



# Energy momentum tensor of a non-minimally coupled scalar from the equivalence of the Einstein and Jordan frames

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Received: 11 April 2023 / Accepted: 14 May 2023 / Published online: 28 May 2023  
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**Abstract** Unlike the minimally coupled gravity theory where matter is coupled with gravity in such a manner so that one can differentiate the matter and gravity sector uniquely, the non-minimally coupled theories (NMCT) are distinguished by the intermingling of two. As a consequence of this the calculation of the energy momentum tensor (EMT) in NMCT is beset with an arbitrariness. In this paper we provide an algorithm based on the well known equivalence between Jordan frame and Einstein frame formulations which enables us to construct the EMT for NMCT in a unique way.

## 1 Introduction

Einstein general relativity (GR) is based on the action of the massive body on the space-time around. The latter again influences the masses self consistently. However the shear of gravitational field itself in the energy momentum tensor has been a controversial point right from the beginning. The principle of equivalence is considered in the weak form in GR [1–3]. So there is no confusion in constructing the energy momentum tensor (EMT) here. What one does is to place the matter in background gravity, watch the response of the system and the limit of this ratio of the changes  $\delta S$  to the change in gravitational field  $g_{\mu\nu}$ , in the limit of the  $g_{\mu\nu}$  tend to  $\eta_{\mu\nu}$ , gives us the EMT.<sup>1</sup> However in the scalar tensor theory initiated by the famous work of Brans and Dicke [4], a coupling

was allowed between the two i.e matter and gravity. This makes the situation complicated because now one cannot be sure whether gravitation will act as a source for itself and if yes, how? As far as we know, this issue is yet not resolved. So interaction of matter with gravity is more difficult to study in such theories [5–9].

The coupling of matter and gravity can be separated in GR. So it is called minimal coupling. With the advent of experimental facilities in cosmology, it is known now that at the order of galactic cluster distance GR must be modified. But this modification would vanish in the solar system order [10–14]. Of the post GR models where the coupling is not separable are called non-minimally coupled theory (NMCT) [15–22]. It appears that for non-minimally coupled theory, it will be difficult to find an algorithm to construct the EMT. This apprehension is corroborated by a plethora of papers on the subject [23–25]. It has been attempted in the past to find theoretically a method of construction based on field theory arguments and along with principle of equivalence [26] which was successful to reveal the inner link between the apparently different empirical methods of obtaining EMT. But the algorithm [26] is so general that it did not serve the purpose of the practical cosmologist. However it gives a lesson that one has to use some extra general principle for this type of construction.

The non-minimal type of coupling has gained popularity in recent past, because observational evidence in favour of the late time cosmic acceleration has opened up possibilities for such alternatives [27–29]. These are the scalar-tensor theories [30–35] and they are adopted as modified theories of gravity in numerous investigations [7, 36]. As the energy momentum tensor (EMT) of the scalar field in these scalar-tensor theories can not be obtained using the standard definition of symmetric EMT [1], so different prescription for writing the EMT are found in the literature. Though all these

<sup>1</sup> The EMT in GR obtained as we have indicated has no reason to be equal to that obtained by Noether's theorem [1].

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EMTs are covariantly conserved, the individual components of them are very different. However it is the individual component which appear in the equations of motion, so it is not clear how the different prescriptions lead to the same physical consequences. Hence an algorithm of obtaining the EMT for a non-minimally coupled theory is very much desirable.

The usual algorithms used to construct the EMT in NMCT is to rearrange the gravitational field equation to mimic the Einstein field equations of GR [23, 37, 38], so that an expression for the EMT may be identified. While such a prescription is completely viable, this process of rearranging the gravitational field equation is not unique and leads to different expressions for the EMT of the same physical theory, as mentioned above. In this paper, we will derive an algorithm to construct an EMT exploiting the equivalence between the conformally connected frames [39].

The new input in this paper is based on the formulation of the theory in two conformally connected frames. The first in which the physical model is defined and it is hence called physical frame or Jordan frame [40–46]. The other frame is connected by the conformal transformation with the physical frame and it is called the Einstein frame. A notable property is the removal of the non-minimality in the Einstein frame. If other forms of matter is present then the non-minimality is shifted to that part. Now the other forms of matter are usually known matter (baryons, radiation and dark matter which can be relatively easy to tackle) and are not important for our general analysis. To simplify our analysis without missing any general connection we don't consider such terms.

The organization of the paper is as follows. In Sect. 2, we explain the process to be followed to obtain a symmetric, covariantly conserved EMT in the Jordan frame in a concise manner. In Sect. 3, we review how a suitable choice of conformal transformation converts our scalar-tensor theory in the Jordan frame to a quintessence scalar field theory in the Einstein frame. In Sect. 4, we examine to what extent the equivalence of these two descriptions of the same physical system works. In Sect. 5, we express the EMT obtained in the Einstein frame in terms of Jordan frame variables and also shading some light on it's nature. In Sect. 6, we assume that the conservation law in the Einstein frame implies the conservation law in the Jordan frame. In view of the universal consensus about the equivalence of the Einstein and Jordan frames( at least in the classical level), these assumptions seems quite reasonable. Using this equivalence, we have shown that we can identify an appropriate EMT from these calculations. We conclude in Sect. 7. The mostly positive signature of the metric is used throughout the paper.

## 2 Our approach

Our purpose is to provide an algorithm for construction of the EMT for NMCT which will depend on the canonical proper-

ties of the system and in no way on any arbitrary assumption or physical intuition. The algorithm we propose is canonical in the sense that it is an action based method. Remember that there exists no such theory till date. So our method if successful will lead to a novel algorithm for such an important physical variables as pressure, energy density etc.

A prototype non-minimally coupled theory will be assumed in a certain Friedmann Lemaître Robertson Walker (FLRW) spacetime which is written as

$$A_J = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} D(\pi) R + \left\{ -\frac{1}{2} g^{\mu\nu} \nabla_\mu \pi \nabla_\nu \pi - V(\pi) \right\} \right] \quad (1)$$

Here

$$D(\pi) = \left( 1 - \frac{\xi B(\pi)}{(8\pi G)^{-1}} \right) \quad (2)$$

It characterizes the non-minimality with  $B(\pi)$  being an arbitrary function of  $\pi$  that can be tuned to give a class of non-minimally coupled theories. Units are chosen such that  $M_{pl}^2 = \frac{1}{8\pi G} = \frac{1}{\kappa^2}$  and for simplicity, *we ignore all other matter fields.*

It has been proved quite generally that a conformal transformation exists [52] that maps the initial theory in FLRW to a flat Minkowski model. Our next task is to provide a conformal transformation which will map our FLRW manifold to a flat Minkowski manifold.

Let  $M$  be an  $n$ -dimensional metric with Lorentzian signature, and  $\Omega$  be a positive definite function then a transformation mapping it to the new space-time with metric,

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \quad (3)$$

is called a conformal transformation.

A conformal transformation in general is thus not equivalent to a diffeomorphism because of it's non-linearity. If the target and projected spaces have identical causal structure then they will be connected by a conformal transformation.

Now we proceed with our construction. Since the target spacetime is a flat Minkowski spacetime (Einstein frame in our problem), it is easy to construct the EMT in this frame using the well known formula

$$\tilde{T}_{\mu\nu} = -\frac{2}{\sqrt{-\tilde{g}}} \frac{\delta(\sqrt{-\tilde{g}} L_\phi)}{\delta \tilde{g}^{\mu\nu}} \quad (4)$$

Automatically [1] this EMT is divergence-less i.e.

$$\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = 0 \quad (5)$$

We propose to transform the l.h.s. of (5) conformally to reach the initial state. This means in practice that the Einstein frame variables are substituted by Jordan frame variables. We can do so because the conformal transformation is invertible. Note carefully, that the operator  $\tilde{\nabla}$  when expressed

in terms of Jordan frame variables following the given conformal connection, the result may not be of the same form as (5). However, from [39], we know that there exists conformal invariance of certain equations involving the metric. Just at this point we float our assumption that the form of the transformed equation (5) may (if necessary by utilizing the equation of motion of Jordan frame where ever required) be put as

$$\nabla_\mu T^{\mu\nu}_J = 0 \quad (6)$$

where all entities are Jordan frame variables.  $T^{\mu\nu}_J$  then can be thought of as the EMT of the NMCT under consideration in the Jordan frame.

### 3 From Jordan frame to Einstein frame

In this section, we discuss the salient features of our scalar-tensor theory as a non-minimally coupled scalar field interacting with gravity in the Jordan frame and briefly review how one can apply a suitable conformal transformation to convert it to a corresponding theory in the Einstein frame where the scalar field is minimally coupled to gravity. This will help fix our notations as well as summarize all the relevant transformation relations that we need for our purpose. We start with the action for a scalar field  $\pi$  non-minimally coupled to gravity in the Jordan frame given by (1). The different symbols are explained therein.

Now let us consider a conformal transformation given by (3) which connects the metric  $g_{\mu\nu}$  of the physical Jordan frame to a metric  $\tilde{g}_{\mu\nu}$  on a different manifold. The determinants of these two matrices are related as

$$\sqrt{-g} = \Omega^{-4} \sqrt{-\tilde{g}} \quad (7)$$

and the affine connection in these two frames are related by

$$\Gamma^\alpha_{\mu\nu} = \tilde{\Gamma}^\alpha_{\mu\nu} - \left[ \tilde{\nabla}_\nu (\ln \Omega) \delta^\alpha_\mu + \tilde{\nabla}_\mu (\ln \Omega) \delta^\alpha_\nu - \tilde{\nabla}^\alpha (\ln \Omega) \tilde{g}_{\mu\nu} \right] = \tilde{\Gamma}^\alpha_{\mu\nu} - \tilde{A}^\alpha_{\mu\nu} \quad (8)$$

For future convenience let us note that although neither  $\tilde{\Gamma}^\alpha_{\mu\nu}$  nor  $\Gamma^\alpha_{\mu\nu}$  are tensor quantities in their respective frames,  $\tilde{A}^\alpha_{\mu\nu}$  is a tensor quantity symmetric under  $\mu \leftrightarrow \nu$ . Moreover, its form is such that it can be readily written either in Jordan frame or in Einstein frame as per our convenience.

$$\begin{aligned} \tilde{A}^\alpha_{\mu\nu} &= \left[ \tilde{\nabla}_\nu (\ln \Omega) \delta^\alpha_\mu + \tilde{\nabla}_\mu (\ln \Omega) \delta^\alpha_\nu - \tilde{\nabla}^\alpha (\ln \Omega) \tilde{g}_{\mu\nu} \right] \\ &= \left[ \nabla_\nu (\ln \Omega) \delta^\alpha_\mu + \nabla_\mu (\ln \Omega) \delta^\alpha_\nu - \nabla^\alpha (\ln \Omega) g_{\mu\nu} \right] \\ &= A^\alpha_{\mu\nu} \end{aligned} \quad (9)$$

Using the above relations (3, 8) we can further relate the curvature tensors defined in the two frames as

$$R^\alpha_{\beta\mu\nu} = \tilde{R}^\alpha_{\beta\mu\nu} + 2\tilde{A}^\alpha_{\beta[\mu;\nu]} + 2\tilde{A}^\alpha_{\lambda[\mu} \tilde{A}^\lambda_{\nu]\beta} \quad (10)$$

and the corresponding Ricci scalars as

$$R = \Omega^2 \left[ \tilde{R} + 6\tilde{\square} (\ln \Omega) - 6\tilde{\nabla}_\mu (\ln \Omega) \tilde{\nabla}^\mu (\ln \Omega) \right] \quad (11)$$

Once we substitute Eqs. (3, 7, 11) the Jordan frame action (1) takes the form

$$\begin{aligned} A_J = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{1}{2\kappa^2} \frac{D(\pi)}{\Omega^2} \{ \tilde{R} + 6\tilde{\square} (\ln \Omega) \right. \\ \left. - 6\tilde{\nabla}_\mu (\ln \Omega) \tilde{\nabla}^\mu (\ln \Omega) \} - \frac{1}{2\Omega^2} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \pi \tilde{\nabla}_\nu \pi - \frac{V(\pi)}{\Omega^4} \right] \end{aligned} \quad (12)$$

Note that since  $\pi$  is a scalar,  $\nabla_\mu \pi = \tilde{\nabla}_\mu \pi$ . Therefore the kinetic term for the scalar field  $\pi$  can be written using the transformed metric.

Now a particular choice of the conformal transformation

$$D(\pi) = \Omega^2 \quad (13)$$

simplifies (12) to

$$\begin{aligned} A_J = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{\tilde{R}}{2\kappa^2} + \frac{3}{\kappa^2} \tilde{\square} (\ln \Omega) \right. \\ \left. - \left( \frac{3}{\kappa^2} \tilde{\nabla}_\mu (\ln \Omega) \tilde{\nabla}^\mu (\ln \Omega) + \frac{1}{2\Omega^2} \tilde{\nabla}_\mu \pi \tilde{\nabla}^\mu \pi \right) - \frac{V(\pi)}{\Omega^4} \right] \end{aligned} \quad (14)$$

where the second term in (14) is a surface term since

$$\tilde{\square} (\ln \Omega) = \frac{1}{\sqrt{-\tilde{g}}} \partial_\alpha \left\{ \sqrt{-\tilde{g}} \tilde{\nabla}^\alpha (\ln \Omega) \right\}$$

Also due to the same choice of conformal transformation (13),

$$\tilde{\nabla}_\mu (\ln \Omega) = \frac{1}{2} \frac{D'}{D} (\tilde{\nabla}_\mu \pi) \quad (15)$$

with  $'$  denoting derivative with respect to  $\pi$ . Using (15) the third term in the parenthesis in equation (14) can be cast as the kinetic energy term of a new scalar field  $\phi$  defined by

$$\left( \frac{d\phi}{d\pi} \right)^2 = \left\{ \frac{3}{2\kappa^2} \left( \frac{D'}{D} \right)^2 + \frac{1}{D} \right\} = f^2(\pi) \quad (16)$$

and finally we have our action (14) converted into

$$A_E = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{\tilde{R}}{2\kappa^2} - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \phi \tilde{\nabla}_\nu \phi - U(\phi) \right] \quad (17)$$

in the transformed manifold where the newly defined scalar field  $\phi$  behaves as a quintessence scalar field with the self-interaction

$$U(\phi) = \frac{V(\pi)}{D^2}. \quad (18)$$

The conformal transformation (3) along with the choice (13) has mapped our non-minimally coupled scalar field theory in the Jordan frame to a minimally-coupled scalar field theory

in this new frame with metric  $\tilde{g}_{\mu\nu}$ . This is referred to as the Einstein frame in standard literature [37]. The issue of the physical equivalence of the Einstein frame to the Jordan frame is a sensitive one in the literature but as far as the classical aspects are concerned the two formulations can be safely assumed to describe the same physical reality. In the next section, let us critically pin-point to what extent this equivalence holds true.

#### 4 Equivalence of Jordan frame vis-à-vis Einstein frame

Let us examine if the equivalence of Jordan frame vis-à-vis Einstein frame holds at the equation of motion level. Because of the minimal coupling the action for the gravitational field and that of the scalar field are readily distinguishable in the Einstein frame action  $A_E$  in (17)

$$A_E = \int d^4x \sqrt{-\tilde{g}} \frac{\tilde{R}}{2\kappa^2} - \int d^4x \sqrt{-\tilde{g}} L_\phi \quad (19)$$

So the standard definition of symmetric EMT (4) for the scalar field  $\phi$  readily applies here and yields

$$\tilde{T}_{\mu\nu} = \tilde{\nabla}_\mu \phi \tilde{\nabla}_\nu \phi - \tilde{g}_{\mu\nu} \left\{ \frac{1}{2} \tilde{g}^{\alpha\beta} \tilde{\nabla}_\alpha \phi \tilde{\nabla}_\beta \phi + U(\phi) \right\} \quad (20)$$

This symmetric EMT appears in the gravitational field equations

$$\tilde{G}_{\mu\nu} = \kappa^2 \tilde{T}_{\mu\nu} \quad (21)$$

obtained by varying the action (17) with respect to the Einstein frame metric  $\tilde{g}_{\mu\nu}$ . To obtain the equation of motion, i.e., the field equation for the scalar field  $\phi$  we can either demand the covariant conservation of its EMT given by (5) or vary the action (17) with respect to  $\phi$  to arrive at

$$\tilde{\square} \phi - \frac{dU}{d\phi} = 0 \quad (22)$$

The equivalence of this Einstein frame description to its Jordan frame counterpart, on-shell, will be verified if starting from the field equation of  $\phi$  (22) we can obtain the field equation for  $\pi$ , the corresponding scalar field in Jordan frame. To do this we remember that both  $\phi$  and  $\pi$  are scalars, therefore their covariant derivatives are related by

$$\tilde{\nabla}_\nu \phi = \partial_\nu \phi = \left( \frac{d\phi}{d\pi} \right) \partial_\nu \pi = \left( \frac{d\phi}{d\pi} \right) \nabla_\nu \pi = f(\pi) \nabla_\nu \pi \quad (23)$$

where in the last equality Eq. (16) is used. Using (23) we can further relate their d'Alembertians as

$$\tilde{\square} \phi = \frac{(fD)'}{D^2} (\nabla_\alpha \pi \nabla^\alpha \pi) + \frac{fD}{D^2} \square \pi \quad (24)$$

and from Eq. (18) we calculate

$$\frac{dU}{d\phi} = \frac{V'(\pi)}{fD^2} - \frac{2VD'}{fD^3} \quad (25)$$

Using these relations (24, 25) in (22) we get

$$\square \pi + \frac{(fD)'}{(fD)} (\nabla_\mu \pi \nabla^\mu \pi) - \frac{V'}{Df^2} - \frac{2VD'}{D^2} = 0 \quad (26)$$

Also note that  $\pi$  is non-minimally coupled, so to get to its equation of motion, we need the gravitational field equation of the Jordan frame as well. The same can be obtained by varying the Jordan frame action (1) w.r.t.  $g_{\alpha\beta}$  that gives

$$D(\pi)G_{\alpha\beta} = \kappa^2 \left[ \nabla_\alpha \pi \nabla_\beta \pi - g_{\alpha\beta} \left\{ \frac{1}{2} g^{\mu\nu} \nabla_\mu \pi \nabla_\nu \pi + V(\pi) \right\} \right] + \{ \nabla_\alpha \nabla_\beta D(\pi) - g_{\alpha\beta} \square D(\pi) \} \quad (27)$$

Using the trace of (27)

$$D(\pi)R = \kappa^2 \left( 1 + \frac{3D''(\pi)}{\kappa^2} \right) (\nabla_\alpha \pi \nabla^\alpha \pi) + 3D'(\pi) \square \pi + 4\kappa^2 V(\pi) \quad (28)$$

and re-expressing  $(fD)'$  using (16), as

$$(fD)' = \frac{(\frac{3D''}{\kappa^2} + 1) \frac{D'}{2}}{(fD)} \quad (29)$$

in (26) we can finally rewrite the  $\phi$ -field equation completely in terms of Jordan frame variables as

$$\square \pi + D' \left[ \frac{R}{2\kappa^2} \right] - V'(\pi) = 0 \quad (30)$$

which is nothing but the equation of motion for the  $\pi$ -field in the Jordan frame, as can be verified by directly varying the Jordan frame action (1) with respect to  $\pi$ . This strengthens our conviction that physical behaviour of the system depicted in both the frames should map into each other exactly. However the geometric quantities of the two frames do not match, as can be seen from Eqs. (10) and (11) that show both curvature tensors and Ricci scalars of the two frames differ by certain tensor quantities. This is not surprising since the basic geometry of the two frames encoded in their respective metrics are different (3). In the next section, let us examine if physical parameters like energy density and pressure in the two frames adhere to this equivalence.

#### 5 Energy-momentum tensor in Einstein frame expressed in terms of Jordan frame variables

The parameters like energy density and pressure of the scalar field are encoded in its EMT. Owing to its minimal coupling to gravity the  $\phi$  field in Einstein frame has an EMT (20) that follows from an unambiguous definition (4). So we start there and use Eqs. (23) and (18) in (20) to re-express it in terms of

the  $\pi$  field, its derivatives and its self-interaction  $V(\pi)$ . This gives

$$\begin{aligned}\tilde{T}^{\alpha\beta} &= \frac{1}{D^2} \left[ (\nabla^\alpha \pi \nabla^\beta \pi) \dot{f}^2(\pi) - \dot{f}^2(\pi) (\nabla^\mu \pi \nabla_\mu \pi) \frac{g^{\alpha\beta}}{2} \right. \\ &\quad \left. - g^{\alpha\beta} \frac{V(\pi)}{D} \right] \\ &\equiv \Theta^{\alpha\beta}(\nabla^\mu \pi, \pi)\end{aligned}\quad (31)$$

We deliberately introduce a different symbol  $\Theta^{\alpha\beta}(\nabla^\mu \pi, \pi)$  here to signify that the right hand side is the Einstein frame EMT, but written in terms of Jordan frame scalar field and its derivatives. Direct calculation shows that  $\Theta^{\alpha\beta}(\nabla^\mu \pi, \pi)$  is not covariantly conserved in the Jordan frame and therefore can not pose as the EMT of the  $\pi$  field. This is not entirely unexpected since the conformal transformation from Jordan to Einstein frames is not a change of variable but that of geometry. Therefore like geometric variables (e.g. curvature tensor) the physical observables in the two frames also do not map into each other.

Before we proceed further let us point out the source of ambiguity of the EMT in any definition depending on the single action (1) – it is due to the term  $D(\pi)$ . So to get an algorithm for EMT we must involve some extra input. In the present case it is the physical equivalence of Einstein and Jordan frame. How it is done will be described in the following.

## 6 Algorithm for the energy momentum tensor in the Jordan frame

We start with the conservation law (5) for the scalar field  $\phi$  in Einstein frame and try to re-express it in terms of Jordan frame variables. To this end, we first expand the covariant divergence of  $\tilde{T}^{\alpha\beta}$

$$\tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta} = \partial_\alpha \tilde{T}^{\alpha\beta} + \tilde{\Gamma}_{\alpha\lambda}^\alpha \tilde{T}^{\lambda\beta} + \tilde{\Gamma}_{\alpha\lambda}^\beta \tilde{T}^{\alpha\lambda} \quad (32)$$

and then using (8), (9) and (31) write the expansion in terms of Jordan frame variables

$$\tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta} = \partial_\alpha \Theta^{\alpha\beta} + (\Gamma_{\alpha\lambda}^\alpha + A_{\alpha\lambda}^\alpha) \Theta^{\lambda\beta} + (\Gamma_{\alpha\lambda}^\beta + A_{\alpha\lambda}^\beta) \Theta^{\alpha\lambda} \quad (33)$$

and finally combine suitable terms to express it as a sum of tensor quantities in the Jordan frame

$$\tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta} = \nabla_\alpha \Theta^{\alpha\beta} + (A_{\alpha\lambda}^\alpha \Theta^{\lambda\beta} + A_{\alpha\lambda}^\beta \Theta^{\alpha\lambda}) \quad (34)$$

so that the conservation Eq. (5) of the Einstein frame, expressed in terms of Jordan frame variables, becomes

$$\nabla_\alpha \Theta^{\alpha\beta} + (A_{\alpha\lambda}^\alpha \Theta^{\lambda\beta} + A_{\alpha\lambda}^\beta \Theta^{\alpha\lambda}) = 0 \quad (35)$$

Our aim is to show that we can extract conserved EMTs from (35) by simple algebraic manipulations and the equation of motion in Jordan frame. In other words we assume that the conformal transformations of the fields will allow us to write the expression

$$\nabla_\alpha \Theta^{\alpha\beta} + (A_{\alpha\lambda}^\alpha \Theta^{\lambda\beta} + A_{\alpha\lambda}^\beta \Theta^{\alpha\lambda}) \quad (36)$$

as a total divergent

$$\nabla_\mu T^{\mu\nu}{}_J \quad (37)$$

where all entities of  $T^{\mu\nu}{}_J$  are Jordan frame variables. The research on non-minimal coupling is an old one with so many papers appearing in the field. The purpose is to explain the late time acceleration [47]. However, there is an opinion that the modifications of gravity can be absorbed in the equations of cosmology to project the theory as written in dark energy paradigm [37]. We do not subscribe to the view [5, 6] because there is deep physical considerations involved in the new modifications of gravity which cannot be just wiped away by a swing of hand.

So finally, we have reached a point from where the answer to the question posed in the very beginning of the paper is explicit.

Using (31) and the expression of  $A_{\mu\nu}^\alpha$  from (9), it is straightforward to compute

$$\begin{aligned}(A_{\alpha\lambda}^\alpha \Theta^{\lambda\beta} + A_{\alpha\lambda}^\beta \Theta^{\alpha\lambda}) &= \left\{ 2 (\nabla_\lambda \pi \nabla^\lambda \pi) \left( \frac{\dot{f}}{D} \right)^2 - \frac{V(\pi)}{D^3} \right\} \\ &\quad \times \left( \frac{D'}{D} \right) \nabla^\beta \pi\end{aligned}\quad (38)$$

Similarly using (31) we can compute

$$\begin{aligned}\nabla_\alpha \Theta^{\alpha\beta} &= \left\{ \nabla_\alpha \left( \frac{\dot{f}}{D} \right)^2 \right\} \left\{ \nabla^\alpha \pi \nabla^\beta \pi - \frac{1}{2} g^{\alpha\beta} (\nabla_\lambda \pi \nabla^\lambda \pi) \right\} \\ &\quad + \left( \frac{\dot{f}}{D} \right)^2 \nabla_\alpha \left\{ \nabla^\alpha \pi \nabla^\beta \pi - \frac{1}{2} g^{\alpha\beta} (\nabla_\lambda \pi \nabla^\lambda \pi) \right\} \\ &\quad - \nabla_\alpha \left( \frac{g^{\alpha\beta} V(\pi)}{D^3} \right)\end{aligned}\quad (39)$$

The second term of Eq. (39) can be written as

$$\begin{aligned}\nabla_\alpha \left\{ \nabla^\alpha \pi \nabla^\beta \pi - \frac{1}{2} g^{\alpha\beta} (\nabla_\lambda \pi \nabla^\lambda \pi) \right\} \\ = \nabla_\alpha \left[ \nabla^\alpha \pi \nabla^\beta \pi - \frac{1}{2} g^{\alpha\beta} (\nabla_\lambda \pi \nabla^\lambda \pi) - g^{\alpha\beta} V(\pi) \right. \\ \left. + \frac{1}{\kappa^2} \{ \nabla^\alpha \nabla^\beta D - g^{\alpha\beta} \square D + (1-D) G^{\alpha\beta} \} \right] \\ + \nabla_\alpha \left[ g^{\alpha\beta} V(\pi) - \frac{1}{\kappa^2} \{ \nabla^\alpha \nabla^\beta D - g^{\alpha\beta} \square D \right. \\ \left. + (1-D) G^{\alpha\beta} \} \right]\end{aligned}\quad (40)$$



We can write the left hand side of (40) as

$$\begin{aligned} \nabla_\alpha \left\{ \nabla^\alpha \pi \nabla^\beta \pi - g^{\alpha\beta} \frac{1}{2} (\nabla_\lambda \pi \nabla^\lambda \pi) \right\} \\ = \nabla_\alpha T^{\alpha\beta} + \nabla_\alpha \left[ g^{\alpha\beta} V(\pi) - \frac{1}{\kappa^2} \{ \nabla^\alpha \nabla^\beta D \right. \\ \left. - g^{\alpha\beta} \square D + (1 - D) G^{\alpha\beta} \} \right] \end{aligned} \quad (41)$$

where

$$\begin{aligned} T^{\alpha\beta} = \left[ \nabla^\alpha \pi \nabla^\beta \pi - \frac{1}{2} g^{\alpha\beta} (\nabla_\lambda \pi \nabla^\lambda \pi) - g^{\alpha\beta} V(\pi) \right. \\ \left. + \frac{1}{\kappa^2} \{ \nabla^\alpha \nabla^\beta D - g^{\alpha\beta} \square D + (1 - D) G^{\alpha\beta} \} \right] \end{aligned} \quad (42)$$

Thus Eq. (39) becomes

$$\begin{aligned} \nabla_\alpha \Theta^{\alpha\beta} = \left( \frac{f}{D} \right)^2 \nabla_\alpha T^{\alpha\beta} + \left\{ \nabla_\alpha \left( \frac{f}{D} \right)^2 \right\} \\ \times \left\{ \nabla^\alpha \pi \nabla^\beta \pi - \frac{1}{2} g^{\alpha\beta} (\nabla_\lambda \pi \nabla^\lambda \pi) \right\} + \left( \frac{f}{D} \right)^2 \nabla_\alpha \\ \times \left[ g^{\alpha\beta} V(\pi) - \frac{1}{\kappa^2} \{ \nabla^\alpha \nabla^\beta D - g^{\alpha\beta} \square D + (1 - D) G^{\alpha\beta} \} \right] \\ - \nabla_\alpha \left( \frac{g^{\alpha\beta} V(\pi)}{D^3} \right) \end{aligned} \quad (43)$$

The fact that  $\nabla_\alpha V(\pi) = V' \nabla_\alpha \pi$  and using the trace of the gravity field equation in Jordan frame (28) and the equation of motion of the  $\pi$  field (30), we can further simplify (43). The same is given by

$$\begin{aligned} \nabla_\alpha \Theta^{\alpha\beta} = \left( \frac{f}{D} \right)^2 \nabla_\alpha T^{\alpha\beta} - \left\{ 2 (\nabla_\lambda \pi \nabla^\lambda \pi) \left( \frac{f}{D} \right)^2 \right. \\ \left. - \frac{V(\pi)}{D^3} \right\} \left( \frac{D'}{D} \right) \nabla^\beta \pi \end{aligned} \quad (44)$$

Putting the values of (38) and (44) in (35), one gets

$$\nabla_\alpha T^{\alpha\beta} = 0 \quad (45)$$

It is gratifying to observed that this is the sought for equation which we expected from our assumption. So we get the algorithm for finding EMT for any NMCT.

For the convenience of the reader we now summarized our algorithm in the following

1. Let the action given in Jordan frame be of the form (1). Then the conformal transformation to the Einstein frame is to be found.
2. Using the Einstein frame action thus obtained and the formula (4) the EMT in the Einstein frame ( $T^{\mu\nu}$ ) can easily be deduced. The corresponding conservation relation read as

$$\tilde{\nabla}_\mu T^{\mu\nu} = 0 \quad (46)$$

Note carefully, that both the operator ( $\tilde{\nabla}_\mu$ ) and EMT ( $T^{\mu\nu}$ ) are obtained as function of Einstein frame parameters.

3. Now using the connection between the Jordan frame and Einstein frame we get the form of the conservation law (46) as

$$\nabla_\alpha \Theta^{\alpha\beta} + (A_{\alpha\lambda}^\alpha \Theta^{\lambda\beta} + A_{\alpha\lambda}^\beta \Theta^{\alpha\lambda}) = 0 \quad (47)$$

Here all the variables are the function of Jordan frame parameters.

4. After the last step we are in the possession of an equation (47) which is of the form  $\chi = 0$ . Now by purely algebraic manipulations and the equation of motion of the Jordan frame we get the desired form (6).

This EMT (42) may be thought of as the conserved EMT in the Jordan frame i.e;  $T^{\alpha\beta}_J$  as mentioned in our approach (see below Eq. (5)). Clearly in our algorithm there is no empirical division of the action or its arbitrary rearrangement.

## 7 Conclusion

The widespread use of scalar-tensor theories [28–32] in cosmology demands a close examination of the ambiguity that is present in the energy momentum tensor (EMT) of the non-minimally coupled scalar. In NMCT, matter and gravity are so coupled that it is impossible to vary the matter action without varying gravity. Thus the form of EMT in this theory are empirically taken without any deep physical foundation. The standard approach in the literature to circumvent this difficulty is to algebraically manipulate the gravity field equation so that the covariant conservation of the Einstein tensor can be used to identify the conserved EMT. However different manipulations lead to different forms of the EMT resulting in the said ambiguity. The importance of EMT in the cosmology can hardly be overemphasized. So such ambiguity in EMT coming from a mere algebraic rearrangement and not from some general physical principle is definitely not welcome. In this paper, we demonstrate how to extract a symmetric, covariantly conserved EMT from the conformal invariance [39,48–52] of Jordan frame and Einstein frame frames. Interestingly, the non-minimally coupled theory in the Jordan frame emerges as a minimally coupled theory in Einstein frame description. Though there remains some difference of opinion about the equivalence of these two formulations in the quantum mechanical level, it is universally accepted that classically the equivalence holds.

In this paper, we employed this equivalence of the Jordan frame and Einstein frame formulation of the theory to explicitly demonstrate that physical features like time evolution of

the fields and conservation laws in one frame implies that in the other. In the process we show that starting from the covariant conservation of the scalar field EMT in the Einstein frame one can arrive at the corresponding conservation law in the Jordan frame.

So this paper serves a two-fold purpose, one is to examine the equivalence of the Einstein frame and Jordan frame descriptions [40, 41, 48, 53, 54] of a class of scalar-tensor theories at the level of the conservation laws and the extraction of EMT of a NMCT in an unambiguous manner. Furthermore, our algorithm is class apart from others in the literature in the sense that we don't look for mathematical rearrangement of Einstein's gravity equation to obtain EMT of a NMCT rather we utilize equivalence of physical relations existing between frames to obtain a symmetric, covariantly conserved EMT.

**Data Availability Statement** This manuscript has no associated data or the data will not be deposited. [Authors' comment: All the required data is in the manuscript.]

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Funded by SCOAP<sup>3</sup>. SCOAP<sup>3</sup> supports the goals of the International Year of Basic Sciences for Sustainable Development.

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