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# Five dimensional analysis of electromagnetism with heat flow in the post-quasi-static approximation

**A. Zahra**[a], **S. A. Mardan**[b]

Department of Mathematics, University of the Management and Technology, C-II, Johar Town, Lahore 54590, Pakistan

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**Abstract** The development of dissipative and electrically charged distributions in five dimensions is presented by using the post-quasistatic approximation. It is an iterative technique for the evolution of self-gravitating spheres of matter. We construct non-adiabatic distributions by means of an equation of state that accounts for the anisotropy based on electric charge. Streaming out and diffusion approximations are used to describe dissipation. In non-comoving coordinates, we match the higher dimensional interior solution with the corresponding Vaidya–Reissner–Nordström exterior solution. Hence, a system of higher dimensional surface equations results from generalized form of the post-quasistatic approximation. Surface equations are essential for understanding physical phenomena such as luminosity, Doppler shift, and red-shift at the boundary surface of gravitating sources.

## 1 Introduction

The study of charged relativistic fluid balls is a topic that physics and astrophysics researchers from many different fields are interested. There is general agreement that astronomical objects with a lot of charge can't exist in the natural world [1,2]. This point of view has been questioned by numerous researchers [3–6]. It is not impossible that matter might pick up significant electric charge during gravitational collapse or accretion onto a compact object. This was considered in Diego et al. [7] and Shvartsman [8].

In this article, we look into account relativistic compact objects with a fluid that dissipates energy and a spherical distribution of charged matter. The generated electric field in self-gravitating systems is thought to be regulated by the requirement that it not be greater than $10^{16}$ V cm$^{-1}$ [9], which is considered the critical field for pair creation. This critical field restriction has been challenged [10–13]. The quasi-static approximation (QSA) [14] is clearly unreliable for intensive dynamical activity with time scales on the order of the hydrostatic time scale. The majority of research on electric charge has been done in static conditions [15–18]. Recent research has focused on charged quasi-black holes [19,20] and their growth into a quasi-spherical world [21]. Electrically charged distributions can be conceived of as anisotropic [22,23] in the real world. The authors combine anisotropy and electric charge using an equation of state, but not as a single entity [24,25].

Spherically symmetric solutions in general relativity (GR) are important in the study of compact objects. The gravitational fields of astronomical bodies can be modeled by using spherically symmetric solutions to the Einstein field equations (EFEs). Indeed, most studied exact solutions to EFEs are spherically symmetric. If the metric components of spherical symmetry are static then exterior space time is taken as Schwarzschild solution [26]. Reissner [27], Weyl [28] and Nordström [29] developed the Reissner–Nordström solution to describe impact of electromagnetic field on gravitating system. The exact vacuum solution of EFEs that describes a rotating, stationary, axially symmetric black hole was discovered by Kerr [30]. This solution describes a black hole because it describes the spacetime generated by a singularity with a curvature hidden by a horizon. Myers and Perry investigated the Schwarzschild, Reissner–Nordström, and Kerr solutions for higher dimensional spacetimes [31]. Shen and Tan [32] discussed Wyman's solution in higher dimensions. Chatterjee [33] obtained an exterior solution for spherically symmetric Kaluza–Klein (KK) type metric.

The development of GR to higher dimensions has gained a lot of attention in recent years. The five dimension and higher manifolds presented by KK are used in various gravity theories that extends Einstein's GR. After a couple of decades of the introduction of special relativity, Kaluza [34]

[a] e-mail: F2019265004@umt.edu.pk (corresponding author)
[b] e-mail: syedalimardanazmi@yahoo.com

and Klein [35] postulated the existence of an extra spatial dimension that can be considered as an extension to relativistic theory. Their motivation for doing so was to give a unified description of electromagnetism and gravity in terms of a five-dimensional metric. The considerable work in this domain after KK was presented by Wesson's [36], he studied the properties of matter in KK theories. The space-time matter theories [37] have become increasingly prominent in gravitation and cosmology in the fifth dimension. Liu and Overduin [38] investigated light deflection and time delay results for massless test particles in higher dimensions. Rahaman et al. [39] studied the usual solar system phenomenon, such as the perihelion shift, light bending, gravitational red-shift, gravitational time delay and motion of test particles that are compatible with the existence of higher spatial dimensions. Bars and Terning [40] introduced the extra time dimension. The solution was based on gauge symmetry. They developed the general framework by using the extra time dimension coordinate and found that the results are consistent with standard models of general relativity. The higher dimensional gravstars were also studied by Rahaman et al. [41]. Moreover, many researchers have worked on higher derivative gravities in connection to extension of GR [42–50].

The Rosen's bimetric field equations in higher dimensions for the static spherically symmetric space-time with charged anisotropic fluid distribution were solved by Pandya and Hasmani [51]. Singh et al. [52] investigated two distinct cosmological models with massive strings in five dimensional relativistic theories. The first produces a five-dimensional model of the Universe, while the second produces the vacuum Universe. The properties of the model Universe are investigated and compared to the properties of the four-dimensional model. Baro et al. [53] investigated a model of the universe that is isotropic throughout its evolution, non-sharing and free from the initial singularity.

Debnath et al. [54] used a higher-dimensional extension of the quasi-spherical Szekeres metrics with a non-zero cosmic constant to study gravitational collapse in $(n+2)$ dimensions. They discovered that the possibility of a naked singularity depends on the initial density of a space-time with more than five dimensions. The results are comparable to the collapse in Tolman–Bondi–Lemaitre space-times with spherically symmetric space-times. Yamada and Shinkai [55] investigated the gravitational collapse of collisionless particles in spheroidal structures in both four and five dimensions of space-time using numerical methods. The collapsing behaviors are quite similar to the cases in four-dimension, but they also found that five-dimensional collapses proceed rapidly than four-dimensional collapses. Khan et al. [56] presented the five-dimensional spherically symmetric anisotropic collapse with a positive cosmological constant. They employ the Schwarzschild–de Sitter and five-dimensional spherically symmetric metrics for the inner and outer regions, respectively. They found that the entire collapse process is impacted by the cosmological constant. The collapse process is slowed down by the cosmological constant.

Barnaföldi et al. [57] investigate the higher dimensional neutron star with compactified fifth-dimensional excitations. They showed that neutron stars with hyperon or extra-dimensional cores are remarkably similar objects in a simple model of a compact star. The Tolman–Oppenheimer–Volkoff (TOV) equation produces a comparable structure with a clearly defined stability area in the extra-dimensional case, where the lowest KK modes may be observed. Additionally, they introduced a new dimension, which contributed about the emergence of new stability areas and the presence of many stable hybrid star configurations. The double neutrino shower from SN 1987A supports this conclusion. Paul [58] studied the relativistic solutions of higher dimension compact star in hydrostatic equilibrium with spherically symmetric space-time. He found that the presence of higher dimensions directly affects the star's central density. The square of the dimensions of space and time causes the density of the star's core to increase roughly proportionally. As a result, if a star is surrounded in dimensions other than the standard four of space-time, its centre density is relatively higher for a given radius. It is also obvious that for a given radius, more space-time dimensions than four allow for a more massive compact star. Bhar et al. [59] provided evidence for the existence of higher dimensional anisotropic compact stars in noncommutative space-time. They found that the physical behaviors of the conservative variables, such as energy density, radial pressure, transverse pressures, anisotropy, and other characteristics, are generally consistent throughout the stellar structure. They also mention that as one goes to higher dimensions, the central densities abruptly decrease, and that the measure of anisotropy gradually increases, reaching its maximum at five dimensions. A star's central density is greatest in four dimensions and lowest in higher dimensions.

Numerical techniques enable mathematicians to study systems that are complicated to handle analytically [60]. Numerical models have proven to be beneficial in the investigation of strong field scenarios and for revealing unexpected occurrences in GR [61]. Nonetheless, it is clearly more straight forward to solve ordinary differential equations (ODEs) than partial differential equations in general. However, numerical solutions frequently make it difficult to express general, qualitative, aspects of this process. The suggested method produces a system of ODEs for quantities determined at the fluid distribution's boundary surface (BS) starting from any interior static spherically symmetric seed solution to the EFEs. The static limit of the numerical solution, which simulates dynamical self-gravitating spheres, is the initial seed solution.

The pioneer work of Oppenheimer and Snyder [62] urged researchers to explore relativistic aspects of gravitating source their formation and inner structure. The motivation for such interest is based on the fact that the relativistic collapse of massive stars is one of the main visible process in which GR is predicted to play a vital role. However, self-gravitating compact objects may experience periods of extreme dynamical activity as they evolve over time. The static or quasi-static (QS) approximation is unreliable for some phenomena, such as the origination of neutron stars as a result of quick collapse. In such conditions, it is necessary to consider concepts that describe departures from equilibrium. Herrera et al. [63] initially presented the post-quasistatic (PQS) approximations essence using radiative Bondi approach. Herrera and his coworkers [64] have made considerable use of it. In Bondi approach, the concept of QS approximation is absent: the system proceeds immediately from static to PQS evolution. The PQS approximation depends on "effective" variables, such as effective pressure and energy density [65]. Because the effective variables of the QS approximation correspond to the physical variables. This approximation can be assumed as an iterative technique, with each successive step representing a greater deviation from equilibrium. More precisely, the authors [61] employed a method for modeling the evolution of compact objects that does not necessitate complete integration of the EFEs with respect to the time coordinate. Zahra et al. presented the general framework of the PQS approximation with heat flow in five dimensional noncomoving coordinates [66].

The purpose of this work is to investigate the evolution of compact objects in the PQS regime in five dimensions. We study, a self-gravitating spherical distribution of charged matter containing dissipative fluid. The emission of photons and/or neutrino particles causes dissipation, which is a common procedure in the evolution of compact objects. In fact, neutrino emission appears to be the only viable technique for removing the majority of the binding energy from a collapsing star. However, there are only two approximations, diffusion and streaming out are frequently employed in the analysis of radiative transport within compact objects. The diffusion approximation assumes that likewise thermal conduction, the energy flux of radiation is approximately equal to the temperature gradient. This assumption is generally viable because the mean free path of the particles responsible for energy transmission in star interiors is usually quite short in comparison to the object's normal length. A star, such as the sun, has a mean free path of massless particles photons on the order of 2 cm. The mean free path of trapped neutrinos is less than the size of the star core in compact cores with densities of about $10^{12}$ g.cm$^{-3}$ [67,68].

The emission of energy through streaming-out radiation is a highly efficient mechanism that results in the redistribution of electric charge throughout the sphere. As the distribution of the sphere continues to collapse over an extended hydrodynamic time scale, this process ensures that the electric charge is evenly distributed. During the initial stages of the collapse, the free streaming process is responsible for this redistribution, while the diffusion approximation becomes increasingly applicable as the collapse progresses towards its final stages [69]. In addition, data from the 1987A supernova show that during the emission phase, the predominant radiation transport regime is closer to the diffusion approximation than the streaming out limit [70].

In this article, we will discuss physical variables such as energy density, pressure, unpolarized radiation of energy density, proper velocity and electric charge. These physical variables predicted to play a vital role in the evolution of self-gravitating objects. Although employing comoving coordinates is the most usual way to solve EFEs, we will use noncomoving coordinates, which means that the velocity of every fluid element must be taken into consideration as a relevant physical variable [71,72]. The layout of this work is as follows. We describe the conventions and provide the higher dimensional Einstein-Maxwell field equations in Sect. 2 and 3. The exterior spacetime is presented in Sect. 4. The Methodology of this paper is discussed in Sect. 5. Finally, conclusion and discussion is presented in Sect. 4 that are followed by a list of references.

## 2 Higher dimensional field equations

We consider non-static spherically symmetric distributions of a collapsing fluid confined by a spherical surface $\sum$, where dissipation occurs due to free-streaming radiation and/or heat flow and anisotropy induced by electric charge. By using five dimensional Schwarzschild-like coordinates [73], the metric is then written as

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + sin^2 d\phi^2) - e^{\mu}dw^2, \quad (1)$$

where $\nu$, $\lambda$ and $\mu$ are general functions of time and radial coordinates. The spacetime coordinates are $x^0 = t, x^1 = r, x^2 = \theta, x^3 = \phi, x^4 = w$. The corresponding system of equations for the Maxwell-EFEs in tensorial form is as follows:

$$G^{\nu}_{\mu} = -8\pi T^{\nu}_{\mu}. \quad (2)$$

That leads to following set of equations

$$-8\pi T^0_0 = -\frac{1}{r^2} + e^{-\lambda}\left(\frac{1}{r^2} + \frac{(\mu' - \lambda')}{r} + \frac{((\mu')^2 - \mu'\lambda')}{4} + \frac{\mu''}{2}\right) + e^{-\nu}\left(\frac{\dot{\mu}\dot{\lambda}}{4}\right), \quad (3)$$

$$-8\pi T^1_1 = -\frac{1}{r^2} + e^{-\lambda}\left(\frac{1}{r^2} + \frac{(\nu' + \mu')}{r} + \frac{\mu'\nu'}{4}\right) - \frac{e^{-\nu}}{4}\left(2\ddot{\mu} + (\dot{\mu})^2 - \dot{\mu}\dot{\nu}\right), \quad (4)$$

$$
\begin{aligned}
-8\pi T_2^2 &= -8\pi T_3^3 \\
&= \frac{1}{4}\Bigg[-e^{-\nu}\Big\{2(\ddot{\mu}+\ddot{\lambda})+\dot{\mu}(\dot{\mu}-\dot{\nu}+\dot{\lambda})+\dot{\lambda}(\dot{\lambda}-\dot{\nu})\Big\} \\
&\quad +e^{-\lambda}\Big\{2(\nu''+\mu'')+(\nu')^2+(\mu')^2-\nu'\lambda'-\mu'(\lambda'+\nu')\Big\}\Bigg] \\
&\quad +\frac{2}{r}\Big(\nu'-\lambda'+\mu'\Big),
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
-8\pi T_4^4 &= \frac{1}{r^2}-e^{-\lambda}\left(\frac{1}{r^2}+\frac{\nu''}{2}+\frac{(\nu')^2}{4}-\frac{\nu'\lambda'}{4}+\frac{(\nu'-\lambda')}{r}\right) \\
&\quad +\frac{e^{-\nu}}{4}\Big(2\ddot{\lambda}+(\dot{\lambda})^2-\dot{\nu}\dot{\lambda}\Big),
\end{aligned} \tag{6}
$$

$$
-8\pi T_{01} = \left(\frac{\mu'\dot{\mu}}{4}-\frac{\dot{\mu}\nu'}{4}-\frac{\mu'\dot{\lambda}}{4}+\frac{\dot{\mu}'}{2}-\frac{\dot{\lambda}}{r}\right), \tag{7}
$$

where (.) and (′) denote partial differentiation in terms of $t$ and $r$ respectively. We use the Bondi technique [74] to give physical meaning to the components of energy stress tensor $T_{\nu}^{\mu}$.

Thus, in the accordance with Bondi, we will introduce Minkowski coordinates $(\tau, x, y, z, h)$, in five dimension as

$$
\begin{aligned}
&d\tau = e^{\nu/2}dt, \quad dx = e^{\lambda/2}dr, \quad dy = rd\theta, \\
&dz = r sin\theta d\phi, \quad dh = e^{\mu/2}dw.
\end{aligned}
$$

Then, using a bar to represent the higher dimensional Minkowski coordinates of the energy stress tensor, we obtain

$$
\begin{aligned}
&\bar{T}_0^0 = T_0^0; \quad \bar{T}_1^1 = T_1^1; \quad \bar{T}_2^2 = T_2^2; \quad \bar{T}_3^3 = T_3^3; \\
&\bar{T}_4^4 = T_4^4; \quad \bar{T}_{01} = e^{-\frac{(\nu+\lambda)}{2}}T_{01}.
\end{aligned}
$$

The five-dimensional Lorentz transformation then demonstrates that

$$
T_0^0 = \bar{T}_0^0 = \frac{\rho+P\omega^2}{1-\omega^2}+\frac{2Q\omega e^{\frac{\lambda}{2}}}{\sqrt{(1-\omega^2)}}+\epsilon, \tag{8}
$$

$$
T_1^1 = \bar{T}_1^1 = -\frac{P+\rho\omega^2}{1-\omega^2}-\frac{2Q\omega e^{\frac{\lambda}{2}}}{\sqrt{(1-\omega^2)}}-\epsilon, \tag{9}
$$

$$
T_2^2 = \bar{T}_2^2 = T_3^3 = \bar{T}_3^3 = T_4^4 = \bar{T}_4^4 = -P, \tag{10}
$$

$$
\begin{aligned}
T_{01} &= e^{\frac{(\nu+\lambda)}{2}}\bar{T}_{01} = -\frac{(\rho+P)\omega e^{\frac{(\nu+\lambda)}{2}}}{1-\omega^2} \\
&\quad -\frac{Qe^{\frac{\nu}{2}}e^{\lambda}(1+\omega^2)}{\sqrt{(1-\omega^2)}}-e^{\frac{(\nu+\lambda)}{2}}\epsilon,
\end{aligned} \tag{11}
$$

with

$$
Q \equiv \frac{\hat{q}e^{\frac{-\lambda}{2}}}{\sqrt{(1-\omega^2)}}, \tag{12}
$$

and

$$
\epsilon \equiv \hat{\epsilon}\frac{(\omega+1)}{(1-\omega)}. \tag{13}
$$

It is worth noting that in $(t, r, \theta, \phi, w)$ system, the coordinate velocity $\frac{dr}{dt}$ is associated with the proper velocity $\omega$ by

$$
\omega = \frac{dr}{dt}e^{\frac{(\lambda-\nu)}{2}}. \tag{14}
$$

Applying Lorentz transformed Eqs. (8–11), in field Eqs. (3–7), we obtain

$$
\begin{aligned}
&\frac{\rho+P\omega^2}{1-\omega^2}+\frac{2Q\omega e^{\frac{\lambda}{2}}}{\sqrt{(1-\omega^2)}}+\epsilon+\frac{s^2e^{-\mu}}{8\pi r^4} \\
&= -\frac{1}{8\pi}\Bigg\{+e^{-\lambda}\left(\frac{1}{r^2}+\frac{(\mu'-\lambda')}{r}+\frac{((\mu')^2-\mu'\lambda')}{4}+\frac{\mu''}{2}\right) \\
&\quad -\frac{1}{r^2}+e^{-\nu}\left(\frac{\dot{\mu}\dot{\lambda}}{4}\right)\Bigg\},
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
&\frac{P+\rho\omega^2}{1-\omega^2}+\frac{2Q\omega e^{\frac{\lambda}{2}}}{\sqrt{(1-\omega^2)}}+\epsilon-\frac{s^2e^{-\mu}}{8\pi r^4} \\
&= -\frac{1}{8\pi}\Bigg\{\frac{1}{r^2}-e^{-\lambda}\left(\frac{1}{r^2}+\frac{(\nu'+\mu')}{r}+\frac{\mu'\nu'}{4}\right) \\
&\quad +\frac{e^{-\nu}}{4}\Big(2\ddot{\mu}+(\dot{\mu})^2-\dot{\mu}\dot{\nu}\Big)\Bigg\},
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
&P+\frac{s^2e^{-\mu}}{8\pi r^4} \\
&= \frac{-1}{32\pi}\Bigg[e^{-\nu}\Big(2(\ddot{\mu}+\ddot{\lambda})+\dot{\mu}(\dot{\mu}-\dot{\nu}+\dot{\lambda})+\dot{\lambda}(\dot{\lambda}-\dot{\nu})\Big) \\
&\quad -e^{-\lambda}\Big(2(\nu''+\mu'')+(\nu')^2+(\mu')^2-\nu'\lambda'-\mu'(\lambda'+\nu')\Big)\Bigg] \\
&\quad -\frac{2}{r}\Big(\nu'-\lambda'+\mu'\Big),
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
&\frac{(\rho+P)\omega e^{\frac{(\nu+\lambda)}{2}}}{1-\omega^2}+\frac{Qe^{\frac{\nu}{2}}e^{\lambda}(1+\omega^2)}{\sqrt{(1-\omega^2)}}+e^{\frac{(\nu+\lambda)}{2}}\epsilon \\
&= -\frac{1}{8\pi}\left\{\left(-\frac{\mu'\dot{\mu}}{4}+\frac{\dot{\mu}\nu'}{4}+\frac{\mu'\dot{\lambda}}{4}-\frac{\dot{\mu}'}{2}+\frac{\dot{\lambda}}{r}\right)\right\}.
\end{aligned} \tag{18}
$$

## 3 Electric charge induced anisotropy in five dimensions

To express the EFEs in terms of an anisotropic fluid, we present

$$
e^{-\lambda} = e^{-\mu} = 1-\frac{\xi}{r^2}, \tag{19}
$$

where

$$
\xi(t, r) = m(t, r)-\frac{s^2}{3r^2}, \tag{20}
$$

$m$ represent the total mass. The corresponding mass function is defined as

$$
m = \int_0^r 4\pi r^2\tilde{\rho}dr. \tag{21}
$$

Thus the field equations (16)–(18) read

$$\tilde{p} = -\frac{1}{8\pi}\left[\frac{1}{r^2} - \frac{(r^2-m)}{r^2}\left(\frac{1}{r^2} + \frac{\nu'}{r} + \frac{m'r-2m}{r^2(r^2-m)} + \frac{\nu'}{4}\left(\frac{m'r-2m}{r(r^2-m)}\right)\right) + \frac{r^2}{4(r^2-m)}\left(\frac{2\ddot{m}}{r^2(r^2-m)} + \frac{2(\dot{m})^2}{(r^2-m)^2}\right)\right], \tag{22}$$

$$p_t = -\frac{1}{32\pi}\left[e^{-\nu}\left(\frac{9(\dot{m})^2}{r^2-m}^2 + (\frac{4(\ddot{m})}{r^2-m})\right) - \frac{(r^2-m)}{r^2}\left(2\nu'' + \frac{2m''r^2-8m'r+12m}{r^2(r62-m)} + \frac{(-2m'r+4m^2)}{r^2(r^2-m)^2} + \nu'^2 - \frac{2\nu'm'r+4\nu'm}{r(r^2-m)}\right) - \frac{2\nu'}{r}\right], \tag{23}$$

and the conservative variables represent as

$$\tilde{\rho} = \frac{\rho + P_r\omega^2\omega}{1-\omega^2} + \frac{2\omega q}{1-\omega^2} + \epsilon\frac{1+\omega}{1-\omega}, \tag{24}$$

$$S = \frac{(\rho+P_r)\omega}{1-\omega^2} + \frac{1+\omega^2}{1-\omega^2}q + \epsilon\frac{1+\omega}{1-\omega}, \tag{25}$$

and the flux variable

$$\tilde{P} = \frac{P_r + \rho\omega^2}{1-\omega^2} + \frac{2\omega q}{1-\omega^2} + \epsilon\frac{1+\omega}{1-\omega}, \tag{26}$$

where the $\tilde{P}$ and $\tilde{\rho}$ represent the effective pressure and energy density in PQS approximation respectively. Formally, Eqs. (21)–(23) correspond to those for an anisotropic fluid, with $\hat{\rho} = \rho + \rho_e$, $p_r = p - p_e$, $p_t = p + p_e$, and the electric energy density $\rho_e = E^2/8\pi$, where $E = s/r^2$ is the local electric field intensity.

## 4 Higher dimensional exterior spacetime

The corresponding Reissner–Nordström–Vaidya exterior geometry for higher dimension electromagnetic field is considered as [75]

$$ds^2 = \left(1 - \frac{2M(u)}{(n-1)R^{n-1}} + \frac{2q^2}{n(n-1)r^{2n-2}}\right)du^2 + 2dudR - R^2\left(d\theta_1^2 + sin^2\theta_1\left(d\theta_2^2 + sin\theta_2^2 d\theta_2^3\right)\right), \tag{27}$$

where $M(u)$ represent the total mass of the system inside the BS denoted as $\Sigma$, $u$ denotes the retarded time and $q$ is the total charge. At the BS and outside it, the two coordinate systems $(t, r, \theta, \phi, h)$ and $(u, R, \theta_1, \theta_2, \phi)$ are connected by

$$u = t - r - 2Mln\left(\frac{r}{2M} - 1\right), \quad R = r_\Sigma, \tag{28}$$

following necessary and sufficient conditions for two metrics (1) and (27) shall be fulfilled to match smoothly.

$$e^{\nu_\Sigma} = e^{-\lambda_\Sigma} = e^{-\mu_\Sigma} = 1 - \frac{M}{R_\Sigma^2} + \frac{q^2}{3R^4}, \tag{29}$$

and

$$\nu_\Sigma = -\lambda_\Sigma = -\mu_\Sigma. \tag{30}$$

The $\Sigma$ subscript shows that the quantity is determined at the BS. This last condition ensures that there are no unusual behaviours on the surface. It is simple to verify this

$$p_\Sigma = q_\Sigma, \tag{31}$$

which expresses the radial pressure's continuity over the distribution's border as $R = r(t)$. The fluid in this study is considered to be anisotropic and dissipative in the form of free streaming radiations and/or heat flow, where $\epsilon$ is the radiation density and $q$ is the heat flow are considered as

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu - Pg_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu + \epsilon l_\nu l_\mu + E_{\mu\nu}, \tag{32}$$

where $u_\alpha$, $l_\alpha$, $q_\alpha$ are the four velocity, the four null vector and the heat flux four vector respectively, which satisfy $u^\alpha u_\alpha = 1$, $q_\alpha u^\alpha = 0$, $l^\alpha l_\alpha = 0$ and $E_{\mu\nu}$ is the electromagnetic energy–momentum tensor

$$E_{\mu\nu} = \frac{1}{4\pi}\left[F^k_\mu F_{\nu k} - \frac{1}{4}g_{\mu\nu}F_{\sigma k}F^{\sigma k}\right], \tag{33}$$

where the Maxwell equations are satisfied by the Maxwell field tensor, $F_{\mu k}$:

$$F_{[\mu\nu;\sigma]} = 0, \tag{34}$$

and

$$(\sqrt{-g}F^{\mu\nu})_{,\nu} = 4\pi\sqrt{-g}J^\mu, \tag{35}$$

where $J^\mu = \sigma u^\mu$ represents for electric current five vector, $\sigma$ stands for electrical conductivity, "," and ";" respectively, represent partial differentiation and covariant derivative with respect to the stated coordinate. Only the radial electric field, $F^{tr} = -F^{rt}$, is nonzero due to spherical symmetry. Conversely, the inhomogeneous Maxwell equations change into

$$s_{,r} = 4\pi r^2 J^t e^{\frac{\nu+\lambda}{2}}, \tag{36}$$

and

$$s_{,t} = -4\pi r^2 J^r e^{\frac{\nu+\lambda}{2}}, \tag{37}$$

where $J^t$ and $J^r$ are the current four vector's respective temporal and radial components. The function $s(t, r)$ naturally yields the charge that is present inside the radius

$r$ at the time $t$. We develop the definition of the function $F^{tr} = se^{\frac{-(\lambda+\nu)}{2}}/r^2$, with

$$s(t,r) = \int 4\pi r^2 J^t e^{-\frac{(\lambda+\nu)}{2}} dr. \tag{38}$$

In a fluid-comoving sphere, the conservation of charge is stated as

$$u^{\alpha}_{s,\alpha} = 0. \tag{39}$$

The conservation equation can be expressed in a way that is more suitable for numerical purposes

$$s_{,t} + \frac{dr}{dt} s_{,r} = 0. \tag{40}$$

The contravariant components of the four velocity, the null outgoing vector and the heat flux four vector are

$$u^{\mu} = \left(\frac{e^{-\nu/2}}{\sqrt{(1-\omega^2)}}, \frac{\omega e^{-\lambda/2}}{\sqrt{(1-\omega^2)}}, 0, 0, 0\right), \tag{41}$$

$$l^{\mu} = \left(e^{-\nu/2},\ e^{-\lambda/2}, 0, 0, 0\right), \tag{42}$$

and

$$q^{u} = Q\left(\omega e^{\frac{\lambda-\nu}{2}}, 1, 0, 0, 0\right). \tag{43}$$

After some lengthy but simple computations, the radial component of conservation law is used to calculate $T^{\mu}_{\nu;\mu} = 0$, we obtain following equation

$$P' = -\left(\frac{\nu' + \mu'}{2}\right)\left(\rho + P\right), \tag{44}$$

which represents the static case of the Tolman–Oppenheimer–Volkoff (TOV) equation.

## 5 The methodology

While dealing with self-gravitating compact objects, the most basic scenario is static equilibrium. This shows that $\omega = \epsilon = Q = 0$, all time-dependent derivatives vanish and a modified TOV equation is obtained. The QS regime, the hydrostatic time scale, which is the typical time scale on which the sphere responds to small changes in the hydrostatic equilibrium, is very long in comparison to the slow rate of change of the sphere. As a result, the system is constantly near to hydrostatic equilibrium in QS regime. Its evolution can be seen as a series of static models linked together by (18). This theory is sensible because the hydrostatic time scale is relatively short for several stages of a star's life. It is approximately 4.5 s for a white dwarf, 27 min for the Sun and $10^{-4}$ s for a neutron star with a mass of one solar mass and a radius of 10 km [76]. Any of the star configurations mentioned above have been observed to change over the period of time that are unusually long in comparison to their respective hydrostatic time scales. As was previously stated, this approximation is no longer accurate in some crucial instances and departures from quasi-equilibrium must be taken into account. We will discuss such departures, in the PQS approximation given in following subsections.

### 5.1 The effective variables and PQS approximation in higher dimension

The effective variables for the PQS approximation are defined as follows:

$$\tilde{\rho} = T^0_0 = \frac{\rho + P\omega^2}{1-\omega^2} + \frac{2Q\omega e^{\frac{\lambda}{2}}}{\sqrt{(1-\omega^2)}} + \epsilon, \tag{45}$$

$$\tilde{P} = -T^1_1 = \frac{P + \rho\omega^2}{1-\omega^2} + \frac{2Q\omega e^{\frac{\lambda}{2}}}{\sqrt{(1-\omega^2)}} + \epsilon. \tag{46}$$

The effective variables in the QS regime satisfy the TOV Eq. (44) as the corresponding physical variables. As a result, effective and physical variables have similar radial dependency in a QS condition (and likely in a static one as well). Substituting Eq. (46) into Eq. (16):

$$\begin{aligned}\nu &= \nu_{\Sigma} \\ &+ \int_{r_{\Sigma}}^{r} \frac{2(8\pi \tilde{P} r^4(r^2-2m) - 2r^3 m' + 6r^2 m - 12m^2 + 4rmm')}{(2r^3(r^2-5m) + 12rm^2 + r^4 m' - 2r^2 mm')} \\ &+ \frac{r^2}{r^2-2m}\left(\frac{\ddot{m}}{(r^2-2m)} + \frac{4\dot{m}^2}{(r^2-2m)^2}\right) dr,\end{aligned} \tag{47}$$

$$\begin{aligned}\mu &= \mu_{\Sigma} \\ &+ \int_{r_{\Sigma}}^{r} \frac{2(8\pi \tilde{P} r^4(r^2-2m) + 2r^3 m' - 2r^2 m + 4m^2 - 4rmm')}{(2r^3(r^2-3m) + 4rm^2 - r^4 m' + 2r^2 mm')} \\ &+ \frac{r^2}{r^2-2m}\left(\frac{\ddot{m}}{(r^2-2m)} + \frac{4\dot{m}^2}{(r^2-2m)^2}\right) dr.\end{aligned} \tag{48}$$

The radial dependency of metric functions is completely determined for a given radial dependency of effective variables. Now, we will discuss the PQS regime as one that corresponds to a system that is not in equilibrium (or quasi-equilibrium), however effective pressure and energy density have similar radial dependency as the associated physical variables in an equilibrium (or quasi-equilibrium) state. As an alternative, metric functions with similar radial dependence as those in the static or QS regime define the system in the PQS regime. The logic behind this formulation is simple: we seek a regime that, while not in an equilibrium condition, represents the closest possible scenario to a QS evolution.

### 5.2 The algorithm in higher dimension

The approach we are going to use is outlined below

1. Consider an analytic interior (seed) solution to the EFEs, which represents a fluid distribution of matter in an equilibrium state, given as $\rho_{st} = \rho(r)$; $P_{st} = P(r)$.

2. Assume that effective pressure $\tilde{P}$ and energy density $\tilde{\rho}$ are dependent on the same $r$ as $P_{st}$ and $\rho_{st}$.
3. One may obtain $m$, $\mu$ and $\nu$ up to some $t$ functions using Eqs. (21), (47) and (48), as well as the radial dependence of $\tilde{P}$ and $\tilde{\rho}$, which will be explored in more detail below.
4. For these $t$ functions, there are three ODEs, which are characterized as surface equations.
   - Evaluate Eq. (14) on $r = r_\Sigma$.
   - The equation that illustrates the relationship between the energy flux ($\hat{E}$) and mass loss rate along the BS.
   - Determine non-static TOV equation on $r = r_\Sigma$.
5. The additional information is needed to close the given system of surface equations to determine some physical variables on the BS.
6. Once it has been closed, the system of surface equations can be integrate for any given set of initial conditions.
7. These two functions are completely determined by substituting the integration results in the expressions for $m$, $\mu$ and $\nu$.
8. The EFEs develop a system of equations for physical variables after appropriately defining metric functions, can be obtained for any kind of fluid distribution.

### 5.3 The surface equations in higher dimension

The system of surface equations is the critical point in the algorithm, as should be obvious from the preceding. For this, dimensionless variables are introduced as

$$A = \frac{r_\Sigma}{m_\Sigma(0)}, \quad F = 1 - \frac{M}{A^2} + \frac{q^2}{3A^4}, \quad M = \frac{m_\Sigma}{m_\Sigma(0)},$$
$$\beta = \frac{t}{m_\Sigma(0)} \quad \Omega = \omega_\Sigma.$$

We obtained the first surface equation with the total initial mass $m_\Sigma(0)$ by evaluating Eq. (14) at $r = r_\Sigma$. As a result,

$$\frac{dA}{d\beta} = F\Omega, \tag{49}$$

by using junction conditions, one may then obtain from (15), (18) and (29) computed at $r = r_\Sigma$, yields

$$\frac{dM}{d\beta} = \frac{-F^2}{\hat{S}}\Big((1+\Omega)(\hat{\epsilon}_\Sigma + \hat{q}_\Sigma) - \frac{\Omega\rho_\Sigma}{2} + \Omega\rho_\Sigma\hat{B}\Big)\hat{E}, \tag{50}$$

where

$$\hat{E} = 8\pi r_\Sigma^5, \tag{51}$$
$$\hat{S} = m'_\Sigma r_\Sigma(3 + r_\Sigma^2 - 2m_\Sigma) + 26m_\Sigma - 16r_\Sigma^2, \tag{52}$$
$$\hat{B} = \frac{3r_\Sigma^2 - 12(r^2{}_\Sigma - 2m_\Sigma) - 12\pi r_\Sigma\rho_\Sigma - 6r_\Sigma + 8\pi r^3{}_\Sigma\rho_\Sigma - 6\pi r^3{}_\Sigma\rho_\Sigma\Omega^2}{3r_\Sigma(r_\Sigma^2 - 2m_\Sigma)}. \tag{53}$$

The gravitational redshift and Doppler shift are represented on the right of Eq. (50). The observer's perceived luminosity at infinity is then defined as

$$L = -\frac{dM}{d\beta}. \tag{54}$$

The second surface equation is

$$\frac{dF}{d\beta} = \frac{2}{A}\left(1 - F - \frac{q^2}{3A^4}\right)F\Omega + \frac{L}{A^2}. \tag{55}$$

Evaluating the law of conservation $T^{\mu}_{\nu;\mu} = 0$ at the BS yields the third surface equation, we obtain

$$\tilde{P} + (\tilde{\rho} + \tilde{P})\left(\frac{\nu' + \mu'}{2}\right) = \frac{e^{-\nu}}{4\pi r(r^2 - 2m)}\Big(2\ddot{m}$$
$$+ \frac{7\dot{m}^2}{r^2 - 2m} - \dot{m}\dot{\nu}\Big) + \frac{2}{r}(\tilde{P} - P_t). \tag{56}$$

This hydrostatic support equation, a generalisation of the higher dimensional TOV equation, is the same as the equation for an anisotropic matter. Equation (56) is the conservative form of the field equation, which leads to the surface's third equation, everything up to this point is completely general within spherical symmetry. To properly describe the dynamics for any given initial conditions and luminosity profile, a third surface equation is needed. For this reason, which is obviously model dependent, we can employ the field Eq. (17) or the conservation Eq. (56) stated in terms of the effective variables.

## 6 Conclusion and discussion

In this work, we study the evolution of a self-gravitating spherical distribution of charged matter with a dissipative fluid in higher dimensions. By using the PQS approximation with noncomoving coordinates, we may analyze electrically charged fluid spheres in the streaming out limits and diffusion approximation as they move away from equilibrium. The PQSA can be considered in this context as a nonlinear perturbation technique for evaluating the stability of solutions in equilibrium. A five-dimensional Schwarzschild-like non-moving coordinate system is used in this paper to develop a general framework for discussing relativistic collapse of spherical systems. To analyze astrophysical scenarios in higher dimensions, mathematicians used the KK theory, the M theory, string theory, and superstring theory. In this paper, we assumed non-comoving five-dimensional Schwarzschild coordinates. The fifth dimension represents radial spatial coordinates. The five-dimensional geometry is used to develop the higher dimension PQS approximation. The physical features of the stellar structure of gravitational objects are analysed, and a general framework for the higher dimensional PQS approximation is developed. We started with an interior (analytical) seed solution to the EFEs. The proposed technique provides a set of ODEs for quantities estimated at the BS. The numerical solution allows for the simulation of self-gravitating spheres.

The inner fluid distribution is assumed to be anisotropically configured, with a heat flow, electrically charged and radiation factor that causes dissipative effects within the gravitating system. Dissipation is a phase of giant star evolution caused by the emission of massless particles. In fact, neutrino emission appears to be the only viable method for removing the bulk of the binding energy from a collapsing star, resulting in the formation of a neutron star or black hole. Electric charge also favours collapse regardless of the transport mechanism, in a similar way to anisotropy with radial pressure less than tangential pressure. In any instance, a huge electric charge is required to alter the gravitational collapse's path. The gradient of the electric charge gradually reduces toward the surface as it is redistributed over time, becoming unexpectedly linear and stationary. Up to a critical total electric charge, the system evolves under the constraints of the Einstein-Maxwell system of field equations (or anisotropy parameter). Outer space is considered as Vaidya–Reissner–Nordström spacetime for smooth matching at the boundary of sphere.

In the discussion of departure from equilibrium there are three possible situations which are (a) static equilibrium, (b) QS equilibrium and (c) PQS equilibrium.

- Static Equilibrium: In case of static equilibrium all the components of the EFEs have radial dependency.
- QS Equilibrium: The system is predicted to evolve slowly enough in this regime to be deemed in equilibrium at any given time. This indicates that the compact object evolves very slowly, on a time scale significantly longer than the time it normally takes for the sphere to react to a small perturbation of hydrostatic equilibrium. The system is assumed to be static between two small variations of time due to very slow evolutionary process.
- PQS Equilibrium: The system which is not in the state of equilibrium or departure from equilibrium is known as PQS equilibrium.

The TOV equation is derived using a higher dimension conservation law. The TOV equation is satisfied by effective variables such as effective pressure and energy density, as well as physical variables. In the static and QS equilibrium, the effective and physical variables have the same radial dependency. This process is iterative, with each successive step reflecting a deeper understanding of the deviation from the equilibrium state.

Motivated by the fact that noncomoving coordinates are frequently used in relativistic collapse research, requiring the definition of the PQS approximation. This method is based on "effective" variables as well as a heuristic approach to the latter, the rationale and justification for which is revealed in the context of the PQS approximation in the five-dimensional regime.

In this work, we restricted ourselves to the five-dimensional PQS level. In higher dimension, we developed a system of surface equations using the PQS approximation algorithm. We established a higher-dimensional surface equation system and studied realistic features of stars like Doppler shift, gravitational redshift, and total mass loss rate, all of which are related to total mass loss energy flux $\hat{E}$ over the BS.

In higher dimensional spherically symmetric gravitational collapse is observed for non-comoving coordinate system in PQS approximation. The effects of heat flux and unpolarized radiation in anisotropic conditions were studied by considering five dimensional Vaidya outer space. The discussion of this phenomena is not available in higher dimensional space-time in any previous work. To comprehend the nature of gravitational collapse in five dimensions, a general framework for the PQS regime must be developed, which necessitates the solution of nonlinear differential equations. Gravitational collapse is a well-known energy-dissipating process that dominates star formation and stellar evolution. We considered dissipation, which is an important factor in the gravitational collapse process. The dissipative model is described by the five-dimensional null outgoing vector in diffusion approximation.

In higher dimensional space-time, compact stars, neutron stars, and hybrid stars exist. These astrophysical phenomena motivate us to study the higher dimensional gravitational collapse. In literature, the gravitational collapse in the PQS approximation for higher dimensions has not been modeled.

In this article, we developed the higher dimensional general framework for the PQS approximation. The PQS approximation can be applied in two distinct ways using two different kinds of models: Schwarzschild-like structures can be found in two different limits: diffusion and free streaming. These models describe the characteristics of electrically charged and dissipative collapse. This work can be extended to the PQS regime's gravitational collapse in co-moving coordinates.

**Data Availability Statement** This manuscript has no associated data or the data will not be deposited. [Authors' comment: This is theoretical study so, no data will not be deposited.]

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