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Fradkin-Bacry-Ruegg-Souriau vector in kappa-deformed space-time

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Abstract. We study the presence of an additional symmetry of a generic central potential in the κ space-time. An explicit construction of Fradkin, Bacry, Ruegg and Souriau (FBRS) for a central potential is carried out and the piecewise conserved nature of the vector is established. We also extend the study to Kepler systems with a drag term, particularly the Gorringe-Leach equation is generalized to the κ-deformed space. The possibility of mapping a Gorringe-Leach equation to an equation without drag term is exploited associating a similar conserved vector to the system with a drag term. An extension of the duality between two classes of central potential is introduced in the κ-deformed space and is used to investigate the duality existing between two classes of Gorringe-Leach equations. All the results obtained can be retraced to the correct commutative limit as we let $a \to 0$.

1 Introduction

Investigation of symmetries plays a pivotal role in our understanding of physical laws as they are intimately related to the existence of integrals of motion. One of the familiar examples is the rotational symmetry enjoyed by central potentials. Noether's theorem shows that the rotational symmetry implies the existence of a constant of motion, namely angular momentum. Historically, it has been noticed that the Kepler system possesses an additional conserved quantity, namely the Laplace-Runge-Lenz vector, which arises from the geometrical nature of the orbit rather than from Noether symmetry. Interestingly, a similar conserved quantity, the Fradkin-Hill tensor is known to exist for the isotropic oscillator [1,2].

It was shown that all dynamical systems in three dimensions are invariant under $SU(3)$ and $O(4)$ algebra [3]. An investigation of these potentials showed that they contain a larger symmetry group than the rotational group. To be specific, the Kepler problem and the isotropic oscillator problem respect $O(4)$ and $SU(3)$ symmetry, respectively [4]. A generalization of the above result showing that all central potentials (in classical systems) enjoy this extended symmetry was obtained in [5] and this symmetry has been ascribed to the existence of a constant plane of orbit [5]. This led to several investigations trying to construct conserved quantities more general than the Laplace-Runge-Lenz vector. A class of conserved quantities, analogous to Runge-Lenz vectors, was obtained for an $O(4)$ or an $O(3,1)$ dynamical algebra in the context of monopole scattering [6]. It is now a well-established fact that any three-dimensional dynamical systems involving central potentials do admit a conserved vector and this general vector has been constructed and analyzed in [4,5]. Shortly afterwards, it was shown that this generalized conserved vector is multi-valued [7,8]. Bacry, Ruegg and Souriau showed that such a vector is exceptionally one-valued in the Kepler case, and corresponds generally to a piecewise conserved quantity known as the Fradkin-Bacry-Ruegg-Souriau (FBRS) perihelion vector [9,10]. An extension of a similar construction on curved manifolds was taken up in [11].

In [12], Peres rederived this constant vector using an approach different from the one adopted in [4,5]. The starting point of [12] was the requirement that a generalization of the Laplace-Runge-Lenz vector with arbitrary coefficients to be an integral of motion. This condition imposed restrictions on these coefficients which are functions of the radial coordinate. These coefficients were shown to satisfy a set of coupled differential equations. So the problem of

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obtaining a conserved vector was reduced to finding solutions to these differential equations. Further, by analyzing these differential equations, it was shown that the conserved vector is multi-valued. This construction was re-visited in [10] for the central potentials in 2-dimensional space, using complex coordinates. The correspondence between Fradkin's and Peres' approaches has been studied by Yoshida [13]. In this paper, we generalize the approach of [10] to analyze the central potentials in the κ-deformed space-time, which is an example of a non-commutative space-time.

It was shown by Grandati *et al.* [14] that any generalized Gorringe-Leach equation admits an associated FBRS vector which is globally conserved for the equations of a particular class. For the dualizable generalized Gorringe-Leach equations, Grandati *et al.* [14] showed that the image sets of the discontinuous FBRS vectors for two classes of power potential problems are dual images of each other. In this paper we also study the non-commutative generalization of the Gorringe-Leach equation in the κ space-time.

Non-commutative space-times and various models on such spaces are being studied in recent times [15–18]. One of the reasons of this interest is the important role of non-commutative space-time in the context of quantum gravity [19]. κ space-time [20] is an example of a non-commutative space-time whose co-ordinates satisfy the conditions

$$[\hat{x}_0, \hat{x}_i] = a\hat{x}_i \quad \text{and} \quad [\hat{x}_i, \hat{x}_j] = 0, \tag{1}$$

which naturally appears in the discussions of deformed special relativity [21]. Various aspects of this space-time have been studied in recent times and one of the approaches used was to map the models defined on the κ-deformed space-time to usual commutative space-time [22]. Recently, there have been many interesting works trying to understand the dynamics of a central potential, both at classical and quantum level, in non-commutative space-time [23–25].

In this paper, we investigate the additional symmetry of central potential problems in kappa-deformed space-time. The FBRS vector for a generic central potential is constructed in the κ-deformed space. It is shown that the vector is piecewise constant as in the case of commutative space and the necessary condition for its constancy is obtained. Kepler-type systems with drag term are also studied in the κ space-time and the existence of an FBRS-like vector is shown. Arnold-Bohlin-Vasiliev duality establishes a correlation between two classes of power potentials and this helps in generalizing the duality existing between Kepler potential and harmonic oscillator potential. The mapping between two classes of Gorringe-Leach equations, which exist in commutative case, is extended to the κ-space using the Arnold-Bohlin-Vasiliev duality.

This paper is organized as follows. In the second section, we set up the platform for describing a generic central force problem in kappa-deformed space-time. We express the Hamiltonian for the corresponding system in terms of commutative variables using a realization which connects the non-commutative variables with commutative variables [26–28]. We obtain the radial motion for the generic central potential system, expressed in terms of complex coordinates, and obtained an expression for the velocity. In the third section, we used the results derived in the previous section to set up the κ-deformed FBRS vector in complex coordinate system. We then show that, in general, such a system admits a piecewise conserved quantity as in the case of a commutative situation. The structure of the conserved quantity is obtained as a general case and it is shown that even in the presence of non-commutativity, the integrability of the system is intact. In the next section, we consider the Hamiltonian for the reparametrised Gorringe-Leach equation. Here we generalize the Gorringe-Leach equations to κ space-time and show that it also possesses a piecewise integral of motion. Section 5 establishes the Bohlin-Arnold-Vassiliev duality for the Gorringe-Leach equation with a power potential. We first summarise the results in the commutative space for the Gorringe-Leach equation with a power potential and this was utilized to confirm the existence of a duality between the class of potentials (which generalize Kepler potential and harmonic potential) in κ space-time. Our concluding remarks are presented in sect. 6.

2 Central force problems in κ-deformed space-time

In this section, we derive the Fradkin-Bacry-Ruegg-Souriau's (FBRS) vector for the central potentials in the κ-deformed space-time. By explicit construction this vector is guaranteed to be a conserved quantity. Our results are valid to all orders in the deformation parameter "a".

The κ-deformed space-time is an example for a Lie-algebraic-type non-commutative space-time which naturally arises in the low energy limit of certain quantum gravity models [19]. We can re-express the coordinates of κ-deformed space-time in terms of commutative phase space variables [26] as

$$\hat{x}^\mu = x^\mu + \alpha x^\mu (a \cdot p) + \beta (a \cdot x) p^\mu + \gamma a^\mu (x \cdot p) \tag{2}$$

and the corresponding momenta are given by

$$\hat{p}^\mu = p^\mu + (\alpha + \beta)(a \cdot p) p^\mu + \gamma a^\mu (p \cdot p). \tag{3}$$

Here, the real constants α, β and γ satisfy the conditions

$$\gamma - \alpha = 1, \quad \alpha, \gamma, \beta \in \mathbb{R}. \tag{4}$$

With the choice, $a^\mu = (a, \vec{0}\,)$ and setting $\beta = 0$, we obtain $\hat{x}_i$ explicitly as

$$\hat{x}^i = x^i + \alpha x^i a p^0. \tag{5}$$

Using this, we find the (square of the) norm of the position vector in κ-space-time as

$$\hat{r}^2 = \hat{x}_i^2 = r^2\left(1 + 2a\alpha E_0 + a^2\alpha^2 E_0^2\right), \tag{6}$$

where we have used the identification of p_0 with E_0.

As in [29], we start with the κ-deformed Hamiltonian

$$H = \frac{p^2}{2\tilde{m}} + U(\hat{r}), \tag{7}$$

where the deformed mass is given by

$$\tilde{m} = \frac{m}{1 + 2\alpha m}\,. \tag{8}$$

Note that the form of Hamiltonian is derived by taking the non-relativistic limit of the *kappa*-deformed Casimir relation, $\hat{p}_\mu \hat{p}^\mu = m^2$ [29]. Using this Hamiltonian, we find the equations of motion to be

$$\tilde{m}\ddot{x}_i = \{\dot{x}_i, H\} = -\frac{\partial U(\hat{r})}{\partial x_i}\,. \tag{9}$$

Using eq. (6), we find the potential, to the first order in the non-commutative parameter as

$$U(\hat{r}) = U\left(\left(1 + \alpha a E^0\right) r\right) = U(r) + \alpha a E^0 r \frac{\partial E}{\partial r} = \left(1 + \alpha a E^0 n\right) U(r), \tag{10}$$

where we made the identification of p_0 with E_0 and $U(r)$ is assumed to be an n-th order polynomial in r.

From eqs. (10) and (9), we find that the modification of the equation of motion due to kappa-deformation can be absorbed into an overall multiplicative coefficient which depends on the deformation parameter "a". Except for this, the equation remains exactly the same as in the commutative space-time [29]. It is also easy to see that in the limit $a \to 0$, we reproduce the result in the commutative space-time.

The construction of FBRS vector can be readily done by re-expressing the central force problem in the complex plane [14]. We briefly summarise the essential steps for re-expressing the central potential problem in the complex plane. For this, we restrict ourselves to two-dimensional motion described by vector

$$\vec{r} = (x(t), y(t)). \tag{11}$$

We introduce the complex coordinate as $z(t) = x(t) + iy(t)$. The potential $U(z, \bar{z})$ can be viewed as a real valued function on complex plane, $\mathbb{C}$. Note that,

$$\nabla \to 2\frac{\partial}{\partial \bar{z}}\,, \tag{12}$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{13}$$

In terms of the complex coordinate, the equation of motion becomes

$$\tilde{m}\ddot{z} + 2\frac{\partial U(z, \bar{z})}{\partial \bar{z}} = 0. \tag{14}$$

It should be noted that the mass that appears in the above equation is given in eq. (8). In the present case, the potentials $U(z, \bar{z})$ satisfy the condition

$$U(z, \bar{z}) = U(|z|) = U(r). \tag{15}$$

Thus it is easy to see that

$$\frac{\partial U}{\partial \bar{z}} = \frac{\partial r}{\partial \bar{z}}\frac{\partial U}{\partial r} = \frac{z}{2r}\frac{\partial U}{\partial r}\,. \tag{16}$$

Recalling that the angular momentum in κ space-time is similar in form to that in commutative space-time except for the modified mass factor [29], we write the angular momentum associated with the above central potential as

$$\vec{L} = \tilde{m}\vec{r} \times \dot{\vec{r}}, \tag{17}$$

where $\tilde{m}$ is the deformed mass. In the complex notation, the above angular momentum has the form

$$L = \mathcal{I}m\bar{z}(t)\dot{z}(t) = \frac{\tilde{m}}{2i}(\bar{z}\dot{z} - \dot{\bar{z}}z). \tag{18}$$

Our major concern in the present paper is the dynamics of a system with radial symmetry. Thus, it would be helpful to review the motion along radial direction. The radial equation we are interested in is written in terms of the complex coordinates. For this, we note that with $r^2 = z\bar{z}$ and thus

$$2r\dot{r} = \dot{z}\bar{z} + z\dot{\bar{z}}, \tag{19}$$

$$\text{and} \quad \dot{r} = \frac{1}{r}\left(z\dot{z} + i\frac{L}{\tilde{m}}\right), \tag{20}$$

where eq. (18) is used in arriving at the last equation above. Using

$$z = r\exp i\theta \quad \text{and} \quad \exp 2i\theta = \frac{z}{\bar{z}}\,, \tag{21}$$

we obtain

$$\dot{\theta} = \frac{1}{2i}\left(\frac{\dot{z}}{z} - \frac{\dot{\bar{z}}}{\bar{z}}\right) = \frac{L}{\tilde{m}r^2}\,, \tag{22}$$

where we have used eq. (18). Using eqs. (20), (21), and (22), we re-express the time derivative of z as

$$\dot{z} = \left(\dot{r} + i\frac{L}{\tilde{m}r}\right)\frac{z}{r}\,, \tag{23}$$

and hence

$$\ddot{z} = \left(\frac{\ddot{r}}{r} - \frac{L^2}{r^4}\right)z. \tag{24}$$

By using (14) along with (24), we re-write the equation of motion as

$$\tilde{m}\ddot{r} = \frac{L^2}{\tilde{m}r^3} - U'(r,a). \tag{25}$$

Multiplying the above equation with $\dot{r}$ and integrating, we obtain

$$\tilde{m}\dot{r}^2 = 2\left(E - \frac{L^2}{2\tilde{m}r^2} - U(r,a)\right), \tag{26}$$

where E is an integration constant which will be identified with total energy, later. The above equation can be expressed as

$$(\dot{r}(t))^2 = \frac{L^2}{\tilde{m}^2r^2}f(r(t),a), \tag{27}$$

where $f(r,a)$ is a generic function of r and the deformation parameter a. Taking the modulus of eq. (23), we find

$$|\dot{z}|^2 = \dot{r}^2 + \frac{L^2}{\tilde{m}^2r^2} = \frac{2}{\tilde{m}}(E - U(r,a)). \tag{28}$$

This clearly shows that the identification of E with the total energy is valid. Equation (27) shows that $\dot{r}(t) = 0$ has two possible solutions given by

$$\dot{r}(t) = \pm\frac{L}{\tilde{m}r}\sqrt{f(r(t),a)}\,. \tag{29}$$

By convention, we take the "+" sign as the solution when r changes from $r_{\min}$ to $r_{\max}$ (call it r_1) and the "−" sign as the solution when r changes from $r_{\max}$ to $r_{\min}$ (r_2).

The motion from $r_{\min}$ to $r_{\max}$ (or vice versa) is termed as the "phase" of the motion. Assuming that the motion starts at $r = r_{\min}$ when $t = 0$, we say that during the motion from $r_{\min}$ to $r_{\max}$ it will be in the first phase of motion, the second phase of motion will be from $r_{\max}$ to $r_{\min}$ and so on. Thus, we index each phase with positive integers which will make it easier for referring the phase of motion under consideration. This convenience is exploited by indexing each phase of motion by an integer $k \in \mathbb{Z}$ with odd k corresponding to r_1 and even k corresponding to r_2, respectively. Using this, we re-express eq. (29) compactly as

$$\dot{r(t)} = (-1)^{k+1} \frac{L}{\tilde{m} r} \sqrt{f(r(t), a)}\,. \tag{30}$$

Using eq. (22) and eq. (29), we find

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = (-1)^{k+1} \frac{1}{r\sqrt{f(r,a)}}\,, \tag{31}$$

whose solution is

$$\theta(r,a) = \theta(r_0,a) + (-1)^{k+1} \int_{r_0}^{r} \frac{1}{\rho\sqrt{f(\rho,a)}} \mathrm{d}\rho. \tag{32}$$

Choosing $r_0 = r_{\min}$ and $\theta(r_0) = 0$, we find

$$\theta_k(r,a) = 2n \int_{r_{\min}}^{r_{\max}} \frac{\mathrm{d}\rho}{\rho\sqrt{f(\rho,a)}} + (-)^{k+1} \int_{r_{\min}}^{r(t)} \frac{\mathrm{d}\rho}{\rho\sqrt{f(\rho,a)}}\,, \tag{33}$$

where n is the index of phases. The above equation can be re-expressed as

$$\theta_k(r,a) = 2n\Phi + (-1)^{k+1} g(r(t),a), \tag{34}$$

with k referring to the phase of the motion and Φ given by

$$\Phi = \int_{r_{\min}}^{r_{\max}} \frac{\mathrm{d}\rho}{\rho\sqrt{f(\rho,a)}} \tag{35}$$

and

$$g(r(t),a) = \int_{r_{\min}}^{r(t)} \frac{\mathrm{d}\rho}{\rho\sqrt{f(\rho,a)}}\,. \tag{36}$$

Using eq. (23) and eq. (29), the instantaneous velocity is given as

$$\dot{z} = \left[(-1)^{k+1}\sqrt{f(r,a)} + i\right] \frac{L}{\tilde{m} r^2} z(t). \tag{37}$$

Since under complexification $\dot{\vec{r}} \times \vec{L} \to iL\dot{z}$, we find

$$\dot{\vec{r}} \times \vec{L} \to iL\dot{z} = -\left(1 + (-1)^k i\sqrt{f(r,a)}\right) \frac{L^2}{\tilde{m} r^2} z(t). \tag{38}$$

This form is used in the next section to complexify the Fradkin-Bacry-Ruegg-Souriau vector.

3 Fradkin-Bacry-Ruegg-Souriau vector in κ space-time

In this section, we generalize the approach of [12] to the κ-deformed space-time. In this approach for an arbitrary central potential, one starts with a generic vector which reduces to the Laplace-Runge-Lenz vector for the Kepler problem. Demanding the constancy of this vector imposes conditions on the arbitrary coefficients appearing in the definition of this vector. These conditions are expressed as coupled differential equations and by analysing these differential equations, a piecewise constant vector has been constructed explicitly. Using the results obtained in the previous section, we now generalize this construction to the κ-deformed space-time.

Our starting point in constructing the Fradkin-Bacry-Ruegg-Souriau vector using Peres approach in the κ-space-time is from the postulate

$$\vec{A} = \tilde{m}^2 \dot{\vec{r}} \times \vec{r} \left(\frac{r^2}{L^2} b(r,a)\right) + c(r,a)\vec{r}, \tag{39}$$

where $b(r,a)$ and $c(r,a)$ are generic functions of r which also depend on the deformation parameter a and in the limit $a \to 0$, reduces to $b(r)$ and $c(r)$. It would be safe to assume that the vector $\vec{A}$ has the same form in non-commutative setting when expressed in terms of commutative variables as all the "a" dependent corrections are absorbed into the definitions of $\tilde{m}$, $b(r,a)$ and $c(r,a)$. Note that the angular momentum appearing in the above is the κ-deformed one defined in eq. (18). After complexification, the above vector becomes

$$\mathcal{A} = \left(\frac{r^2}{L^2} b(r,a)\right) i \frac{L}{\tilde{m}} \dot{z} + c(r,a) z. \tag{40}$$

Using (37), we re-write the $\mathcal{A}$ (for the k-th phase) as

$$\mathcal{A}_k = \left[c(r) - b(r) + (-1)^{k+1}\, i \sqrt{f(r,a)}\, b(r,a)\right] z. \tag{41}$$

The requirement that the above vector is a constant of motion (*i.e.*, $\dot{\mathcal{A}}_k = 0$) results in two coupled differential equations on $c(r,a)$ and $b(r,a)$. They are

$$c'(r,a) - b'(r,a) + \frac{2c(r,a) - b(r,a)}{r} = 0, \tag{42}$$

$$r f(r) c'(r,a) + c(r,a)\left(\frac{r f'(r,a)}{r} + f(r,a) - 1\right) + b(r,a) = 0, \tag{43}$$

where we have used eq. (37). After rearranging eq. (42) as

$$(rc(r,a))' = (rb)'(r,a) - c(r,a), \tag{44}$$

and using

$$c(r,a) = u'(r,a); \qquad c'(r,a) = u''(r,a), \tag{45}$$

we get

$$(rb(r,a))' = ru''(r,a) + 2u'(r,a) = (ru(r,a))''. \tag{46}$$

From the above equation, we find $b(r,a) = b_0 + u'(r,a) + \frac{u(r,a)}{r}$, where the b_0 is the integration constant, which we set to zero. Thus the above equations reduces to

$$b(r,a) = u'(r,a) + \frac{u(r,a)}{r}\,. \tag{47}$$

Now using eq. (45) and eq. (47), eq. (43) is re-expressed as

$$r f(r,a) u''(r,a) + u'(r,a)\left(\frac{r f'(r,a)}{r} + f(r,a)\right) + \frac{u(r,a)}{r} = 0, \tag{48}$$

which leads to

$$u''(r,a) + u'(r,a)\left(\frac{1}{r} + \frac{f'(r,a)}{f(r,a)}\right) + \frac{1}{r^2 f(r,a)} u(r,a) = 0. \tag{49}$$

The above equation can be re-cast as

$$u''(r,a) - u'(r,a)\left[\frac{1}{\log r\sqrt{f(r,a)}}\right]' u'(r,a) + \left[\frac{1}{r\sqrt{f(r,a)}}\right]^2 u(r,a) = 0. \tag{50}$$

Defining $u(r,a) = v(g(r,a))$ where $g(r,a)$ is given in eq. (36), we find

$$u'(r,a) = v^{(1)}(g(r,a)) g'(r,a), \tag{51}$$

$$u''(r,a) = v^{(1)}(g(r,a)) g''(r,a) + v^{(2)}(g(r,a))(g'(r,a))^2, \tag{52}$$

where v^1 and v^2 are the first and second derivatives of v with respect to the argument. Using these, eq. (50) is expressed as

$$v^{(2)}(g(r,a)) + v(g(r,a)) = 0. \tag{53}$$

This leads to the solution for u given by

$$u(r,a) = v(g(r,a)) = A\cos(g(r,a)) + B\sin(g(r,a)). \tag{54}$$

Using this, we find

$$b(r,a) = g'(r,a)(B\cos(g(r,a)) - A\sin(g(r,a))), \tag{55}$$

$$c(r,a) = b(r,a) + \frac{1}{r}(A\cos(g(r,a)) + B\sin(g(r,a))), \tag{56}$$

where eqs. (45) and (47) are used. Thus we see that the condition $\dot{\mathcal{A}} = 0$ gives equations which can be solved analytically (for b and c) and thus, establish the existence of such a constant vector for generic central potentials.

Thus we have derived the conditions required for the conservation of Fradkin-Bacry-Ruegg-Souriau vector in the κ-deformed space-time and have showed the existence of an integral of motion for a generic central potential in kappa-deformed space-time.

4 Gorringe-Leach equation in κ space-time

The existence of a conserved vector, FBRS vector, for a central potential naturally leads to attempts for constructing similar conserved quantities for dissipative systems also. The idea that a dissipative system can be mapped to a system with no drag term is well known in the commutative case [14]. This method has been used to obtain a conserved vector for dissipative systems. In this section, we generalize the procedure of [14] by Grandati *et al.* developed for the Gorringe-Leach equation to the kappa-deformed space-time and establish the existence of a piecewise conserved quantity. This is a demonstration for the existence of conserved quantities for dissipative systems in the κ-deformed space-time. We start with a Hamiltonian in the kappa-deformed space-time for a potential that would give rise to a generic form of Gorringe-Leach equation. In addition, we achieved the conditions for the existence of a conserved quantity. Further we will show that, as in the undeformed case, this vector is piecewise conserved.

We start with the Hamiltonian given in eq. (7). When choosing the form of the potential, we should take care of the following points:

- The corresponding equation of motion should transform into the Gorringe-Leach equation under a reparameterisation given by $\mathrm{d}s = e^{+K(z,\bar{z})}\mathrm{d}t$, where s is the new parameter, t is the old parameter and K is an arbitrary function.
- Since the map includes an exponential, our potential should contain the same exponential term for a neat cancellation.

With the above points, we consider a specific class of potentials of the form

$$\tilde{U}(\hat{r}) = U(r,a) = \int r\mathrm{d}r(1+\alpha a E_0)^2 g_1(r,a)e^{2K(r,a)}, \tag{57}$$

where $g_1(r,a)$ and $K(r,a)$ are two arbitrary functions of radial coordinate which also depend on the deformation parameter a. Note that these functions reduce to the correct commutative limit as we set a to zero. Without loss of generality, we rewrite K as $K(r,a) = K(r(1+a\alpha E_0)) = K_1(r) + K_2(r,a)$ and using this define $g(r,a)$ as

$$g(r,a) = g_1(r,a)(1+\alpha a E_0)^2 e^{2K_2(r,a)}. \tag{58}$$

Now using this equation and the above Hamiltonian, we obtain the equations of motion as

$$\tilde{m}\ddot{\vec{r}} + g(r,a)e^{2K_1(r)}\vec{r} = 0, \tag{59}$$

where $\dot{\vec{r}} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$. Since we have restricted to two-dimensional space, as in the previous section, we re-express the above equation in terms of complex coordinates as

$$\tilde{m}\ddot{z} + g(z,\bar{z},a)e^{2K_1(z,\bar{z})}z = 0. \tag{60}$$

From now on we will drop the suffix in K_1 for simplicity and denote it as just K. We would like to point out that the dependence on deformation comes through the functions $g(z,\bar{z},a)$ and $\tilde{m}$, while $K_1(z,\bar{z})$ is independent of deformation parameter.

We now introduce a change of parameter from t to s such that

$$\mathrm{d}s = e^{+K(z,\bar{z})}\mathrm{d}t. \tag{61}$$

It is interesting to see that the reparameterisation does not depend on the deformation. We have a one-to-one correspondence between t and $s = s(t, z, \bar{z})$. Using this reparameterization, we re-write the equation of motion as

$$\tilde{m}z'' + \tilde{m}K'(z,\bar{z})z' + g(z,\bar{z},a)z = 0, \tag{62}$$

where we denote $\frac{\mathrm{d}z}{\mathrm{d}s}$ as z'. Here, we now restrict our attention to those K which depend only on $r = (z\bar{z})^{\frac{1}{2}}$. We note here that the entire effect of κ-deformation is present in the deformed mass, $\tilde{m}$ and in the $g(z,\bar{z},a)$, whereas the velocity-dependent term is independent of the deformation and thus remains the same as in commutative space. One readily sees that the above equation, when expressed in terms of Cartesian coordinate system would be the generalized Gorringe-Leach equations, given by

$$\tilde{m}\vec{r}'' + \tilde{m}h\vec{r}' + g\vec{r} = 0, \tag{63}$$

where h and g are arbitrary scalar functions. Such an equation takes into account the drag effect on the dynamics of systems in the presence of a central potential and it has been shown that they possess conserved quantities, as in the case of central potential systems in ordinary space-time. Comparing eqs. (62) and (63), we find that h is a total derivative, *i.e.*,

$$h = K'. \tag{64}$$

This is crucial when we apply the inverse map to the Gorringe-Leach equation to obtain an equation of the form given by eq. (60). If there is no such total derivative in the expression, we would not be able to achieve this mapping and this can be easily observed from the reparameterisation given in eq. (61).

Thus we see that eq. (60) is mapped to the Gorringe-Leach equation in the κ space-time, given by eq. (62). Now it is easy to see that the angular momentum and energy are the integrals of motion for the system describing eq. (60).

The integrals of motion, angular momentum and energy, can be expressed as

$$\mathcal{L} = \frac{\tilde{m}}{2i}(\dot{z}\bar{z} - z\dot{\bar{z}}), \tag{65}$$

and

$$\mathcal{E} = \frac{\tilde{m}}{2}|\dot{z}|^2 + U(r,a), \tag{66}$$

respectively. Note that all the effects of non-commutativity are included in our equations through the factors depending on the parameter "a" and the structure of equations is similar to the usual commutative case apart from the modified coefficients and mass.

As in the previous section, one introduces a piecewise connected vector given by

$$\mathcal{A}_k = (c(r,a) - b(r,a) + (-1)^{k+1}\, i\sqrt{f(r,a)}\, b(r,a))\, z, \tag{67}$$

and it is straightforward to see that all the previous calculations remain valid for this case also. Thus we conclude that a central potential system with a drag term in κ-deformed also possesses a piecewise connected integral of motion as in the case of commuting space-time.

5 Bohlin-Arnold-Vassiliev duality and κ-deformed generalized Gorringe-Leach equations

Theorem 5.1. *Suppose the motion of a point in the complex plane is given by $w(t)$ which satisfies*

$$\ddot{w} = -cw|w|^{\mu-1}, \qquad \dot{\omega} = \frac{\mathrm{d}\omega}{\mathrm{d}t}\,. \tag{68}$$

Then the orbit of this equation can be mapped to an orbit undergoing

$$z'' = -\tilde{c}z|z|^{-\frac{4(\mu+2)}{\mu+3}} \quad \text{where} \quad z' = \frac{\mathrm{d}z}{\mathrm{d}\tau}\,, \tag{69}$$

under the transformation $z = w^\nu$ and Euler-Sundman's reparameterization of the type $\mathrm{d}\tau = |w(t)|^{2(\nu-1)}\mathrm{d}t$.

Proof. Let us start from (69)

$$\begin{aligned}\frac{\mathrm{d}^2 z}{\mathrm{d}\tau^2} &= \frac{1}{|w|^{2(\nu-1)}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{|w|^{2(\nu-1)}}\frac{\mathrm{d}w^\nu}{\mathrm{d}t}\right) = \frac{\nu}{|w|^{2(\nu-1)}}\frac{1}{\bar{w}^{\nu-1}}\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} - \frac{\nu(\nu-1)}{|w|^{2(\nu-1)}}\frac{1}{\bar{w}^\nu}|\dot{w}|^2\\ &= -\left(\frac{\nu c|w|^{1+\mu}}{\bar{w}^\nu|w|^{2(\nu-1)}} + \frac{\nu(\nu-1)}{\bar{w}^\nu|w|^{2(\nu-1)}}|\dot{w}|^2\right) = -\frac{2\nu(\nu-1)w^\nu}{|w|^{2(2\nu-1)}}\left(\frac{1}{2}|\dot{w}|^2 + \frac{c}{2(\nu-1)}|w|^{1+\mu}\right).\end{aligned}$$

From the second term of the last expression it is clear that, if $(\frac{1}{2}|\dot{w}|^2+\frac{c}{2(\nu-1)}|w|^{1+\mu})$ has to be the energy function of w equation, then $2(\nu-1)=1+\mu$ or $\nu=(\mu+3)/2$. It is clear that $|z|=|w|^{\nu}$ and $z=w^{\nu}$. If we express the denominator of the last expression $|w|^{2(2\nu-1)}=|z|^{\nu(1-a)}$, we obtain $2(2\nu-1)=\nu(1-a)$ or $\nu(3+a)=2$. By substituting $\nu=(\mu+3)/2$, we obtain

$$(\mu+3)(a+3)=4 \quad \text{and} \quad a-1=-\frac{4(\mu+2)}{\mu+3}\,.$$

If we consider the potential as a power law potential $V(w,\bar{w})=c|w|^{1+\mu}$, then the associated generalized Gorringe-Leach equation is given by

$$\ddot{w}+\dot{K}(w,\bar{w})\dot{w}+c|w|^{\mu-1}\exp(-2K(w,\bar{w}))w(t)=0. \tag{70}$$

Thus by conformal transformation of coordinates $z=w^{\nu}$ and Euler-Sundman reparametrization $\mathrm{d}\tau=|w(t)|^{2(\nu-1)}\mathrm{d}t$ we obtain the following equation:

$$z''+\dot{h}(z,\bar{z})\dot{z}+\tilde{c}|z|^{a-1}\exp(-2h(z,\bar{z})z)=0, \tag{71}$$

where $\tilde{c}$ is an arbitrary constant which gives the coupling strength of the corresponding potential.

We note that the above calculations remain true for the case where the functions such as K have a dependence on the deformation parameter "a". In order to see this, note that as we use expressions in the κ-deformed space, written in terms of commuting coordinates, the constant c and the exponential term in eq. (70) get an "a" dependence but the general structure of the equation is not modified. Thus, the results of commutative space are perfectly valid for the case of our interest *i.e.* in the κ-deformed space.

5.1 Duality and κ-deformed generalized Gorringe-Leach equation

In this subsection, we use the above result to establish the duality between the Kepler type potential and harmonic oscillator type potential [30]. We first consider Gorringe-Leach equation with power potential in the kappa-deformed case and this is mapped to a Gorringe-Leach equation with a different power potential.

As in the previous section, take a power potential of the form $V(w,\bar{w})=c|w|^{1+\mu}(1+a\alpha E_0)^{1+\mu}$. The associated generalized Gorringe-Leach equation is

$$\ddot{w}+\dot{K}(w,\bar{w},a)\dot{w}+c|w|^{\mu-1}\exp(-2K(w,\bar{w},a))w(t)=0, \tag{72}$$

where $c=c'(1+a\alpha E_0)^{\mu-1}$ with c' being an arbitrary constant. In the special case, where $\exp(-2K(w,\bar{w},a))=\exp(-2K(|w|,a))=\exp(-2K(r(1+a\alpha E_0)))$, we seperate the a dependence as

$$\exp(-2K(|w|,a))=\exp(-2K_0(|w|))+\exp(-2K_1(|w|,a)), \tag{73}$$

$$g(r,a)=c\exp(-2K_1(|w|,a)), \tag{74}$$

where K_0 and K_1 are arbitrary functions of $|w|$ and $(|w|,a)$ respectively.

Using this, for the above special case, the κ-deformed generalized Gorringe-Leach equation can be re-written as

$$\ddot{w}+\left(\dot{K}_0(|w|)+\dot{K}_1(|w|,a)\right)\dot{w}+g(r,a)|w|^{\mu-1}\exp(-2K_0(|w|))w(t)=0. \tag{75}$$

In the above expression, it is easy to see that the κ-deformed equation has additional terms with a specific a dependence. Further, note that in the limit $a\to 0$, we reproduce the undeformed (generalized) Gorringe-Leach equation.

We have two classes of potentials, namely the class of Harmonic-type potentials ($\mathcal{H}$ class) and the class of Kepler-type potentials ($\mathcal{K}$).

The $\mathcal{H}$ class is given by

$$g_{\mathcal{H}}(r,a)=C_{\mathcal{H}}\exp(-2K_1(r)) \tag{76}$$

and the $\mathcal{K}$ class is given by

$$g_{\mathcal{K}}(r,a)=\frac{C_{\mathcal{K}}}{r^3}\exp(-2K_1(r)), \tag{77}$$

where $C_{\mathcal{H}}$ and $C_{\mathcal{K}}$ have the form

$$C_{\mathcal{H}}=c(1+a\alpha E_0)^2, \tag{78}$$

$$C_{\mathcal{K}}=\frac{c}{(1+a\alpha E_0)}\,. \tag{79}$$

The labels $\mathcal{H}$ and $\mathcal{K}$ denote that they are the generic versions of the harmonic potential and the Kepler potential, respectively. Having defined the generic classes for the harmonic potentials and the Kepler potentials, we are all set to validate the duality between these two classes.

Using eq. (75) and the above definitions, we have the following equations for $\mathcal{H}$ class:

$$\ddot{w} + \dot{K}(|w|)\dot{w} + C_{\mathcal{K}} \exp(-2K(|w|))w(t) = 0 \tag{80}$$

and for the $\mathcal{K}$ class we have

$$\ddot{w} + \dot{K}(|w|)\dot{w} + C_{\mathcal{K}}|w|^{-3} \exp(-2K(|w|))w(t) = 0. \tag{81}$$

Comparing this with the general expression (75), we have $\mu_{\mathcal{H}} = 1$ for the $\mathcal{H}$ class and $\mu_{\mathcal{K}} = -2$ for the $\mathcal{K}$ class. One can clearly verify that these values clearly satisfy the condition for duality given by

$$(\mu_{\mathcal{H}} + 3)(\mu_{\mathcal{K}} + 3) = 4. \tag{82}$$

This shows the duality relation between the Kepler-type and harmonic-oscillator-type Gorringe-Leach equations.

6 Discussion and conclusions

In this paper, we have constructed conserved quantities associated with generic potentials in the κ-deformed space. The existence of a conserved vector is first established for central potentials. We have also shown the existence of a conserved vector for dissipative systems in the κ-deformed space. We have used the approach of mapping non-commuting coordinates to corresponding commuting coordinates and their functions. Thus we map the central force problem and the dissipative system defined in terms of κ-deformed phase space variables to that of commutative phase space variables.

In sect. 2, we have presented a brief summary of the construction of κ-deformed Hamiltonian in terms of commutative variables. Using this, we then obtain the equation of motion valid in the κ-deformed space and re-cast it in the complex notation and obtain the expression for conserved angular momentum of this κ-deformed system. We then analyze the radial motion of this system using complex coordinates. We have shown that the radial motion can be classified into different phases, as in the commutative case. We have obtained the instantaneous velocity in terms of complex coordinate and in the limit of a going to zero, we get back the commutative result. These results are used in the next section. In sect. 3, we have constructed the FBRS vector for a generic κ-deformed central potential. We start with the most general vector which reduces to the correct commutative result. This vector, for the Kepler potential, is exactly the same as the Laplace-Runge-Lenz vector. Using the results of the previous section, the requirement that this vector is a conserved quantity is expressed as a set of coupled differential equations. We have obtained the solution to these equations, thereby explicitly constructing the FBRS vector for the general central potential in κ-deformed space. In the next section, we investigated the dynamical symmetry of the system with the drag term in the equation of motion in κ-deformed space. We derived the equation of motion, which under reparameterization leads to a generalisation of the Gorringe-Leach equation in the κ-space-time. We have shown that even in the presence of a drag term, a generalized version of the Fradkin-Barcy-Ruegg-Souriau vector is an integral of motion in the κ space-time. It is clear from the analysis presented here that the κ-deformed system will have all the dynamical symmetries of its commutative counterpart. Further, we have established the Bohlin-Arnold-Vassiliev duality for the Gorringe-Leach equations with power potentials. This result shows that the Gorringe-Leach equations include classes of potentials which are the generalised versions of the Kepler potential and isotropic oscillator potentials in the κ-deformed space also.

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