^{(\mu\nu)} + F_{\mu\nu}^{(dil)} F_{\sigma\rho}^{(dil)} g^{\mu\sigma} g^{\nu\rho}\,,\end{aligned} \tag{365}$$

where $F^{(dil)}_{\sigma\rho} = \partial_\rho A^{(dil)}_\sigma - \partial_\sigma A^{(dil)}_\rho$.
Note that we have chosen a Lagrangian linear on $\mathcal{F}^{(\rho\sigma)}_{\mu\nu}$, as in standard Gravity, but quadratic on $\mathcal{F}^{(dil)}_{\mu\nu}$, as in Electromagnetism.

The equations of motion for k turn out to acquire the expression:

$$F^{(\nu\sigma)}_{\mu\sigma} - \frac{1}{2}\delta^\nu_\mu F^{(\sigma\lambda)}_{\sigma\lambda} = T^{(dil)\nu}_\mu\,, \tag{366}$$

where the right-hand side is the energy-momentum tensor for some sort of dark energy:

$$T^{(dil)\nu}_\mu \equiv -F^{(dil)\nu}_\sigma F^{(dil)\sigma}_\mu + \frac{1}{2}\delta^\nu_\mu F^{(dil)}_{\sigma\lambda} F^{(dil)\sigma\lambda}\,. \tag{367}$$

Exercise: Is there any configuration for $A^{(dil)}_\mu$ allowing for a cosmological constant term?

9.4 Extending Diffeomorphism invariance: New approach to Teleparallelism

We shall mimic the extension of the gauge group $G(M)$, $G^1(M)$, which gave rise to non-trivial symmetries, that is, symmetries with non-null Noether invariants.

Let us remember that $G^1(M)$ was constructed out of $J^1(G(M))$, the group of 1-jets of the mappings $\varphi^\alpha : M \to G$, the local group.

Now, the role of the gauge group is played by $T^4(M)$ or, roughly speaking, Diff(M). This group is gauge, in the strict sense that the corresponding Noether invariants are trivial, except for the subgroup of "rigid transformations," which give rise to quantities like energy or angular momentum.

We then define, in an analogous way to the case of jet bundle of Variational Calculus, the 1-jets of the diffeomorphisms of M, considered as mappings $\xi : M \to M$:

$$J^1(\mathrm{Diff}(M)) \equiv \frac{\mathrm{Diff}(M)\times M}{\overset{1}{\sim}}\,, \tag{368}$$

where the equivalence $\overset{1}{\sim}$ is defined by (to be compared with (89)):

$$(\xi^\mu, x) \overset{1}{\sim} (\xi'^\mu, x') \Longleftrightarrow \begin{cases} x = x' \\ \xi^\mu(x) = \xi'^\mu(x) \\ \partial_\nu\xi^\mu(x) = \partial_\nu\xi'^\mu(x), \end{cases} \tag{369}$$

$\forall\, (\xi^\mu, x),\ (\xi'^\mu, x') \in \mathrm{Diff}(M)\times M$.

A coordinate system on $J^1(\mathrm{Diff}(M))$ is $(x^\mu, \xi^\mu, \xi^\mu_\nu)$, where the objects ξ^μ_ν parameterize those transformations on $T(M)$ which are non-necessarily the tangent mapping of a transformation ξ on M; that is, $\xi^\mu_\nu \neq \xi^\mu_{,\nu}$, except for the jet extensions of ξ, $j^1(\xi)$, for which $\xi^\mu_\nu = \partial_\nu\xi^\mu$.

The relevant, infinite-dimensional, symmetry group consists in the "local" $J^1(\mathrm{Diff}(M))$ group:

$$\mathrm{Diff}^1(M) \equiv \Gamma(J^1(\mathrm{Diff}(M))) = \{M \to J^1(\mathrm{Diff}(M))\}\,. \tag{370}$$

It contains Diff(M), as jet extensions, in a natural way!!. In fact, any generator in the Lie algebra diff(M), $X_f = f^\mu(x)\frac{\partial}{\partial x^\mu}$, can be canonically lifted to diff$^1(M)$:

$$X^{\mathrm{Lift}}_f = f^\mu\frac{\partial}{\partial x^\mu} + \partial_\rho f^\mu \xi^\rho_\nu \frac{\partial}{\partial \xi^\mu_\nu}\,, \tag{371}$$

in such a way that the Lie algebra commutator $[X_f, X_g] = (f^\mu \partial_\mu g^\nu - g^\mu \partial_\mu f^\nu)\partial_\nu$, is preserved, that is:

$$[X_f, X_g]^{\mathrm{Lift}} = [X_f^{\mathrm{Lift}},\ X_g^{\mathrm{Lift}}]\ .\ !! \tag{372}$$

9.4.1 Invariance under Diff(M): Standard Gauge Symmetry

Before going into the new, extended symmetry let us recover the standard Teleparallelism Theory from Diff(M).

We could seek Lagrangians $\mathcal{L}_0(x^\mu, \xi^\nu, \xi^\sigma_\rho, \partial_\mu \xi^\nu, \partial_\mu \xi^\sigma_\rho)$, invariant under Diff$(M)$ although the dependence on $\{\xi^\nu, \partial_\mu \xi^\nu\}$ can be dropped out for the sake of simplicity. We then look for Lagrangians $\mathcal{L}(x^\mu, \xi^\mu_\nu, \xi^\mu_{\nu,\sigma})$ invariant under the jet extension (in the sense of variational calculus) of the lifted X_f^{Lift}, that is

$$\bar{X}_f^{\mathrm{Lift}} \mathcal{L} = 0\ . \tag{373}$$

Explicitly,

$$\begin{aligned} & f^\mu(x) \frac{\partial \mathcal{L}}{\partial x^\mu} + \partial_\rho f^\mu(x) \xi^\rho_\nu \frac{\partial \mathcal{L}}{\partial \xi^\mu_\nu} \\ & \quad + \left(\xi^\rho_\nu \partial_\sigma \partial_\rho f^\mu(x) + \xi^\rho_{\nu,\sigma} \partial_\rho f^\mu(x) - \xi^\mu_{\nu,\rho} \partial_\sigma f^\rho(x)\right) \frac{\partial \mathcal{L}}{\partial \xi^\mu_{\nu,\sigma}} = 0 \end{aligned} \tag{374}$$

$\Rightarrow$

$$a)\ \ \partial_\nu f^\mu(x):\quad \xi^\nu_\epsilon \frac{\partial \mathcal{L}}{\partial \xi^\mu_\epsilon} + (\xi^\nu_{\epsilon,\sigma} \delta^\rho_\mu - \xi^\rho_{\epsilon,\mu} \delta^\nu_\sigma) \frac{\partial \mathcal{L}}{\partial \xi^\rho_{\epsilon,\sigma}} = 0 \tag{375}$$

$$b)\ \ \partial_\sigma \partial_\nu f^\mu(x):\quad \xi^\nu_\epsilon \frac{\partial \mathcal{L}}{\partial \xi^\mu_{\epsilon,\sigma}} + \xi^\sigma_\epsilon \frac{\partial \mathcal{L}}{\partial \xi^\mu_{\epsilon,\nu}} = 0\ . \tag{376}$$

b) $\Rightarrow$ the Lagrangian $\mathcal{L}_0$ must depend on $(\xi^\mu_\nu, \xi^\mu_{\nu,\sigma})$ only through the combination

$$\tau^\rho_{\mu\nu} \equiv \xi^\rho_{\nu,\theta} \xi^\theta_\mu - \xi^\rho_{\mu,\theta} \xi^\theta_\nu\ . \tag{377}$$

Then, a) acquires the form

$$\xi^\nu_\lambda \xi^\rho_{\theta,\mu} \frac{\partial \mathcal{L}}{\partial \tau^\rho_{\lambda\theta}} + \xi^\sigma_\theta (\xi^\nu_{\lambda,\sigma} \delta^\rho_\mu - \xi^\rho_{\lambda,\mu} \delta^\nu_\sigma) \frac{\partial \mathcal{L}}{\partial \tau^\rho_{\theta\lambda}} = 0\ \Rightarrow \tag{378}$$

$$\mathcal{L}_0 = \mathcal{L}_0(T^\sigma{}_{\mu\nu}) \tag{379}$$

$$T^\sigma{}_{\mu\nu} \equiv \zeta^\sigma_\rho \tau^\rho_{\mu\nu} = \zeta^\sigma_\rho (\xi^\rho_{\nu,\theta} \xi^\theta_\mu - \xi^\rho_{\mu,\theta} \xi^\theta_\nu)\ \ \begin{cases} \zeta^\sigma_\rho \xi^\rho_\nu = \delta^\sigma_\nu \\ \zeta^\sigma_\rho \xi^\mu_\sigma = \delta^\mu_\rho\ . \end{cases} \tag{380}$$

Using $L_0 \equiv \Lambda \mathcal{L}_0$, $\Lambda \equiv \det(\theta^\sigma_\rho)$, we arrive at exactly the same situation as in the gauge theory of T^4 with the trivial identification

$$\xi^\mu_\nu \equiv k^\mu_\nu\ ,\ \theta^\nu_\mu \equiv q^\nu_\mu\ , \tag{381}$$

allowing for the special choice for the Lagrangian, $\mathcal{L}_0^{(Tel)}$.

However, the actual form of $\mathcal{L}$ still remains to be determined by a symmetry group.

9.4.2 *Invariance under Diff$^1(M)$: Einstein Theory in vacuum*

Let us resort to the additional symmetry in the group Diff$^1(M)$ which is not the jet extension of Diff(M). Among the possible generators of this type of symmetry we shall select the following set:

$$X_l^1 \equiv l_\nu{}^\sigma(x)\xi^\mu_\sigma \frac{\partial}{\partial \xi^\mu_\nu} \ , \tag{382}$$

where $l^\sigma_\nu(x)$ are "infinitesimal" parameters that are not the derivative of diffeomorphisms and satisfy $l_{\mu\nu}(x) = -l_{\nu\mu}(x),\ l_{\mu\nu} = \eta_{\mu\sigma} l^\sigma_\nu$.

The imposition of invariance under Diff(M) has already been done with the result that $\mathcal{L}_0 = \mathcal{L}_0(T)$, though arbitrary. Now we impose the rest of the symmetry in two steps:

1) Invariance under the rigid $X^1_{l(\text{global})}$, that is, with constant $l's$

$$\bar{X}^1_{l(\text{global})}\mathcal{L}_0 = l_\nu{}^\sigma \left(\xi^\mu_\sigma \frac{\partial \mathcal{L}}{\partial \xi^\mu_\nu} + \xi^\mu_{\sigma,\rho} \frac{\partial \mathcal{L}}{\partial \xi^\mu_{\nu,\rho}} \right) = 0 \tag{383}$$

The simplest solution is

$$\mathcal{L}_0^{T^2}(T) = A T^\sigma{}_{\mu\nu} T_\sigma{}^{\mu\nu} + B T^\sigma{}_{\mu\nu} T^{\nu\mu}{}_\sigma + C T^\sigma{}_{\sigma\mu} T_\nu{}^{\nu\mu}$$
$$\text{(indices move with the metric) } \eta \tag{384}$$

2) Invariance under local $X^1_{l(x)}$ fixes A, B, C, although we must demand only semi-invariance (just like in the free Galilean particle):

$$\bar{X}^1_l(\Lambda \mathcal{L}_0^{T^2}) = \Lambda \partial_\sigma l_\nu{}^\rho \xi^\mu_\rho \frac{\partial \mathcal{L}_0^{T^2}}{\partial \xi^\mu_{\nu,\sigma}} = \text{div}\lambda_l \ . \tag{385}$$

Equation (385) can be explicitly solved, giving:

$$A = \frac{B}{2} \ , \ B = -\frac{C}{2} \ , \ \lambda^\mu_l = -4C\Lambda \xi^\mu_\nu \partial_\sigma l^{\sigma\nu} \ . \tag{386}$$

By choosing $C = -1$, we arrive at a Lagrangian equivalent (up to a total derivative) to the Hilbert–Einstein Lagrangian associated with the metric $g_{\mu\nu} \equiv \zeta^\sigma_\mu \zeta^\rho_\nu \eta_{\sigma\rho}$.

As an extra bonus, the extended Diff$^1(M)$ symmetry provides infinitely many non-null Noether invariants:

$$J^\mu_{X^1_l} = \Lambda l^\sigma_\nu(x)(\xi^\mu_\rho T^\rho_\sigma{}^\nu - 2\xi^\mu_\sigma T^\rho_\rho{}^\nu) - \lambda^\mu_l \ . \tag{387}$$

Final comments: We have got an infinite set of Noether invariants defining coordinates on the solution manifold of Einstein equations in the vacuum. The completeness of this set is still lacking, but it is worth noticing the similarity of the generators $X^1_{l(x)}$ with those used in the Klein–Gordon field to parameterize its solution manifold.

In fact, there, the generators $X_{a^*(x)} = e^{ikx}\frac{\partial}{\partial \phi}$, provided the Noether invariants $a(k)$ (and $X_{a(x)} \to a^*(k)$), and they can be viewed as $X_l = l(x)\frac{\partial}{\partial \phi}$, where $l(x)$ is a solution of the Klein–Gordon equation. Comparing with $X^1_l \equiv l_\nu{}^\sigma(x)\xi^\mu_\sigma \frac{\partial}{\partial \xi^\mu_\nu}$, l^σ_ν are suggested to be solutions of Einstein equations.

The question naturally arises of whether a particular solution submanifold, corresponding to a set of particular solutions, could be parameterized by such Noether invariants. If this really happens, we could proceed to the group quantization of this submanifold !!.

10 The general case and unification

10.1 No-go theorems on symmetry mixing

The possibility of unifying internal gauge interactions with Gravity, as a gauge theory associated with a space-time symmetry group, was tied to the existence of a finite-dimensional global symmetry group containing the Poincaré group and an internal unitary (compact) group in a non-trivial way, that is, not a tensor product. This possibility was soon discarded by the publication of a series of papers establishing the now known as "No-Go theorems" on symmetries (see, in particular, [59,60]). The situation is quite different in dealing directly with infinite-dimensional groups where those theorems do not apply.

10.2 Electrogravity mixing

Thinking of Quantum Theory as a more exact theory than Classical Theory, and starting from the rigid symmetry of "quantum matter" we arrive at a non-trivial consequence consisting in a non-trivial mixing of space-time and internal gauge interactions. A first attempt was given at the Quantum Mechanical level [61], and then this idea was extended to field theory in the form of a generalized gauge theory [58].

Let us substitute the $U(1)$-extended Poincaré group, $\tilde{\mathcal{P}}$, by the standard Poincaré group $\mathcal{P}$. The Lie algebra of $\tilde{\mathcal{P}}$ is:

$$[\widetilde{M}_{\mu\nu}, \widetilde{P}_\rho] = \eta_{\nu\rho}\widetilde{P}_\mu - \eta_{\mu\rho}\widetilde{P}_\nu - (\lambda_\mu\eta_{\nu\rho} - \lambda_\nu\eta_{\mu\rho})\Xi \equiv C_{\mu\nu,\ \rho}^{\ \sigma}\widetilde{P}_\sigma + C_{\mu\nu,\ \rho}^{\ \Phi}\Xi\ , \quad (388)$$

where

$$C_{\mu\nu,\ \rho}^{\ \Phi} \equiv \lambda_\nu\eta_{\mu\rho} - \lambda_\mu\eta_{\nu\rho}\ , \quad (389)$$

Ξ is the (central) generator of $U(1)$, and λ_μ is a vector in the Poincaré co-algebra belonging to a certain co-adjoint orbit.

We shall take λ_μ in the simplest, though non-covariant, way:

$$\lambda_\mu = -\kappa\delta^0_\mu\ , \quad (390)$$

the constant κ being the mixing parameter. Then, the new structure constants are $C_{\mu,\ \sigma\rho}^{\ \Phi} \equiv -\kappa(\eta_{\rho\mu}\delta^0_\sigma - \eta_{\sigma\mu}\delta^0_\rho)$, and give rise to the following curvature components:

$$F^{(\lambda\rho)}_{\mu\nu} = A^{(\lambda\rho)}_{\mu,\nu} - A^{(\lambda\rho)}_{\nu,\mu} - \eta_{\theta\sigma}(A^{(\lambda\theta)}_\mu A^{(\sigma\rho)}_\nu - A^{(\lambda\theta)}_\nu A^{(\sigma\rho)}_\mu)\ , \quad (391)$$

$$F^{(\Phi)}_{\mu\nu} = A^{(\Phi)}_{\mu,\nu} - A^{(\Phi)}_{\nu,\mu} + \kappa\eta_{ij}(A^{(j)}_\mu A^{(0i)}_\nu - A^{(j)}_\nu A^{(0i)}_\mu)\ . \quad (392)$$

Note that $F^{(\Phi)}_{\mu\nu}$ involves, apart from the free term $A_{\mu,\nu} - A_{\nu,\mu}$, the potentials A^j_μ associated with translations, which are omitted in the standard theory. Besides, the electromagnetic strength of gravitational origin find its source in the Coriolis-like gravitational potentials; that is to say, those of rotating massive bodies.

The geodesic motion, for instance, can be derived by considering matter Lagrangian corresponding to a single particle: $\mathcal{L}_{matt} = \frac{1}{2m}p_\mu p_\nu\eta^{\mu\nu}$. We easily arrive at:

$$g_{\mu\sigma}\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = -u^\mu u^\nu\Gamma^{(L-C)}_{\mu\nu,\sigma} - \frac{e}{m}u^\mu F^{(elec)}_{\mu\sigma} - \frac{\kappa e}{m}u^\mu(B^{(grav)}_{\mu,\sigma} - B^{(grav)}_{\sigma,\mu})\ , \quad (393)$$

where we have separated $A^{(\Phi)}_\mu$ into two different pieces: $A^{(\Phi)}_\mu = A^{(elect)}_\mu + \kappa B^{(grav)}_\mu$, corresponding to the ordinary electromagnetic field added with the new mixing term. We refer the reader to Ref. [58] for specific details.

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11 Appendix: Derivation of the Euler–Lagrange and Poincaré–Cartan equations

E–L equations:

$$
\begin{aligned}
L_{\bar{X}}(\mathcal{L}\omega) &= (L_{\bar{X}}\mathcal{L})\omega + \mathcal{L}L_{\bar{X}}\omega = \left\{X^\mu \frac{\partial\mathcal{L}}{\partial x^\mu} + X^\alpha \frac{\partial\mathcal{L}}{\partial\psi^\alpha}\right\}\omega + \bar{X}^\alpha_\nu \frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\mathrm{d}x^\nu \wedge \theta_\mu \\
&\quad + \mathcal{L}\mathrm{d}X^\mu \wedge \theta_\mu \\
&= \left\{X^\mu \frac{\partial\mathcal{L}}{\partial x^\mu} + X^\alpha \frac{\partial\mathcal{L}}{\partial\psi^\alpha}\right\}\omega + \left\{\frac{\partial X^\alpha}{\partial x^\sigma} - \psi^\alpha_\nu \frac{\partial X^\nu}{\partial x^\sigma}\right\}\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\mathrm{d}x^\sigma \wedge \theta_\mu \\
&\quad + \frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\left\{\frac{\partial X^\alpha}{\partial\psi^\beta} - \psi^\alpha_\nu \frac{\partial X^\nu}{\partial\psi^\beta}\right\}(\mathrm{d}\psi^\beta - \theta^\beta)\wedge\theta_\mu + \mathcal{L}\mathrm{d}X^\mu \wedge \theta_\mu \\
&= \left\{X^\mu \frac{\partial\mathcal{L}}{\partial x^\mu} + X^\alpha \frac{\partial\mathcal{L}}{\partial\psi^\alpha}\right\}\omega + \frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}(\mathrm{d}X^\alpha - \psi^\alpha_\nu \mathrm{d}X^\nu)\wedge\theta_\mu \\
&\quad - \frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\left\{\frac{\partial X^\alpha}{\partial\psi^\beta} - \psi^\alpha_\nu \frac{\partial X^\nu}{\partial\psi^\beta}\right\}\theta^\beta\wedge\theta_\mu + \mathcal{L}\mathrm{d}X^\mu \wedge \theta_\mu \\
\int_{j^1(\psi)(M)} L_{\bar{X}}(\mathcal{L}\omega) &= \int_M \left[\left\{X^\mu \frac{\partial\mathcal{L}}{\partial x^\mu} + X^\alpha \frac{\partial\mathcal{L}}{\partial\psi^\alpha}\right\}\omega - X^\alpha \mathrm{d}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\theta_\mu\right) + \mathrm{d}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}X^\alpha\theta_\mu\right)\right. \\
&\quad \left. + X^\nu \mathrm{d}\left(\psi^\alpha_\nu \frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\theta_\mu\right) - \mathrm{d}\left(\psi^\alpha_\nu X^\nu \frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\theta_\mu\right) - X^\mu \mathrm{d}(\mathcal{L}\theta_\mu) + \mathrm{d}(X^\mu\mathcal{L}\theta_\mu)\right] \\
&= \int_M X^\mu \left\{\frac{\partial\mathcal{L}}{\partial x^\mu}\omega + \mathrm{d}\left(\psi^\alpha_\mu \frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\theta_\nu\right) - \mathrm{d}(\mathcal{L}\theta_\mu)\right\} \\
&\quad + \int_M X^\alpha \left\{\frac{\partial\mathcal{L}}{\partial\psi^\alpha}\omega - \mathrm{d}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\theta_\mu\right)\right\} \\
&= \int_M \left[X^\mu \psi^\alpha_\mu \left\{\frac{\mathrm{d}}{\mathrm{d}x^\nu}\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\nu} - \frac{\partial\mathcal{L}}{\psi^\alpha}\right\}\omega + X^\alpha\left\{\frac{\mathrm{d}}{\mathrm{d}x^\nu}\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\nu} - \frac{\partial\mathcal{L}}{\psi^\alpha}\right\}\omega\right] = 0 \\
&\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x^\nu}\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\nu} - \frac{\partial\mathcal{L}}{\psi^\alpha} = 0\,.
\end{aligned}
$$

As can be seen, the same result is obtained varying the Lagrangian with vector fields $X^\alpha \frac{\partial}{\partial\psi^\alpha}$ (with $X^\mu = 0$), and we avoid a lot of calculations.
Cartan-like equations (Modified Hamilton Principle):

$$\theta_{PC} = \frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}(\mathrm{d}\psi^\alpha - \psi^\alpha_\nu \mathrm{d}x^\nu) + \mathcal{L}\omega = \frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\mathrm{d}\psi^\alpha \wedge \theta_\mu + (\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\psi^\alpha_\mu)\omega$$

$$\begin{aligned}
\mathrm{d}\Theta_{PC} &= \mathrm{d}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right) \wedge \mathrm{d}\psi^\alpha \wedge \theta_\mu + \mathrm{d}\left(\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\psi^\alpha_\mu\right)\wedge\omega \\
&= \left\{\frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right)\mathrm{d}x^\mu + \frac{\partial}{\partial\psi^\beta}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right)\mathrm{d}\psi^\beta\right. \\
&\quad \left. + \frac{\partial}{\partial\psi^\beta_\nu}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right)\mathrm{d}\psi^\beta_\nu\right\} \wedge \mathrm{d}\psi^\alpha \wedge \theta_\mu \\
&\quad + \left[\frac{\partial\mathcal{L}}{\partial x^\nu}\mathrm{d}x^\nu + \frac{\partial\mathcal{L}}{\partial\psi^\beta}\mathrm{d}\psi^\beta + \frac{\partial\mathcal{L}}{\partial\psi^\beta_\nu}\mathrm{d}\psi^\beta_\nu - \psi^\alpha_\mu\left\{\frac{\partial}{\partial x^\nu}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right)\mathrm{d}x^\nu\right.\right. \\
&\quad \left.\left. + \frac{\partial}{\partial\psi^\beta}\left(\partial\mathcal{L}\partial\psi^\alpha_\mu\right)\mathrm{d}\psi^\beta + \frac{\partial}{\partial\psi^\beta_\nu}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right)\mathrm{d}\psi^\beta_\nu\right\} - \frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\mathrm{d}\psi^\alpha_\mu\right]\wedge\omega \\
&= \left\{-\frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right) + \frac{\partial\mathcal{L}}{\partial\psi^\alpha} - \psi^\beta_\mu\frac{\partial}{\partial\psi^\alpha}\left(\frac{\partial\mathcal{L}}{\partial\psi^\beta_\mu}\right)\right\}\mathrm{d}\psi^\alpha\wedge\omega + \frac{\partial}{\partial\psi^\beta}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right)\mathrm{d}\psi^\beta\wedge\mathrm{d}\psi^\alpha\wedge\theta_\mu \\
&\quad + \frac{\partial}{\partial\psi^\beta_\nu}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right)\mathrm{d}\psi^\beta_\nu\wedge\mathrm{d}\psi^\alpha\wedge\theta_\mu - \psi^\alpha_\mu\frac{\partial}{\partial\psi^\beta_\nu}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right)\mathrm{d}\psi^\beta_\nu\wedge\omega
\end{aligned}$$

$$\begin{aligned}
i_{X^1}\mathrm{d}\theta_{PC} &= \left\{\frac{\partial\mathcal{L}}{\partial\psi^\alpha} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right) - \psi^\beta_\mu\frac{\partial}{\partial\psi^\alpha}\left(\frac{\partial\mathcal{L}}{\partial\psi^\beta_\mu}\right)\right\}X^\alpha\omega \\
&\quad - \left\{\frac{\partial\mathcal{L}}{\partial\psi^\alpha} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right) - \psi^\beta_\mu\frac{\partial}{\partial\psi^\alpha}\left(\frac{\partial\mathcal{L}}{\partial\psi^\beta_\mu}\right)\right\}X^\nu\mathrm{d}\psi^\alpha\wedge\theta_\nu + \frac{\partial}{\partial\psi^\beta}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right)X^\beta\mathrm{d}\psi^\alpha\wedge\theta_\mu \\
&\quad - \frac{\partial}{\partial\psi^\beta}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right)X^\alpha\mathrm{d}\psi^\beta\wedge\theta_\mu + \frac{\partial}{\partial\psi^\beta}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right)\mathrm{d}\psi^\beta\wedge\mathrm{d}\psi^\alpha\wedge i_{X^1}\theta_\mu + \frac{\partial}{\partial\psi^\beta_\nu}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right)X^\beta_\nu\mathrm{d}\psi^\alpha\wedge\theta_\mu \\
&\quad - \frac{\partial}{\partial\psi^\beta_\nu}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right)X^\alpha\mathrm{d}\psi^\beta_\nu\wedge\theta_\mu + \frac{\partial}{\partial\psi^\beta_\nu}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right)\mathrm{d}\psi^\beta_\nu\wedge\mathrm{d}\psi^\alpha\wedge i_{X^1}\theta_\mu - \psi^\alpha_\mu\frac{\partial}{\partial\psi^\beta_\nu}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right)X^\beta_\nu\omega \\
&\quad + \psi^\alpha_\mu\frac{\partial}{\partial\psi^\beta_\nu}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right)X^\sigma\mathrm{d}\psi^\beta_\nu\wedge\theta_\sigma \\
&\qquad (\mathrm{d}x^\nu\wedge i_{X^1}\theta_\mu = X^\nu\theta_\mu - \delta^\nu_\mu X^\sigma\theta_\sigma)
\end{aligned}$$

$$\begin{aligned}
i_{X^1}\mathrm{d}\Theta_{PC}|_{\psi^1(M)} &= \left[\frac{\partial\mathcal{L}}{\partial\psi^\alpha} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right) - \psi^\beta_\mu\frac{\partial}{\partial\psi^\alpha}\left(\frac{\partial\mathcal{L}}{\partial\psi^\beta_\mu}\right) + \frac{\partial}{\partial\psi^\alpha}\left(\frac{\partial\mathcal{L}}{\partial\psi^\beta_\mu}\right)\frac{\partial\psi^\beta}{\partial x^\mu}\right. \\
&\quad \left. - \frac{\partial}{\partial\psi^\beta}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right)\frac{\partial\psi^\beta}{\partial x^\mu} - \frac{\partial}{\partial\psi^\beta_\nu}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right)\frac{\partial\psi^\beta_\nu}{\partial x^\mu}\right]X^\alpha\omega \\
&\quad \left[-\frac{\partial\psi^\alpha}{\partial x^\nu}\left\{\frac{\partial\mathcal{L}}{\partial\psi^\alpha} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right) - \psi^\beta_\mu\frac{\partial}{\partial\psi^\alpha}\left(\frac{\partial\mathcal{L}}{\partial\psi^\beta_\mu}\right)\right\} + \frac{\partial\psi^\alpha}{\partial x^\nu}\frac{\partial\psi^\beta}{\partial x^\mu}\frac{\partial}{\partial\psi^\beta}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right)\right. \\
&\quad - \frac{\partial}{\partial\psi^\beta}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right)\frac{\partial\psi^\alpha}{\partial x^\mu}\frac{\partial\psi^\beta}{\partial x^\nu} + \psi^\alpha_\mu\frac{\partial}{\partial\psi^\beta_\sigma}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right)\frac{\partial\psi^\beta_\sigma}{\partial x^\nu} + \frac{\partial}{\partial\psi^\beta_\sigma}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right)\frac{\partial\psi^\alpha}{\partial x^\nu}\frac{\partial\psi^\beta_\sigma}{\partial x^\mu} \\
&\quad \left. - \frac{\partial}{\partial\psi^\beta_\sigma}\left(\frac{\partial\mathcal{L}}{\partial\psi^\alpha_\mu}\right)\frac{\partial\psi^\alpha}{\partial x^\mu}\frac{\partial\psi^\beta_\sigma}{\partial x^\nu}\right]X^\nu\omega
\end{aligned}$$

$$
\begin{aligned}
&+\left[\frac{\partial}{\partial \psi_\nu^\beta}\left(\frac{\partial \mathcal{L}}{\partial \psi_\mu^\alpha}\right)\frac{\partial \psi^\alpha}{\partial x^\mu} - \frac{\partial}{\partial \psi_\nu^\beta}\left(\frac{\partial \mathcal{L}}{\partial \psi_\mu^\alpha}\right)\psi_\mu^\alpha\right] X_\nu^\beta \omega \quad \Rightarrow \\
&X_\nu^\beta \quad : \frac{\partial^2 \mathcal{L}}{\partial \psi_\nu^\beta \partial \psi_\mu^\alpha}\left[\frac{\partial \psi^\alpha}{\partial x^\mu} - \psi_\mu^\alpha\right] = 0 \\
&X^\alpha \quad : \left\{\frac{\partial \mathcal{L}}{\partial \phi^\alpha} - \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \psi_\mu^\alpha}\right)\right\} + \frac{\partial}{\partial \psi^\alpha}\left(\frac{\partial \mathcal{L}}{\partial \psi_\mu^\beta}\right)\left[\frac{\partial \psi^\beta}{\partial x^\mu} - \psi_\mu^\beta\right] = 0 \\
&X^\nu \quad : \frac{\partial \psi^\alpha}{\partial x^\nu}\left\{\frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \psi_\mu^\beta}\right) - \frac{\partial \mathcal{L}}{\partial \psi^\alpha}\right\} = 0\,.
\end{aligned}
$$

Obviously, for regular Lagrangians, that is, those satisfying $\left|\frac{\partial^2 \mathcal{L}}{\partial \psi_\nu^\beta \partial \psi_\mu^\alpha}\right| \neq 0$, we arrive at the solutions of the Ordinary Hamilton Principle.

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