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Supersymmetrization of deformed BMS algebras

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Abstract The $W(a,b)$ and $W(a,b;\bar{a},\bar{b})$ algebras are deformations of $\mathfrak{bms}_3$ and $\mathfrak{bms}_4$ algebras, respectively. We present a $\mathcal{N}=2$ supersymmetric extension of both algebras in the presence of Rsymmetry generators that rotates the two supercharges. Our construction provides the most generic central extensions of the $W(a,b)$ algebra. In particular, we find that $\mathcal{N}=2$ $\mathfrak{bms}_3$ algebra admits a new central extension. On the other hand, we explicitly demonstrate that an infinite $U(1)_V \times U(1)_A$ extension of the $W(a,b;\bar{a},\bar{b})$ algebra corresponding to the $R-$symmetry is not possible for linear and quadratic structure constants with generic deformation parameters. This also implies that the infinite R symmetry considered in our analysis is broken for the $\mathcal{N}=2$ $\mathfrak{bms}_4$ algebra.

1 Introduction and summary

For any theory, its asymptotic symmetries are of immense physical significance. The symmetries at the asymptotic boundary of a theory depend on the boundary fall-off of its constituent fields. In most examples, the asymptotic symmetry is generally increased compared to the bulk symmetry of the theory. However, the bulk symmetry must be contained in the asymptotic symmetry group algebra as a subalgebra. For a theory in asymptotically flat spacetime, if one recedes from sources towards null infinity, at any finite radial distance from the source, one expects the symmetry algebra to be just Poincaré. However, at null infinity, the asymptotic symmetry algebra in the Bondi gauge is enhanced to the Bondi–Metzner–Sachs or the $\mathfrak{bms}$ algebra generated by an infinite number of generators known as supertranslations [1,2]. The $\mathfrak{bms}$ algebra can also be realized in other asymptotic regions, such as at spatial or time-like infinity [3,4]. One can further extend the $\mathfrak{bms}$ algebra by including superrotations [5], which manifest as a double copy of Virasoro algebra. The finite-dimensional Poincaré algebra is a subalgebra of the extended $\mathfrak{bms}$ algebra. These infinite-dimensional $\mathfrak{bms}$ algebras have gained renewed importance due to recent developments on the relations between soft theorems and asymptotic symmetries in analyzing the vacuum of gauge theories and gravity [6–11]. It is well understood that at any null boundary in two- or three-dimensional spacetime, one can obtain an infinite-dimensional algebra by constructing the conserved charges [12].[1] Recently, this has also been realized in general dimensions [14]. It is interesting to understand how these $\mathfrak{bms}$ algebras are modified in the presence of extended supersymmetries in a theory of gravity. Furthermore, in the presence of internal gauge fields, namely the $R-$ symmetry fields, the supercharges rotate non-trivially among themselves. This brings interesting dynamics to the system such as modification of the Bogomol'nyi–Prasad–Sommerfield (BPS) condition in the presence of $R-$charges [15,16]. The effects of extended supersymmetries and $R-$symmetries have been studied extensively in the context of asymptotic symmetries of three-dimensional supergravity theories. The supersymmetric deformations of $\mathfrak{bms}_3$ algebras and their consequences were detailed in [16–25]. In particular, it has been shown that the $R-$charges also obtain infinite extensions at the null infinity, and the space of the asymptotically flat cosmological solutions is extended considerably [16,22]. There has been no similar study in the context of four-dimensional

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[1] One can impose more general boundary conditions compared to boundary fall-offs consistent with Bondi gauge to obtain an infinite number of non-conserved asymptotic charges. They can, however, be made integrable by appropriate field redefinitions [13].

asymptotically flat extended supergravity theory. This leads us to look for deformations of $\mathfrak{bms}_4$ algebra.

There are two distinct possible ways of generating a new algebra starting from one, namely deformations and contractions of an algebra. Deformation of a Lie algebra can be viewed as an inverse procedure of Inönü–Wigner [26] contraction. While physicists have tackled more with contraction of Lie algebras, deformations of various well-known Lie algebras in physics have been recently considered in the literature [27–32]. In contraction prescription, one tries to obtain a new non-isomorphic algebra through specific limits of a known algebra, whereas in the deformation prescription one deforms a Lie algebra to obtain new (more stable) algebras by turning on structure constants in some commutators [33,34]. For instance, one may take the limit of the Poincaré algebra by sending the speed of light to infinity (or to zero) to obtain Galilean (or Carroll) algebra, and conversely, the Galilean (or the Carroll) algebra may be deformed into the Poincaré algebra [27,35]. In recent works [36,37], it has been proven that the three- and four-dimensional pure $\mathfrak{bms}$ algebras can be deformed, in their non-ideal part, into two families of new non-isomorphic infinite-dimensional algebras called W algebras. In the context of three spacetime dimensions, these are known as $W(a, b)$ algebras, where $\mathfrak{bms}_3$ corresponds to $W(0, -1)$. In the context of four spacetime dimensions, these are known as $W(a, b; \bar{a}, \bar{b})$ algebras, where $\mathfrak{bms}_4$ corresponds to $W(-1/2. -1/2; -1/2, -1/2)$. It has been shown that by imposing appropriate boundary conditions, both $W(0, b)$ and $W(b, b; b, b)$ algebras are obtained as near-horizon symmetry algebras of three- and four-dimensional black holes [38]. Also, $W(b, b; b, b)$ has been obtained as the asymptotic symmetry algebra of flat Friedmann–Lemaître–Robertson–Walker (FLRW) spacetimes [39]. On the other hand, $W(0, 1)$, $W(0, 0)$ and $W(0, -1)$ have appeared as asymptotic symmetry algebras in various gravitational theories [40–42].

In this paper, our primary goal is to find the supersymmetric extension of $\mathfrak{bms}_4$ algebras in the presence of $R-$symmetry charges. To achieve this, we need to extend the $W(a, b)$ and $W(a, b; \bar{a}, \bar{b})$ algebras with two (fermionic) supercharges. We further consider that the two supercharges rotate among themselves due to an internal R-symmetry. Our construction is purely group-theoretic with only two inputs, namely, (a) we demand the consistency of the possible extended algebra with the Jacobi identities, and (b) we demand that the extended algebra contains the super-Poincaré algebra as its subalgebra, for particular values of deformation parameters. The explicit construction is as follows:

- In three spacetime dimensions, we first introduce a set of infinite fermionic generators to grade the known $W(a, b)$ algebras and ensure that the resulting superalgebra satisfies graded Jacobi identities. Next, we perform a similar construction with infinite bosonic $R-$charge generators. We further extend our analysis to include the central charges in the algebras. As we have stated above, $W(0, -1)$ gives the usual $\mathfrak{bms}_3$, and various supersymmetric extensions of $\mathfrak{bms}_3$ algebras are well investigated [22,23]. Our construction of supersymmetric centrally extended $W(a, b)$ algebras in this paper reproduces the known results for $a = 0, b = -1$, although we obtain a new possible central extension.
- So far in four spacetime dimensions, we know generic bosonic $W(a, b; \bar{a}, \bar{b})$ algebra with central extensions [37]. As stated above, $\mathfrak{bms}_4$ is a special case of these for $a = b = \bar{a} = \bar{b} = -1/2$. Furthermore, in [43], minimal supersymmetric generalization of $\mathfrak{bms}_4$ with one supercharge has been obtained. In this paper, we first extend bosonic $W(a, b; \bar{a}, \bar{b})$ with a set of infinite supercharges. Next, we perform further extension with infinite $R-$charges. In this case, the resulting algebra has not been centrally extended. Interestingly, we find that for $\mathfrak{bms}_4$ with two supercharges, one cannot infinitely extend the $R-$charge sectors. We have shown this rigidity for both linear and quadratic dependence of the structure constants. The Jacobi identities are only satisfied within the global sector, i.e., for $\mathcal{N} = 2$ super-Poincaré algebra with global $R-$charges. This is one of the primary results of this paper.

Here we must mention that both the three- and four-dimensional algebras constructed in this paper are purely mathematical. In both cases, the corresponding super-Poincaré algebras are embedded in them as subalgebras for appropriate values of the deformation parameters. Thus, in principle, these algebras might show up as the asymptotic symmetry algebras for three- and four-dimensional asymptotically flat theories. In particular, the $\mathcal{N} = 2$ extension of $\mathfrak{bms}_4$ is a probable candidate for four-dimensional $\mathcal{N} = 2$ supergravity theories with $R-$charges.

Let us summarize the organization and results of the paper below:

- In Sect. 2, we begin with a brief review of the basic properties of the $W(a, b)$ algebra. Next, we present a new analysis on a $\mathcal{N} = 2$ supersymmetric extension in the presence of R-charges which rotate the supercharges among themselves. We conclude this section with the central extension of the supersymmetric $W(a, b)$ algebra. Equation (2.43) represents the $\mathcal{N} = 2,\ W(a, b)$ algebra, whereas Eq. (2.45) along with the table below it represents its most generic central extensions.

- Section 3 discusses the basic properties of $W(a, b, \bar{a}, \bar{b})$ algebra and may be skipped by the experts. We have added it for establishing the notations used in the later sections.
- In Sect. 4, we extend the $W(a, b, \bar{a}, \bar{b})$ algebra to include two supercharges, but any internal symmetry. This section forms the base of the main results of the paper, which are presented in Sect. 5. Equation (4.5) presents the key results of this section, which is an infinite extension of $\mathfrak{bms}_4$ in the presence of two supercharges.
- Section 5 contains the most important results of this paper. In this section, we introduce two sets of R-charges along with other $\mathcal{N} = 2$ $W(a, b, \bar{a}, \bar{b})$ generators and studied the possibility to find an infinite extension of the algebra. Here we consider the cases for structure constants being both linear and nonlinear in their arguments and perform a detailed analysis. We find a non-affirmative result (unlike the case of Sect. 4), as discussed at the end of the section.
- In Sect. 6, we conclude with a discussion on the main results and possible future directions.

2 Three-dimensional supersymmetric $W(a, b)$ and R-extended $W(a, b)$

Earlier works such as [36,37] discussed aspects of deformation and stability of $\mathfrak{bms}_3$ and $\mathfrak{bms}_4$ algebras. In this section, we briefly describe their results and observations for $\mathfrak{bms}_3$. The centerless $\mathfrak{bms}_3$ algebra can be written as

$$\begin{aligned} &[J_m, J_n] = (m-n)J_{m+n} \\ &[J_m, P_n] = (m-n)P_{m+n}, \\ &[P_m, P_n] = 0. \end{aligned} \tag{2.1}$$

Physically, the J_ms are identified with superrotations, while the P_ns are supertranslations. This algebra can be deformed into the two-parameter family algebra called $W(a, b)$, where a, b are arbitrary real parameters [37]. Explicitly, the $W(a, b)$ algebra is given by

$$\begin{aligned} &[J_m, J_n] = (m-n)J_{m+n}, \\ &[J_m, P_n] = -(n+bm+a)P_{m+n}, \\ &[P_m, P_n] = 0. \end{aligned} \tag{2.2}$$

It is straightforward to see that $W(0, -1)$ corresponds to $\mathfrak{bms}_3$.

2.1 Supersymmetric $W(a, b)$ algebra

In this section, we write down a *supersymmetric* version of the $W(a, b)$ algebra. Subsequently, we will introduce R- and S-charges and also determine the central extension to the algebra. We start by introducing fermionic generators G_s in the $W(a, b)$ algebra where s runs over half-integers. Our goal would be to write down an extended algebra starting with the centerless $W(a, b)$ algebra as given above by demanding the consistency of Jacobi identities. For the time being, unlike $\mathfrak{bms}_3$, we do not search for the realization of the algebra as the asymptotic symmetry algebra of a supersymmetric theory at null infinity in three spacetime dimensions.

Along with the usual $W(a, b)$ algebra as given in (2.2), we introduce the following three commutators

$$\{G_r, G_s\} = P_{r+s}, \tag{2.3}$$

$$[J_m, G_s] = \alpha(m, s)G_{m+s}, \tag{2.4}$$

$$[P_m, G_s] = \beta(m, s)G_{l(m,s)}. \tag{2.5}$$

The above extension is motivated by various super-$\mathfrak{bms}_3$ algebras written in [21–23]. We choose to normalize the super-current generators G_s in such a way that the structure constant appearing in (2.3) is unity. It is expected that any deformation of the $\mathfrak{bms}_3$ algebra by the parameters a and b will not change the index structure appearing on the RHS of (2.4). For $\mathfrak{bms}_3$, it is known that $[P_m, G_r] = 0$. However, it is possible that a deformation gives a nontrivial commutator between the supercurrents and supertranslation which vanishes when $a = 0, b = -1$.[2] This motivates us to propose (2.5), where $l(m, s)$ is a linear function in m and s. The structure constants α and β appearing above are also assumed to be linear functions of its arguments. Our strategy will be to fix these structure constants and $l(m, s)$ by demanding the consistency of certain relevant Jacobi identities.

The Jacobi identity involving the generators G_r, G_s and P_m is given by

$$[\{G_s, G_r\}, P_m] = \{G_s, [G_r, P_m]\} + \{G_r, [G_s, P_m]\}. \tag{2.6}$$

Using (2.2) and (2.3)–(2.5), we obtain

$$\beta(m, r)P_{s+l(m,r)} + \beta(m, s)P_{r+l(m,s)} = 0 \tag{2.7}$$

Assuming linearity of $l(m, s)$ and the structure constant $\beta(m, s)$ in both of their arguments, the above equation is satisfied if $l(m, s) = l_0 + l_1 m + s$, where l_0 and l_1 are constants and $\beta(m, r) = -\beta(m, s)$ for any r, s. This hence yields $\beta(m, s) = 0$.

Next, we use Jacobi identities on the operators J_m, G_s and G_r to determine $\alpha(m, s)$. The corresponding Jacobi identity

[2] One can think of the expression $a + b + 1$. Clearly, for $\mathfrak{bms}_3$, this combination identically vanishes and can be a possible candidate structure constant in (2.5). However, if $l(m, s)$ is indeed a linear function of its argument, we will see that such a commutator does not satisfy the Jacobi identity.

is

$$[\{G_s, G_r\}, J_m] = \{G_s, [G_r, J_m]\} + \{G_r, [G_s, J_m]\}. \quad (2.8)$$

Using (2.2), (2.3)–(2.5), we obtain

$$-\alpha(m,r)P_{m+r+s} - \alpha(m,s)P_{m+r+s} = (r+s+bm+a)P_{m+r+s}. \quad (2.9)$$

Clearly, equating the coefficients of P_{m+r+s}, we recover

$$\alpha(m,s) = -\left(\frac{bm+a}{2} + s\right). \quad (2.10)$$

The Jacobi identity applied on J_m, J_n and G_s reads as

$$[[G_s, J_m], J_n] + [[J_m, J_n], G_s] + [[J_n, G_s], J_m] = 0. \quad (2.11)$$

A similar exercise on the above Jacobi identity followed by equating the coefficient of G_{m+n+s} yields

$$\alpha(m,s)\alpha(n,m+s) + (m-n)\alpha(m+s,s) = \alpha(n,s)\alpha(m,n+s). \quad (2.12)$$

It can be easily seen that the structure constant $\alpha(m,s)$ as determined in (2.10) indeed satisfies the above equality. We thus end up with a possible $\mathcal{N}=1$ supersymmetric extension of $W(a,b)$ algebra given by

$$\begin{aligned}
&[J_m, J_n] = (m-n)J_{m+n},\\
&[J_m, P_n] = -(n+bm+a)P_{m+n},\\
&[P_m, P_n] = 0,\\
&\{G_r, G_s\} = P_{r+s},\\
&[J_m, G_s] = -\left(\frac{bm+a}{2} + s\right)G_{m+s},\\
&[P_m, G_s] = 0
\end{aligned} \quad (2.13)$$

Now that we have obtained a possible $\mathcal{N}=1$ extension of the $W(a,b)$ algebra, we can consider including another copy of fermionic supercharges which we denote by H_s, where the index s can take half-integer values. They satisfy the following commutators with the superrotation and supertranslation generators of the $W(a,b)$ algebra

$$\begin{aligned}
&\{H_r, H_s\} = P_{r+s}, \quad [P_m, H_s] = 0,\\
&[J_m, H_s] = -\left(\frac{bm+a}{2} + s\right)H_{m+s}.
\end{aligned} \quad (2.14)$$

The supercurrent generators G_s and H_s can be used to define the following linear combinations

$$Q_r^1 = \frac{1}{2}(G_r + iH_r), \quad Q_r^2 = \frac{1}{2}(G_r - iH_r). \quad (2.15)$$

The newly defined generators Q_r^1 and Q_r^2 satisfy

$$\begin{aligned}
&\{Q_r^1, Q_s^2\} = P_{r+s}, \quad \{Q_r^1, Q_s^1\} = 0, \quad \{Q_r^2, Q_s^2\} = 0,\\
&[J_m, Q_s^i] = -\left(\frac{bm+a}{2} + s\right)Q_{m+s}^i, \quad [P_m, Q_s^i] = 0.
\end{aligned} \quad (2.16)$$

2.2 R-extension of supersymmetric $W(a,b)$ algebra

Our next aim is to write the generalized algebra in the presence of R-charges. R-charge generators rotate the supercharge generators and thus introduce additional non-trivialities in the algebra. It is known that the introduction of $R-$charge generators necessitates the introduction of S-charge generators [44] in the context of $\mathfrak{bms}_3$ symmetries. Motivated by the $\mathcal{N}=2$ $\mathfrak{bms}_3$ algebra as discussed in [23], we begin our analysis by proposing the following relations involving the R-charge and S-charge generators

$$[R_n, Q_r^1] = \beta(n,r)\,Q_{n+r}^1, \quad [R_n, Q_r^2] = -\beta'(n,r)Q_{n+r}^2, \quad (2.17)$$

$$[P_n, R_m] = \sigma(n,m)S_{n+m}, \; \{Q_r^1, Q_s^2\} = P_{r+s} + \eta(r,s)S_{r+s}, \quad (2.18)$$

$$[R_n, J_m] = \gamma(n,m)R_{n+m}, \qquad [S_n, J_m] = \kappa(n,m)S_{n+m}. \quad (2.19)$$

In writing the above ansatz, we have assumed that the addition of $R-$ symmetry generators to the $\mathcal{N}=2$ super-$W(a,b)$ algebra will not affect the index structure of the undeformed algebra. The Jacobi identity for the operators Q_m^1, Q_n^2 and R_s is given by

$$[\{Q_r^1, Q_s^2\}, R_m] = \{Q_r^1, [Q_s^2, R_m]\} + \{Q_s^2, [Q_r^1, R_m]\}.$$

Using (2.17) and (2.18) in the above, we obtain

$$\begin{aligned}
[P_{r+s} + \eta(r,s)S_{r+s}, R_m] &= (\beta'(m,s) - \beta(m,r))P_{r+s+m}\\
&+ (\beta'(m,s)\eta(r,m+s) - \beta(m,r)\eta(m+r,s))S_{m+r+s}.
\end{aligned} \quad (2.20)$$

Further, noting that $[S_m, R_n] = 0$ implies that the LHS of (2.20) is independent of the translation generator P_m, in order to make this Jacobi identity consistent, the coefficient of the translation generator on the RHS must vanish identically, implying

$$\beta'(m,s) = \beta(m,r). \quad (2.21)$$

Since the above has to be true for arbitrary half-integer values of r, s and integer values of m, we conclude that both $\beta'(m,s)$ and $\beta(m,r)$ depend only on m, assuming they are linear in their argument. For simplicity, we denote the structure constant appearing in (2.17) as $\beta(m)$.

Demanding further consistency of (2.20) yields

$$\sigma(r+s,m)=\beta(m)(\eta(r,s+m)-\eta(r+m,s)). \tag{2.22}$$

The two Jacobi identities involving Q^1_r, J_n and R_m, and R_l, J_m and J_n,

$$[[Q^1_r,J_n],R_m]+[[J_n,R_m],Q^1_r]+[[R_m,Q^1_r],J_n]=0,$$
$$[[R_l,J_m],J_n]+[[J_m,J_n],R_l]+[[J_n,R_l],J_m]=0,$$

lead to

$$m\beta(m)=\gamma(m,n)\beta(m+n), \tag{2.23}$$
$$\gamma(l,m)\gamma(l+m,n)-(m-n)\gamma(l,m+n)$$
$$-\gamma(l,n)\gamma(l+n,m)=0. \tag{2.24}$$

Assuming the structure constants $\gamma(m,n)$ and $\beta(m)$ to be linear in m and n, we explicitly take them to be of the form

$$\beta(m)=\beta_0+m\beta_1, \tag{2.25}$$
$$\gamma(m,n)=\gamma_0+\gamma_1 m+\gamma_2 n. \tag{2.26}$$

The above ansatz for $\gamma(m,n)$ along with (2.24) ensures that

$$\gamma_1=1. \tag{2.27}$$

(2.23) then gives us five equations satisfied by four parameters β_0, β_1, γ_0 and γ_2, where

$$\gamma_0\beta_0=0,\quad \beta_0\gamma_2+\beta_1\gamma_0=0,$$
$$\gamma_0\beta_1=0,\quad \beta_1(\gamma_2+1)=0,$$
$$\gamma_2\beta_1=0. \tag{2.28}$$

Clearly, the above set of equations are over-constrained but admit the following consistent solution

$$\gamma_0=0,\quad \gamma_2=0,\quad \beta_1=0, \tag{2.29}$$

while β_0 is a nonzero constant that cannot be further fixed. One can easily check that this is also consistent with the Jacobi identity for R_m, R_n and Q^i_r. Thus, we can write the following commutation relations

$$[R_n,Q^1_r]=\beta_0 Q^1_{n+r},\quad [R_n,Q^2_r]=-\beta_0 Q^2_{n+r},$$
$$[R_n,J_m]=nR_{n+m}. \tag{2.30}$$

The Jacobi identity involving J_m, J_n and S_l,

$$[[J_m,J_n],S_l]+[[J_n,S_l],J_m]+[[S_l,J_m],J_n]=0$$

yields

$$\kappa(l,m)\kappa(l+m,n)-(m-n)\kappa(l,m+n)-\kappa(l,n)\kappa(n+l,m)=0. \tag{2.31}$$

Similar to the ansatz for $\gamma(m,n)$, we assume the following ansatz for $\kappa(m,n)$

$$\kappa(m,n)=\kappa_0+\kappa_1 m+\kappa_2 n. \tag{2.32}$$

Plugging the above ansatz into (2.31), we obtain $\kappa_1=1$. The Jacobi identity for the operators J_l, P_m and R_n leads to the relation

$$-(m+bl+a)\sigma(m+l,n)+\sigma(m,n)\kappa(m+n,l)$$
$$-n\sigma(m,n+l)=0. \tag{2.33}$$

Assuming a linear ansatz for $\sigma(m,n)$, i.e.,

$$\sigma(m,n)=\sigma_0+\sigma_1 m+\sigma_2 n, \tag{2.34}$$

one can substitute it back into (2.33) to obtain the set of seven relations:

$$b\sigma_1=0,\quad \sigma_0(\kappa_0-a)=0,$$
$$\sigma_1(\kappa_2-b-1)=0,\quad \sigma_2(\kappa_2-b-1)=0,$$
$$\sigma_1(\kappa_0-a)=0,\quad \sigma_2(\kappa_0-a)=0,$$
$$\sigma_0(\kappa_2-b)-a\sigma_1=0. \tag{2.35}$$

The above system of equations has two consistent solutions:

- Case I: $\sigma_0=\sigma_1=0$; $\kappa_0=a$, $\kappa_2=b+1$ while σ_2 is arbitrary.
- Case II: $\sigma_1=\sigma_2=0$; $\kappa_0=a$, $\kappa_2=b$ while σ_0 is arbitrary.

In light of the above, we can rewrite (2.18) and (2.19) as

$$\text{Case I:}\quad [P_n,R_m]=\sigma_2 mS_{n+m},$$
$$[S_n,J_m]=(a+n+(b+1)m)S_{n+m};$$
$$\text{Case II:}\quad [P_n,R_m]=\sigma_0 S_{n+m},$$
$$[S_n,J_m]=(a+n+bm)S_{n+m}. \tag{2.36}$$

Finally, we need to find the structure constant $\eta(r,s)$ appearing in $\{Q^1_r,Q^2_s\}$ in (2.18). Assuming a linear form of $\eta(r,s)$ i.e. $\eta(r,s)=\eta_0+\eta_1 r+\eta_2 s$, we use (2.22) to see that we must have a relation of the form

$$\sigma(r+s,m)=m\beta_0(\eta_2-\eta_1). \tag{2.37}$$

It is quite evident that a choice of parameters as defined in Case II in our preceding analysis is inconsistent with the above equation since the LHS is a constant and independent of m. However, from the structure constants of Case I, we arrive at the relation

$$\sigma_2 = \beta_0(\eta_2 - \eta_1). \tag{2.38}$$

Finally, the Jacobi identity for Q^1_r, Q^2_s and J_m can be written as

$$[\{Q^1_r, Q^2_s\}, J_m] = \{Q^1_r, [Q^2_s, J_m]\} + \{Q^2_s, [Q^1_r, J_m]\}. \tag{2.39}$$

Using (2.18) and (2.19), the above gives rise to

$$\begin{aligned}\eta(r,s)\kappa(r+s,m) &= \left(\tfrac{bm+a}{2}+s\right)\eta(r,s+m)\\ &\quad+\left(\tfrac{bm+a}{2}+r\right)\eta(r+m,s).\end{aligned} \tag{2.40}$$

Since, $\kappa(m,n) = a + m + (b+1)n$, the above equation will be satisfied if,

$$\eta_0 = 0 \quad \text{and} \quad \eta_1 = -\eta_2. \tag{2.41}$$

Thus, the R- and S-charge sector algebra under a deformation reads:[3]

$$\begin{aligned}
&[R_n, Q^1_r] = \beta_0 Q^1_{n+r},\\
&[R_n, Q^2_r] = -\beta_0 Q^2_{n+r},\\
&[P_n, R_m] = -2\beta_0\eta_1 m S_{n+m},\\
&\{Q^1_r, Q^2_s\} = P_{r+s} + \eta_1(r-s)S_{r+s},\\
&[R_n, J_m] = nR_{n+m},\\
&[S_n, J_m] = (a+n+(b+1)m)S_{n+m}.
\end{aligned} \tag{2.42}$$

Redefining $S_n \to \mathcal{S}_n/\eta_1$ and $R_n \to \beta_0\mathcal{R}_n$, we arrive at the full $W(a,b)$ algebra including R- and S-charge generators

$$\begin{aligned}
&[J_m, J_n] = (m-n)J_{m+n},\\
&[J_m, P_n] = -(n+bm+a)P_{m+n},\\
&[J_m, Q^1_r] = -\left(\frac{bm+a}{2}+r\right)Q^1_{m+r},\\
&[J_m, Q^2_r] = -\left(\frac{bm+a}{2}+r\right)Q^2_{m+r},\\
&[J_m, \mathcal{R}_n] = -n\mathcal{R}_{n+m},\\
&[J_m, \mathcal{S}_n] = -(a+n+(b+1)m)\mathcal{S}_{n+m},\\
&[\mathcal{R}_m, Q^1_r] = Q^1_{m+r},\\
&[\mathcal{R}_m, Q^2_r] = -Q^2_{m+r},\\
&\{Q^1_r, Q^2_s\} = P_{r+s} + (r-s)\mathcal{S}_{r+s},\\
&[P_m, \mathcal{R}_n] = -2n S_{n+m}.
\end{aligned} \tag{2.43}$$

where indices $\{m,n,p,q\} \in \mathbb{Z}$, while $\{r,s\} \in \mathbb{Z}+\frac{1}{2}$ and $i \in \{1,2\}$. All other commutators vanish. The conformal weight of the generators P_m and S_m are $-b+1$ and $-b$, respectively, while the weight of Q^1, Q^2 is $-\frac{b}{2}+1$. For the specific case $a=0$ and $b=-1$ which corresponds to supersymmetric-$\mathfrak{bms}_3$, we recover the same algebra as given in [23].[4]

2.3 Central extensions of supersymmetric $W(a,b)$

One can show that the $W(a,b)$ algebra for generic values of its parameters just admits one central term in its Witt part, but for certain specific values of a and b, it admits various central extensions. The most general centrally extended supersymmetric $W(a,b)$ algebra can be written as

$$\begin{aligned}
&[J_m, J_n] = (m-n)J_{m+n} + u(m,n),\\
&[J_m, P_n] = -(n+bm+a)P_{m+n} + v(m,n),\\
&[J_m, Q^1_r] = -\left(\frac{bm+a}{2}+r\right)Q^1_{m+r} + x_1(m,r),\\
&[J_m, Q^2_r] = -\left(\frac{bm+a}{2}+r\right)Q^2_{m+r} + x_2(m,r),\\
&[J_m, \mathcal{R}_n] = -n\mathcal{R}_{n+m} + y(m,n),\\
&[J_m, \mathcal{S}_n] = -(a+n+(b+1)m)\mathcal{S}_{n+m} + z(m,n),\\
&[\mathcal{R}_m, Q^1_r] = Q^1_{m+r} + g_1(m,r),\\
&[\mathcal{R}_m, Q^2_r] = -Q^2_{m+r} + g_2(m,r),\\
&\{Q^1_r, Q^2_s\} = P_{r+s} + (r-s)\mathcal{S}_{r+s} + f(r,s),\\
&[P_m, \mathcal{R}_n] = -2n\mathcal{S}_{n+m} + h(m,n),\\
&\{Q^1_r, Q^1_s\} = w_1(r,s), \quad \{Q^2_r, Q^2_s\} = w_2(r,s),\\
&[P_m, P_n] = t_1(m,n),\ [\mathcal{R}_m, \mathcal{R}_n] = w(m,n),\\
&[\mathcal{R}_m, \mathcal{S}_n] = k(m,n),\ [\mathcal{S}_m, \mathcal{S}_n] = s(m,n)\\
&[P_m, \mathcal{S}_n] = t_2(m,n),\ [P_m, Q^i_r] = h^i(m,r),\\
&[\mathcal{S}_n, Q^i_r] = f^i(n,r),
\end{aligned} \tag{2.44}$$

where the Jacobi identity between the generators will put constraints on unknown functions $u, v, x_i, y, z, w, g_i, f, h, w_i, t_i, k, s, h^i, f^i$, which denote the possible central extensions.

The analysis in Appendix A gives a complete classification of all possible central extensions to the $W(a,b)$ algebra for arbitrary values of a and b. Although our ansatz (2.44) was very general, we eventually ended up with only a few nonzero central extensions for certain specific values of a and b. To summarize our findings, we tabulate the nontrivial central charges obtained for other specific domains of the deformation parameters a and b in the following (Table 1):

[3] $\eta_1 = 0$ or $\beta_0 = 0$ are also viable choices of parameters that satisfy the Jacobi identities. But for interpreting R_0 as the $R-$symmetry generator, we must consider nonzero values of those parameters.

[4] There is a typo in Eq. 3.19 and Eq. 8.59 of [23]. The structure constant in the $[M_n, R_m]$ commutator will be $-2m$ instead of $-4m$; otherwise the $(\mathcal{G}^1_r, \mathcal{G}^2_s, R_m)$ will not be satisfied.

Table 1 Central extensions

Central extensions	$a=0, b=-1$	$a=0, b=1$	$a=0, b=2$	$a\neq 0, b\neq 0$
$v(m,n)$	$C^{(5)}_{jp}m^3\delta_{m+n,0}$	0	0	0
$f(r,s)$	$2C^{(5)}_{jp}r^2\delta_{r+s,0}$	$C^{(1)}_{qq}\delta_{r+s,0}$	$C^{(2)}_{qq}r^2\delta_{r+s,0}$	0
$k(m,n)$	$2C^{(5)}_{jp}m\delta_{m+n,0}$	0	$C^{(2)}_{qq}m\delta_{m+n,0}$	0

Note that in the above, we have not included the central extension in the $[J_m, J_n]$ commutator, which is the usual Virasoro central extension $C_{jj}m^3\delta_{m+n}$ that is present for any values of a and b. In addition, the central terms in the $[J_m, \mathcal{R}_n]$ and $[\mathcal{R}_m, \mathcal{R}_n]$ commutator also exist for all values of a and b and are given by $y(m,n) = C^{(0)}_{jr}m^2\delta_{m+n,0}$ and $w(m,n) = C_{rr}m\delta_{m+n,0}$, respectively. For all cases other than those above, the central extensions vanish.

Here, we tabulate the $W(a,b)$ for $a=0$ and $b=-1$, which is the same as the super-$\mathfrak{bms}_3$ algebra

$$\begin{aligned}
&[J_m, J_n] = (m-n)J_{m+n} + C^{(1)}_{jj}\, m^3\, \delta_{m+n,0},\\
&[J_m, P_n] = -(n-m)P_{m+n} + C^{(5)}_{jp}\, m^3\, \delta_{m+n,0},\\
&[J_m, Q^1_r] = -\left(-\frac{m}{2}+r\right) Q^1_{m+r},\\
&[J_m, Q^2_r] = -\left(-\frac{m}{2}+r\right) Q^2_{m+r},\\
&[J_m, \mathcal{R}_n] = -n\mathcal{R}_{n+m} + C^{(0)}_{jr}\, m^2\, \delta_{m+n,0},\\
&[J_m, \mathcal{S}_n] = -n\mathcal{S}_{n+m},\\
&[\mathcal{R}_m, Q^1_r] = Q^1_{m+r},\\
&[\mathcal{R}_m, Q^2_r] = -Q^2_{m+r},\\
&\{Q^1_r, Q^2_s\} = P_{r+s} + (r-s)\mathcal{S}_{r+s} + 2C^{(5)}_{jp}\, r^2\, \delta_{r+s,0},\\
&[P_m, \mathcal{R}_n] = -2n\mathcal{S}_{n+m},\\
&[\mathcal{R}_m, \mathcal{R}_n] = C_{rr}\, m\, \delta_{m+n,0},\\
&[\mathcal{R}_m, \mathcal{S}_n] = 2C^{(5)}_{jp} m\delta_{m+n,0}.
\end{aligned} \tag{2.45}$$

The above is largely in agreement with the results of [23] except for the $[J_m, \mathcal{R}_n]$ case, where we find a new central term. It is interesting to understand the source of this central term in the three-dimensional asymptotically flat bulk supergravity theory. From the analysis of [23], one can observe that such a central term would not arise with the usual Barnich–Compere boundary conditions. Nevertheless, the study of [45] clearly indicates that an asymptotically Rindler-like behavior will modify the asymptotic symmetry algebra with such central extensions. Therefore, with the Rindler-like boundary condition, one can expect to obtain a BMS-like symmetry with this new central term.

3 $\mathfrak{bms}_4$ group and $W(a, b; \bar{a}, \bar{b})$ algebra

Having established a realization of the extended $W(a,b)$ algebra, we now move on to generalize the above analysis in four spacetime dimensions. Like before, our starting point will be the $\mathfrak{bms}_4$ algebra, which can be thought of as a special case of the more general $W(a,b;\bar{a},\bar{b})$ algebra [37,46]. In the early 1960s, [1,2] attempted to understand and study the radiation that would be detected by a distant observer. Interestingly, they found that the full set of symmetries for an asymptotically flat spacetime[5] is an infinite-dimensional group spanned by the so-called supertranslation and superrotation generators, dubbed the $\mathfrak{bms}_4$ group.

The infinite-dimensional centerless asymptotic symmetry algebra of four-dimensional flat spacetime, conventionally known as the $\mathfrak{bms}_4$ algebra [37,49], is given by

$$\begin{aligned}
&[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n}\\
&\left[\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n\right] = (m-n)\bar{\mathcal{L}}_{m+n}\\
&\left[\mathcal{L}_m, \bar{\mathcal{L}}_n\right] = 0\\
&\left[\mathcal{L}_m, T_{p,q}\right] = \left(\frac{m+1}{2} - p\right) T_{p+m,q}\\
&\left[\bar{\mathcal{L}}_m, T_{p,q}\right] = \left(\frac{m+1}{2} - q\right) T_{p,q+m}\\
&\left[T_{p,q}, T_{k,l}\right] = 0
\end{aligned} \tag{3.1}$$

where the indices $m, n, p, q, k, l \in \mathbb{Z}$. The generators $\mathcal{L}_m$ and $\bar{\mathcal{L}}_m$ forming two independent copies of the Witt algebra are known to correspond to *superrotations*, while the generators $T_{p,q}$ are known to correspond to *supertranslations*. Following [37,50], we briefly describe the map between the global sector of $\mathfrak{bms}_4$ algebra and Poincaré algebra.

Denoting the Lorentz generators as $M_{\mu\nu}$ and the translations as P_μ, we know that in four spacetime dimensions, they satisfy the algebra

$$\begin{aligned}
&\left[M_{\mu\nu}, M_{\rho\sigma}\right] = i(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\sigma\mu}M_{\rho\nu} - \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\sigma\nu}M_{\rho\mu})\\
&\left[M_{\mu\nu}, P_\sigma\right] = i(\eta_{\sigma\mu}P_\nu - \eta_{\sigma\nu}P_\mu)\\
&\left[P_\mu, P_\nu\right] = 0
\end{aligned} \tag{3.2}$$

[5] There are various equivalent ways in which one can specify asymptotic behavior of spacetimes. For a detailed exposition, the reader is urged to consult [47,48] and references therein.

where the indices $\mu, \nu, \rho, \sigma \in \{0, 1, 2, 3\}$ and $\eta_{\mu\nu} \equiv \mathrm{diag}(-1, +1, +1, +1)$ is the flat Minkowski metric. We can define the generator of rotations and boosts as

$$J_i = \frac{1}{2}\epsilon_{ijk}M^{jk} \quad \text{and} \quad K_i = M^{0i}, \tag{3.3}$$

respectively, where ϵ_{ijk} is the Levi-Civita tensor and the indices $i, j, k \in \{1, 2, 3\}$. Further, we define the quantities

$$\begin{aligned} \mathcal{L}_{\pm 1} &= iS_1 \pm S_2, \quad \bar{\mathcal{L}}_{\pm 1} = iR_1 \pm R_2 \\ \mathcal{L}_0 &= S_3, \quad \bar{\mathcal{L}}_0 = R_3 \end{aligned} \tag{3.4}$$

where

$$R_i = \frac{1}{2}(J_i + iK_i) \quad \text{and} \quad S_i = \frac{1}{2}(J_i - iK_i). \tag{3.5}$$

It can be easily verified that the set of operators $\{\mathcal{L}_{\pm 1}, \mathcal{L}_0, \bar{\mathcal{L}}_{\pm 1}, \bar{\mathcal{L}}_0\}$ satisfy the algebra

$$\begin{aligned} &[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n}, \\ &\left[\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n\right] = (m-n)\bar{\mathcal{L}}_{m+n}, \\ &\left[\mathcal{L}_m, \bar{\mathcal{L}}_n\right] = 0 \end{aligned}$$

for $(m, n) \in \{\pm 1, 0\}$, thus showing that the set of operators defined in (3.4) indeed correspond to the global part of the infinite-dimensional $\mathfrak{bms}_4$ algebra.

The translation generators P_μ can be mapped to linear combinations of $T_{p,q}$, where $(p, q) \in \{0, 1\}$, as follows:

$$\begin{aligned} P^0 &= H = (T_{1,0} - T_{0,1}) \\ P^1 &= (-i)(T_{1,1} + T_{0,0}) \\ P^2 &= T_{1,1} - T_{0,0} \\ P^3 &= T_{1,0} + T_{0,1} \end{aligned} \tag{3.6}$$

This demonstrates that appropriate combinations of the global part of $\mathfrak{bms}_4$ algebra consisting of the operators $\{\mathcal{L}_{\pm 1}, \mathcal{L}_0, \bar{\mathcal{L}}_{\pm 1}, \bar{\mathcal{L}}_0, T_{1,0}, T_{0,1}, T_{0,0}, T_{1,1}\}$ can be suitably repackaged to give the Poincaré algebra. It is noteworthy that the $T_{0,1}, T_{1,0}, T_{0,0}$ and $T_{1,1}$ gives rise to the translation generators, while a certain combination of $(\mathcal{L}_m, \bar{\mathcal{L}}_m)$ where $(m, n) \in \{\pm 1, 0\}$ gives rise to the Lorentz generators.

Similarly, for four spacetime dimensions, it has been proved that $\mathfrak{bms}_4$ algebra is not rigid and can be deformed into four-parameter family algebra called $W(a, b; \bar{a}, \bar{b})$ algebra, with commutators as

$$\begin{aligned} &[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n}, \\ &[\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n] = (m-n)\bar{\mathcal{L}}_{m+n}, \\ &[\mathcal{L}_m, \bar{\mathcal{L}}_n] = 0. \\ &[\mathcal{L}_m, T_{p,q}] = -(a + bm + p)T_{m+p,q}, \\ &[\bar{\mathcal{L}}_n, T_{p,q}] = -(\bar{a} + \bar{b}n + q)T_{p,n+q}, \\ &[T_{p,q}, T_{k,l}] = 0, \end{aligned} \tag{3.7}$$

where $a, b, \bar{a}$ and $\bar{b}$ are arbitrary real parameters. In this way, $\mathfrak{bms}_4$ algebra (3.1) can be viewed as $W(-\frac{1}{2}, -\frac{1}{2}; -\frac{1}{2}, -\frac{1}{2})$. This algebra can be viewed as two copies of $W(a, b)$ and $W(\bar{a}, \bar{b})$, with the identification of $T_{p,q}$ as a product of supertranslation generators of both algebras. However, as we see in the next section, this structure does not extend to the supersymmetric extensions of the algebra. Another interesting case is $W(0, 0; 0, 0)$, which represents an infinite-dimensional algebra of the symmetries of the near-horizon geometry of non-extremal black holes [51]. The algebra with $a = b = \bar{a} = \bar{b} = -\frac{1+s}{2}$ for $0 < s < 1$ describes the asymptotic symmetry algebra of decelerating FLRW spacetime [39].

4 Supersymmetric $W(a, b; \bar{a}, \bar{b})$ algebra from $\mathcal{N} = 2$ super-$\mathfrak{bms}_4$

In this section, we write down a supersymmetric extension of $W(a, b; \bar{a}, \bar{b})$ algebra with two supercharges. To get to this, like the $\mathfrak{bms}_3$ algebra, our first goal is to write the supersymmetrized $\mathfrak{bms}_4$ algebra. The first attempt towards the construction of a super-$\mathfrak{bms}$ algebra was carried out in [52], which however did not consider superrotation generators. [53] further explored asymptotic fermionic charges in $\mathcal{N} = 1$ supergravity on a four-dimensional asymptotically flat background.[6] [43,56] have derived such an algebra by analyzing operator product expansions (OPEs) of appropriate operators of Einstein–Yang–Mills theory at the celestial sphere. However, their convention for indices on the supertranslation generators is different from ours. This changes the index structure that appears in the commutators. We fix the index structure of the supersymmetrized $\mathfrak{bms}_4$ algebra by demanding consistency between its global part and the four-dimensional super-Poincaré algebra, which along with (3.2) now also contains

$$\begin{aligned} &\left\{\mathcal{Q}_A, \bar{\mathcal{Q}}_{\dot{B}}\right\} = 2(\sigma^\mu)_{A\dot{B}}P_\mu, \quad \left[M^{\mu\nu}, \mathcal{Q}_A\right] = i(\sigma^{\mu\nu})_A^{\ B}\mathcal{Q}_B, \\ &[M^{\mu\nu}, \bar{\mathcal{Q}}^{\dot{A}}] = i(\bar{\sigma}^{\mu\nu})^{\dot{A}}_{\ \dot{B}}\bar{\mathcal{Q}}^{\dot{B}}. \end{aligned} \tag{4.1}$$

Our starting point in the current context is the algebra stated at (3.1). However, as mentioned earlier, we need to determine the index structure once we include the super-current generators, which we denote by Q_r^i and $\bar{Q}_r^i$, where $i = 1, 2$ while $r \in \mathbb{Z} + \frac{1}{2}$. We begin with the global algebra to determine the indices. Hence, we propose the following ansatz involving the supertranslation and superrotation generators with the fermionic supercurrent generators

$$\{Q_r^i, \bar{Q}_s^j\} = \delta^{ij}T_{f(r,s),g(r,s)}; \quad [\mathcal{L}_m, Q_r^i] = \alpha(m, r)Q^i_{h(m,r)};$$

[6] There also exists a realization of super-$\mathfrak{bms}$ algebra at spatial infinity [54,55].

$$[\bar{\mathcal{L}}_m, \bar{Q}^i_r] = \bar{\alpha}(m,r)\bar{Q}^i_{\bar{h}(m,r)}\,, \tag{4.2}$$

and all other elements in the super-algebra are zero. We must note here that it is not a priori necessary that the other possible (anti)-commutators are zero for a four-dimensional asymptotically flat supergravity theory; however, we consider this simplified deformation for the purpose of this paper. Our approach is pragmatic—we simply want to write a possible supersymmetric extension of $\mathfrak{bms}_4$ algebra such that its global part coincides with the super-Poincaré algebra. The OPE analysis of [43,56] found that the only nonzero commutators in super-$\mathfrak{bms}_4$ algebra are those mentioned above in (4.2). This motivates us to propose the ansatz as written above. We further make the simplifying assumption that the functions parameterizing the indices $f, g, h, \bar{h}$ are all linear in their arguments.[7]

We *demand* that the map between supercurrent modes and the fermionic generators of the super-Poincaré algebra be

$$Q^i_1 \to Q^i_{+\frac{1}{2}}, \quad Q^i_2 \to Q^i_{-\frac{1}{2}},$$
$$\bar{Q}^i_{\dot 1} \to \bar{Q}^i_{+\frac{1}{2}}, \quad \bar{Q}^i_{\dot 2} \to \bar{Q}^i_{-\frac{1}{2}}.$$

Assuming the indices f and g are linear in their arguments, one can fix the explicit form using the global sector, i.e., the super-Poincaré algebra, to obtain

$$\begin{aligned}
\{Q^i_r, \bar{Q}^j_s\} &= \delta^{ij} T_{r+\frac{1}{2},-s+\frac{1}{2}},\\
\{Q^i_r, Q^j_s\} &= \{\bar{Q}^i_r, \bar{Q}^j_s\} = 0,\\
[\mathcal{L}_m, Q^i_r] &= \left(\frac{m}{2} - r\right) Q^i_{m+r},\\
[\bar{\mathcal{L}}_m, \bar{Q}^j_s] &= \left(\frac{m}{2} + s\right) \bar{Q}^j_{-m+s}.
\end{aligned} \tag{4.3}$$

The details of the derivation are provided in Appendix 1.

Thus, a particular realization of the $\mathcal{N} = 2$ super-$\mathfrak{bms}_4$ algebra can be written as

$$\begin{aligned}
&[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n},\\
&[\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n] = (m-n)\bar{\mathcal{L}}_{m+n},\\
&[\mathcal{L}_m, T_{p,q}] = \left(\frac{m+1}{2} - p\right) T_{m+p,q},\\
&[\bar{\mathcal{L}}_m, T_{p,q}] = \left(\frac{m+1}{2} - q\right) T_{p,m+q},\\
&\{Q^i_r, \bar{Q}^j_s\} = \delta^{ij} T_{r+1/2,-s+1/2},\\
&[\mathcal{L}_m, Q^i_r] = \left(\frac{m}{2} - r\right) Q^i_{m+r},\\
&[\bar{\mathcal{L}}_m, \bar{Q}^i_r] = \left(\frac{m}{2} + r\right) \bar{Q}^i_{-m+r}.
\end{aligned} \tag{4.4}$$

while the other (anti)-commutators are identically zero. One can easily check that all the Jacobi identities are satisfied for the above algebra. Now that we have fixed the indices in the super-$\mathfrak{bms}_4$ algebra, we assume that deformations do not change that and thus will carry over to the $W(a, b; \bar{a}, \bar{b})$ algebra. Thus, the supersymmetrized $W(a, b; \bar{a}, \bar{b})$ algebra is

$$\begin{aligned}
&[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n},\\
&[\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n] = (m-n)\bar{\mathcal{L}}_{m+n},\\
&[\mathcal{L}_m, T_{p,q}] = -(a + bm + p)\, T_{m+p,q},\\
&[\bar{\mathcal{L}}_m, T_{p,q}] = -\left(\bar{a} + \bar{b}m + q\right) T_{p,m+q},\\
&\{Q^i_r, \bar{Q}^j_s\} = \delta^{ij} T_{r+1/2,-s+1/2},\\
&[\mathcal{L}_m, Q^i_r] = \alpha(m,r) Q^i_{m+r} = -\left(a + bm + r + \frac{1}{2}\right) Q^i_{m+r},\\
&[\bar{\mathcal{L}}_m, \bar{Q}^i_r] = -\bar{\alpha}(m,r) \bar{Q}^i_{m+r} = -\left(\bar{a} + \bar{b}m - r + \frac{1}{2}\right) \bar{Q}^i_{-m+r},
\end{aligned} \tag{4.5}$$

where all other commutators are zero. $\alpha(m, r)$ is fixed by the Jacobi identity for Q^i_r, $\bar{Q}^j_s$ and $\mathcal{L}_m$, which is given by

$$[\{Q^i_r, \bar{Q}^j_s\}, \mathcal{L}_m] = \{Q^i_r, [\bar{Q}^j_s, \mathcal{L}_m]\} + \{\bar{Q}^j_s, [Q^i_r, \mathcal{L}_m]\} \tag{4.6}$$

Similarly. we fix $\bar{\alpha}(m, r)$ using the Jacobi identities of Q^i_r, $\bar{Q}^j_s$ and $\bar{\mathcal{L}}_m$.

5 R-extended supersymmetric $W(a, b; \bar{a}, \bar{b})$ algebra

Before going to the R-extension of the $W(a, b; \bar{a}, \bar{b})$ algebra, let us briefly recall some important points about super-Poincaré algebras in the presence of $R-$symmetry. R-charge rotates the SUSY generators among themselves. The $R-$extension is the largest subgroup of the automorphism group of the supersymmetry algebra which commutes with the Lorentz group. As discussed in [58], generic $\mathcal{N} = 2$ super-Poincaré algebra contains two species of R-symmetry generators—vectorial and axial—which act on the supercharges as

$$\begin{aligned}
&[Q^i_{\pm\frac{1}{2}}, R_0] = Q^i_{\pm\frac{1}{2}}, \quad [\bar{Q}^i_{\pm\frac{1}{2}}, R_0] = -\bar{Q}^i_{\pm\frac{1}{2}},\\
&[Q^i_{\pm\frac{1}{2}}, \bar{R}_0] = \pm Q^i_{\pm\frac{1}{2}}, \quad [\bar{Q}^i_{\pm\frac{1}{2}}, \bar{R}_0] = \mp \bar{Q}^i_{\pm\frac{1}{2}}
\end{aligned} \tag{5.1}$$

As a matter of fact, theories with extended supersymmetries are extremely rich precisely due to the presence of these two kinds of supercharges. These R-symmetries can in fact be used as a powerful tool to define various *twistings* in a theory (A-type or B-type), resulting in what is known as topological field theories whose correlators happen to be independent of the background metric. Although, conventionally, theories with either kind of R-symmetry are considered, in general, the full theory does contain both kinds of R-symmetries. In the current context, since we are specifically interested in

[7] This structure does not hold for certain symmetry algebras such as that discussed in [57].

$\mathcal{N}=2$ SUSY, the R-symmetry generator in fact generates the group $U(1)_V \times U(1)_A$. Following [58], we will associate R_0 with the vectorial R-symmetry while $\bar{R}_0$ will be associated as the axial R-symmetry. Another important aspect of super-Poincaré algebras is that the $R-$charges commute with the bosonic Poincaré generators. We utilize these facts in our constructions. To be precise, in the following we demand that our R-extended $W(a,b;\bar{a},\bar{b})$ algebra must have vanishing commutators of the $R-$charge generators with other bosonic generators in the global sector for the deformation $W(-1/2,-1/2;-1/2,-1/2)$[8]

In order to realize the extension of R-charges in the context of $W(a,b;\bar{a},\bar{b})$ algebra, we will start with a general ansatz,

$$[Q^i_r, R_n] = \beta^{(i)}(r,n) Q^i_{\theta_1(r,n)}, \quad [\bar{Q}^i_r, R_n] = \bar{\beta}^{(i)}(r,n)\bar{Q}^i_{\bar{\theta}_1(r,n)},$$
$$[Q^i_r, \bar{R}_n] = \kappa^{(i)}(r,n) Q^i_{\theta_2(r,n)}, \quad [\bar{Q}^i_r, \bar{R}_n] = \bar{\kappa}^{(i)}(r,n)\bar{Q}^i_{\bar{\theta}_2(r,n)}. \tag{5.2}$$

It must be noted that in the above ansatz, although i is a repeated index on the RHS, there is no sum over i. While we will be primarily interested in $\mathcal{N}=2$ algebras, the discussion in this section is valid for an arbitrary value of $\mathcal{N}$. Thus we consider $i=1,2,\ldots,\mathcal{N}$. Furthermore, since we are working in a very general setting, we assume that β^i is different for $i=1,2,\ldots,\mathcal{N}$. This is of course not the most general extension one can think of, but rather a simpler starting point, which is also consistent with the global subsector (5.1).

In the following analysis, we will assume that the indices appearing in the above ansatz are linear in their arguments, which is a basic feature of most algebras. Consistency with the global subsector, i.e., (5.1), fixes the form of the indices as

$$[Q^i_r, R_n] = \beta^{(i)}(r,n) Q^i_{cn+r}, \; [\bar{Q}^i_r, R_n] = \bar{\beta}^{(i)}(r,n)\bar{Q}^i_{\bar{c}n+r}$$
$$[Q^i_r, \bar{R}_n] = \kappa^{(i)}(r,n) Q^i_{kn+r}, \; [\bar{Q}^i_r, \bar{R}_n] = \bar{\kappa}^{(i)}(r,n)\bar{Q}^i_{\bar{k}n+r}. \tag{5.3}$$

where $c, \bar{c}, k$ and $\bar{k}$ are integers, and the structure constants are nonzero at least in the global subsector, i.e., when $r=\pm\frac{1}{2}$ and $n=0$. We have to find their form away from the global sector. Subsequently, we will concentrate on the commutators $[R_n, T_{p,q}]$ and $[\bar{R}_n, T_{p,q}]$. Consider the Jacobi identity for the operators $Q^i_r, \bar{Q}^j_s$ and R_n, which gives us

$$[\{Q^i_r, \bar{Q}^j_s\}, R_n] = \{Q^i_r, [\bar{Q}^j_s, R_n]\} + \{\bar{Q}^j_s, [Q^i_r, R_n]\}.$$

Using (5.3), in the above, we get

$$\delta^{ij}[T_{p,q}, R_n] = \delta^{ij}\left[\bar{\beta}^{(j)}\left(-q+\frac{1}{2}, n\right) T_{p,-\bar{c}n+q} + \beta^{(i)}\left(p-\frac{1}{2}, n\right) T_{cn+p,q}\right].$$

In the above equation, i, j are free indices. In particular for $\mathcal{N}=2$ SUSY , we see that

$$\begin{aligned}[T_{p,q}, R_n] &= \bar{\beta}^{(1)}\left(-q+\frac{1}{2}, n\right) T_{p,-\bar{c}n+q} + \beta^{(1)}\left(p-\frac{1}{2}, n\right) T_{cn+p,q} \\ &= \bar{\beta}^{(2)}\left(-q+\frac{1}{2}, n\right) T_{p,-\bar{c}n+q} + \beta^{(2)}\left(p-\frac{1}{2}, n\right) T_{cn+p,q}.\end{aligned} \tag{5.4}$$

which also implies $\beta^{(1)}(r,n)=\beta^{(2)}(r,n)$ and $\bar{\beta}^{(1)}(r,n)=\bar{\beta}^{(2)}(r,n)$. An identical exercise with $Q^i_r, \bar{Q}^j_s$ and $\bar{R}_n$ gives

$$\begin{aligned}[T_{p,q}, \bar{R}_n] &= \bar{\kappa}^{(1)}\left(-q+\frac{1}{2}, n\right) T_{p,-\bar{k}n+q} + \kappa^{(1)}\left(p-\frac{1}{2}, n\right) T_{kn+p,q} \\ &= \bar{\kappa}^{(2)}\left(-q+\frac{1}{2}, n\right) T_{p,-\bar{k}n+q} + \kappa^{(2)}\left(p-\frac{1}{2}, n\right) T_{kn+p,q},\end{aligned} \tag{5.5}$$

leading to the condition that $\kappa^{(1)}(r,n) = \kappa^{(2)}(r,n)$ and $\bar{\kappa}^{(1)}(r,n)=\bar{\kappa}^{(2)}(r,n)$. Given the relation between the structure constants for $i=1,2$, we see that the index is extraneous, and hence we will subsequently simply be dropping it from our notation. Thus, we may write more simply

$$\begin{aligned}
&[Q^i_r, R_n] = \beta(r,n) Q^i_{cn+r}, \; [\bar{Q}^i_r, R_n] = \bar{\beta}(r,n)\bar{Q}^i_{\bar{c}n+r}, \\
&[Q^i_r, \bar{R}_n] = \kappa(r,n) Q^i_{kn+r}, \; [\bar{Q}^i_r, \bar{R}_n] = \bar{\kappa}(r,n)\bar{Q}^i_{\bar{k}n+r}, \\
&[T_{p,q}, R_n] = \bar{\beta}\left(-q+\frac{1}{2}, n\right) T_{p,-\bar{c}n+q} + \beta\left(p-\frac{1}{2}, n\right) T_{cn+p,q}, \\
&[T_{p,q}, \bar{R}_n] = \bar{\kappa}\left(-q+\frac{1}{2}, n\right) T_{p,-\bar{k}n+q} + \kappa\left(p-\frac{1}{2}, n\right) T_{kn+p,q}\;.
\end{aligned} \tag{5.6}$$

Note that the above commutation relations also ensure that the Jacobi identities between $T_{p,q}, T_{m,n}, R_l$ as well as $T_{p,q}, Q^i_r, R_n$ (and its corresponding counterparts with Q^i_r replaced with $\bar{Q}^j_s$ and R_n replaced with $\bar{R}_n$) are also satisfied.

[8] $W(-1/2,-1/2;-1/2,-1/2)$ is the $\mathfrak{bms}_4$ algebra whose global sector coincides with the $\mathcal{N}=2$ super-Poincaré algebra.

Now that we have identified R_n and $\bar{R}_n$ to the vectorial and axial R-supercurrents, we make a further assumption that

$$[R_n, \bar{R}_m] = 0. \tag{5.7}$$

The above assumptions applied to the Jacobi identity of R_n, $\bar{R}_m$ and Q^i_r leads to a relation between the structure constants

$$\kappa(cn + r, m)\beta(r, n) = \kappa(r, m)\beta(km + r, n), \tag{5.8}$$

and analogously the Jacobi identity of R_n, $\bar{R}_m$ and $\bar{Q}^i_r$ gives rise to

$$\bar{\kappa}(\bar{c}n + r, m)\bar{\beta}(r, n) = \bar{\kappa}(r, m)\bar{\beta}(\bar{k}m + r, n). \tag{5.9}$$

The above two relations seem to put certain constraints on the free parameters $c, \bar{c}, k$ and $\bar{k}$. However, we will explore this subsequently.

Finally, we need to fix the algebra between the R-supercurrents and the superrotations $\mathcal{L}_m$ and $\bar{\mathcal{L}}_m$. For this purpose we consider a set of Jacobi identities detailed in Appendix C between C.1 and C.28. This analysis helps us to fix the form of the commutators between the R-charge supercurrents and the superrotation generators. Thus, by simply imposing Jacobi identities in a systematic manner we have obtained the following simplified algebra involving the R-supercurrents

$$\begin{aligned}
&[Q^i_r, R_n] = \beta(r, n)Q^i_{cn+r},\ [\bar{Q}^i_r, R_n] = \bar{\beta}(r, n)\bar{Q}^i_{\xi cn+r},\\
&[Q^i_r, \bar{R}_n] = \kappa(r, n)Q^i_{kn+r},\ [\bar{Q}^i_r, \bar{R}_n] = \bar{\kappa}(r, n)\bar{Q}^i_{\xi kn+r},\\
&[T_{p,q}, R_n] = \bar{\beta}\left(-q + \frac{1}{2}, n\right)T_{p,-\xi cn+q}\\
&\quad + \beta\left(p - \frac{1}{2}, n\right)T_{cn+p,q},\\
&[T_{p,q}, \bar{R}_n] = \bar{\kappa}\left(-q + \frac{1}{2}, n\right)T_{p,-\xi kn+q}\\
&\quad + \kappa\left(p - \frac{1}{2}, n\right)T_{kn+p,q},\\
&[R_n, \mathcal{L}_m] = w_1(n, m)\mathcal{L}_{cn+m} + h_1(n, m)R_{n+cm}\\
&\quad + \bar{h}_1(n, m)\bar{R}_{kcn+km},\\
&[R_n, \bar{\mathcal{L}}_m] = w_2(n, m)\bar{\mathcal{L}}_{-\xi cn+m}\\
&\quad + h_2(n, m)R_{n-\xi cm} + \bar{h}_2(n, m)\bar{R}_{kcn-\xi km},\\
&[\bar{R}_n, \mathcal{L}_m] = w_3(n, m)\mathcal{L}_{kn+m}\\
&\quad + h_3(n, m)R_{kcn+cm} + \bar{h}_3(n, m)\bar{R}_{n+km},\\
&[\bar{R}_n, \bar{\mathcal{L}}_m] = w_4(n, m)\bar{\mathcal{L}}_{-\xi kn+m}\\
&\quad + h_4(n, m)R_{kcn-\xi cm} + \bar{h}_4(n, m)\bar{R}_{n-\xi km}.
\end{aligned} \tag{5.10}$$

In Appendix C we have further listed the Jacobi identities involving the R-supercurrent generator and superrotation generator which will help to fix the form of w and h.

5.1 Algebra with linear structure constants

In our analysis so far, the structure constants have been kept arbitrary. However, at this point, we make two powerful simplifying assumptions which will somewhat reduce the complexity of the problem. The assumptions are as follows:

- The structure constants appearing in the proposed algebra are linear in their arguments.
- The global subsector of $W(-1/2, -1/2; -1/2, -1/2)$ algebra must coincide with the R-extended $\mathcal{N} = 2$ super-Poincarë algebra.

This leads us to propose an ansatz of the form

$$\mu_i(n, p) = \omega_{i0} + \omega_{i1}n + \omega_{i2}p \quad (\text{for } i = 1, 2, 3, 4), \tag{5.11}$$

where μ_i denotes the structure constants $w(n, p), h(n, p)$ or $\bar{h}(n, p)$. Since, for the global subsector, we must have $[R_0, \mathcal{L}_m] = [\bar{R}_0, \mathcal{L}_m] = [R_0, \bar{\mathcal{L}}_m] = [\bar{R}_0, \bar{\mathcal{L}}_m] = 0$, it implies that $\mu_i(0, 0) = \mu_i(0, 1) = \mu_i(0, -1) = 0$. This along with the proposed ansatz leads us to conclude that these structure constants must be of the form

$$\mu_i(n, p) = \omega n, \tag{5.12}$$

where ω is an arbitrary constant.

The global sector of the R-supercurrent, i.e., R_0 and $\bar{R}_0$, is supposed to commute with the generators of the Poincaré algebra $M_{\mu\nu}$, P_μ, while its commutator with the SUSY generators $Q^i_{\pm\frac{1}{2}}$, $\bar{Q}^i_{\pm\frac{1}{2}}$ must be nonzero. Specifically, imposing this constraint on the translation generators P_μ, and using the identification (3.6)–(3.6), we obtain

$$[T_{0,0}, R_0] = [T_{0,1}, R_0] = [T_{1,0}, R_0] = [T_{1,1}, R_0] = 0. \tag{5.13}$$

An identical set of relations is true for $\bar{R}_0$. The above implies that the structure constants are related by

$$\bar{\beta}\left(+\frac{1}{2}, 0\right) = \bar{\beta}\left(-\frac{1}{2}, 0\right) = -\beta\left(+\frac{1}{2}, 0\right) = -\beta\left(-\frac{1}{2}, 0\right) \neq 0. \tag{5.14}$$

Now, assuming that both $\beta(r, n)$ and $\bar{\beta}(r, n)$ are linear in its argument, the above relation immediately implies that they must be of the form

$$\beta(r, n) = \beta_1 n + \beta_0 \quad \text{and} \quad \bar{\beta}(r, n) = \bar{\beta}_1 n - \beta_0, \tag{5.15}$$

where $\beta_1, \beta_2, \bar{\beta}_1$ and $\bar{\beta}_2$ are constants. One can repeat the same exercise with the structure constants $\kappa(r, n)$ and $\bar{\kappa}(r, n)$

to see that they are also independent of the first index r and hence can be written as

$$\kappa(r,n)=\kappa_1 n+\kappa_0 \quad \text{and} \quad \bar{\kappa}(r,n)=\bar{\kappa}_1 n-\kappa_0. \tag{5.16}$$

One immediate consequence of the above is that (5.8) and (5.9) is now trivially satisfied. With all of these conclusions, we have a more simplified algebra, given by

$$\begin{aligned}
&[Q^i_r, R_n]=\beta(n)Q^i_{cn+r}, \quad [\bar{Q}^i_r, R_n]=\bar{\beta}(n)\bar{Q}^i_{\xi cn+r},\\
&[Q^i_r, \bar{R}_n]=\kappa(n)Q^i_{kn+r}, \quad [\bar{Q}^i_r, \bar{R}_n]=\bar{\kappa}(n)\bar{Q}^i_{\xi kn+r},\\
&[T_{p,q}, R_n]=\bar{\beta}\,(n)\,T_{p,-\xi cn+q}+\beta\,(n)\,T_{cn+p,q},\\
&[T_{p,q}, \bar{R}_n]=\bar{\kappa}\,(n)\,T_{p,-\xi kn+q}+\kappa\,(n)\,T_{kn+p,q},\\
&[R_n, \mathcal{L}_m]=w_1(n)\mathcal{L}_{cn+m}+h_1(n)R_{n+cm}+\bar{h}_1(n)\bar{R}_{kcn+km},\\
&[R_n, \bar{\mathcal{L}}_m]=w_2(n)\bar{\mathcal{L}}_{-\xi cn+m}+h_2(n)R_{n-\xi cm}+\bar{h}_2(n)\bar{R}_{kcn-\xi km},\\
&[\bar{R}_n, \mathcal{L}_m]=w_3(n)\mathcal{L}_{kn+m}+h_3(n)R_{kcn+cm}+\bar{h}_3(n)\bar{R}_{n+km},\\
&[\bar{R}_n, \bar{\mathcal{L}}_m]=w_4(n)\bar{\mathcal{L}}_{-\xi kn+m}+h_4(n)R_{kcn-\xi cm}+\bar{h}_4(n)\bar{R}_{n-\xi km}.
\end{aligned} \tag{5.17}$$

Now, we will focus on (C.22) specifically, which yields two equations given by

$$\begin{aligned}
&w_1(n)(a+bcn+bm+p)-h_1(n)\beta\,(n+cm)\\
&\quad-\bar{h}_1(n)\kappa\,(ckn+km)+cn\beta\,(n)=0,
\end{aligned}\tag{5.18}$$

$$-h_1(n)\bar{\beta}\,(n+cm)-\bar{h}_1(n)\bar{\kappa}\,(ckn+km)=0. \tag{5.19}$$

The first equation written above must be true for arbitrary integral values of m, n and p. However, it contains a term of the form $w_1(n)p$ which must vanish, implying that $w_1(n)=0$ identically. A similar argument can be applied to the Jacobi identities (C.24), (C.26) and (C.28) to reach the conclusion that $w_2=w_3=w_4=0$ too. Plugging in the linear forms of $h_1(n)$, $\bar{h}_1(n)$ and $\beta(n)$ in the equations (5.18) and (5.19), we obtain

$$\begin{aligned}
c\beta_1-\omega_1\beta_1-ck\kappa_1\bar{\omega}_1&=0,\\
-c\omega_1\beta_1-k\kappa_1\bar{\omega}_1&=0,\\
c\beta_0-\omega_1\beta_0-\kappa_0\bar{\omega}_1&=0.
\end{aligned}\tag{5.20}$$

The above set of equations implies that $\beta_1=0$. Applying the exact same argument to (C.24), (C.26) and (C.28) yields $\bar{\beta}_1=\kappa_1=\bar{\kappa}_1=0$. Essentially, this shows that irrespective of the arguments, $\beta, \bar{\beta}, \kappa$ and $\bar{\kappa}$ are constants and related as $\beta=-\bar{\beta}$ and $\kappa=-\bar{\kappa}$. However, (5.18) and (5.19) reduce to simply

$$-\omega_1\beta_0-\bar{\omega}_1\kappa_0+c\beta_0=0 \quad \text{and} \quad \omega_1\beta_0+\bar{\omega}_1\kappa_0=0, \tag{5.21}$$

which leads us to conclude that the structure constant β must be identically zero, which is clearly in contradiction with (5.14). (C.24), (C.26). Similarly, (C.28) leads to the conclusion that $\bar{\beta}, \kappa$ and $\bar{\kappa}$ must also be vanishing. Thus we essentially find that an infinite-dimensional extension of R-charges in $\mathcal{N}=2\,\mathfrak{bms}_4$ algebra is impossible with linear structure constants. Therefore we conclude that a generic $\mathcal{N}=2$ $W(a,b;\bar{a},\bar{b})$ algebra (4.5) of 4 cannot have infinite $R-$extension with linear structure constants. In Appendix D we have further assumed that the structure constants are quadratic in the arguments. Still, one cannot have infinite $R-$extension.

6 Conclusion

As mentioned earlier, symmetry algebras are powerful tools which severely constrain the dynamics and vacua of gauge and gravity theories. In this work, we have concentrated on supersymmetric $W(a,b)$ and $W(a,b;\bar{a},\bar{b})$ algebras which are deformations of the asymptotic symmetry algebra of supergravity theories in three and four spacetime dimensions, respectively. Earlier works [36,37] have established that generic deformations of $\mathfrak{bms}_3$ algebras involve two parameters a and b, while generic deformations of $\mathfrak{bms}_4$ algebras involve $a, b, \bar{a}$ and $\bar{b}$. The $a=0, b=-1$ centerless R-extension of supersymmetric $W(a,b)$ algebra given by (2.43) indeed matches with earlier results of [23], where the authors performed an asymptotic symmetry analysis to obtain the super-$\mathfrak{bms}_3$ algebra. We also classified and wrote down the possible central extensions of the supersymmetric, R-extended $W(a,b)$ algebra. We observed interesting and novel central charges appearing in the $\{Q^1_r, Q^2_s\}$ anticommutator, denoted by $f(r,s)$, for various values of a and b. We also found that $[J_m, \mathcal{R}_n]$ admits a quadratic central charge which was not realized through the asymptotic symmetry analysis performed in [23], although it seems to be present for arbitrary values of a and b. Thus, it remains an interesting open problem to find the importance of this central term in the context of three-dimensional asymptotically flat supergravity theory in more generic contexts. As mentioned earlier, the $W(0,0)$ and $W(0,1)$ algebras have also appeared as asymptotic symmetry algebras of gravity theories [40,41], so it is worth exploring appropriate boundary/fall-off conditions to obtain supersymmetric $W(0,0)$ and $W(0,1)$ algebras as asymptotic symmetry algebras in some supergravity theory. To conclude, the analysis of the present work, being mathematically rigorous, provides new asymptotic algebras and hence opens up the possibility for finding new boundary conditions for supergravity fields. We hope to report on these possibilities in future works.

The construction of the R-extended supersymmetric $W(a,b;\bar{a},\bar{b})$ algebra turned out to be more involved. Physically, the R-charge generators are supposed to rotate the global SUSY-generators, which motivates us to propose an ansatz of the form (5.2). We essentially tried to extend the super-$W(a,b;\bar{a},\bar{b})$ algebra (written explicitly in (4.5)) by a

$U(1)_V \times U(1)_A$ group where each sector is represented by infinitely many generators. One of the sectors of the $U(1)_V$ symmetry can be thought of as *vectorial* R-symmetry, while the other copy of the $U(1)_A$ symmetry can be thought of as *axial* R-symmetry. We considered two primary guiding principles to fix the algebra:

- For $a = b = \bar{a} = \bar{b} = -\frac{1}{2}$, the global subalgebra must be identical to the R-extended super-Poincaré algebra.
- The indices appearing in all the proposed commutators involving the R-charges must be linear in their arguments.

In order to simplify our calculations, we considered both linear and quadratic structure constants. In either case, we realized that having infinitely many R-charges is in contradiction with one or more Jacobi identities that must be followed by such a graded Lie algebra. Essentially, it seems there is an obstruction in the $u(1) \times u(1)$ extension of super-$W(a, b; \bar{a}, \bar{b})$ algebra—which will naturally hinder the construction of an R-extended super-$\mathfrak{bms}_4$ algebra. Recent work [59] has carried out $u(1)$ and $u(N)$ extensions of $\mathfrak{bms}_4$ by analyzing celestial amplitudes of Einstein–Maxwell and Einstein–Yang–Mills theories, and has indeed obtained nontrivial asymptotic symmetry algebras at the boundary which does include infinitely many generators parameterizing the $u(1)$ or $u(N)$ symmetry. This is, however, not in contradiction with our results. Our demand on the behavior of R-charges, i.e., it must non-trivially rotate the global SUSY generators, forces us to demand (5.14), which ensures the $[Q^i_r, R_m]$ commutator to be nonzero for the global sector. Such a constraint need not be followed for the $u(1)$ or $u(N)$ gauge groups that enter the analysis of [59]. Relaxing (5.14) in our current work does indeed recover the symmetry algebras derived in [59]. Finally, the methodology of our construction by thoroughly analyzing all possible Jacobi identities while imposing consistency with the global subsector is quite general. It is possible to adapt this algorithm to construct $u(1)$ or $u(N)$ extension for other exotic symmetry algebras. We must emphasize that in this paper, we have studied the R-extended super-$W(a, b)$ and super-$W(a, b; \bar{a}, \bar{b})$ algebras for generic permissible values of a and b. However, it is only for specific values of a and b, physical theories of gravity or supergravity are known where these are realized as boundary symmetry algebras. It will be interesting to explore what kind of supergravity theories give rise to these wide ranges of W-algebras for more generic values of $a, b, \bar{a}$ and $\bar{b}$.

Let us conclude the paper with the importance of the study of supersymmetric extensions of the deformations of $\mathfrak{bms}$ algebras. As is well understood, $\mathfrak{bms}$ algebras are symmetries of asymptotically flat gravity theories at their null boundaries. Some of their deformations have also been realized as the symmetry algebra at the horizon of certain black hole backgrounds. In the context of three spacetime dimensions, the presence of extended supersymmetries and internal $R-$symmetries plays crucial roles in characterizing the soft hair modes (that give nontrivial cosmological solutions) and their thermodynamics [16,22,60,61]. A similar study has not been performed for four spacetime dimensions, where nontrivial black hole and gravitational wave solutions exist. In the context of $\mathfrak{bms}_4$, the soft hair modes contribute to black hole entropy, although they do not correspond to the entire microscopic degeneracy. The microscopic degeneracy for a class of four-dimensional $\mathcal{N} = 2, 4, 8$ supersymmetric BPS black holes is very well understood [62–67]. It would be interesting to understand how much of this entropy is contributed by the soft hairs. Such a study will be involved and is not within the scope of the present work. However, our present results suggest that, for black holes appearing in $\mathcal{N} = 2$ supergravity theory, where the internal $R-$symmetry (that only scales the supercharges) will not have any contributions to the soft hairs. A similar study for other supergravity theories with exotic internal symmetries remains an open problem for the future.

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Appendix A: Fixing the structure constants of centrally extended supersymmetric $W(a, b)$

In this section we have provided a detailed analysis to find the form of structure constants involved in the central extension of $\mathcal{N} = 2$ supersymmetric $W(a, b)$ algebra. The nontrivial central extensions of $\mathcal{N} = 2$ supersymmetric $W(a, b)$ algebra belong to its second real cohomology $\mathcal{H}^2(W, \mathbb{R})$. To compute it, we must take two steps:

- Consider all possible central terms in various commutators
- Eliminate the trivial central terms.

For the first step, we set some unknown functions in various commutators, and try to find the (nontrivial) form of the functions by using the constraints obtained from Jacobi identity analysis (also known as 2-cocycle conditions). Then, it should be checked which of these functions cannot be absorbed by the redefinition of the generators. These terms are nontrivial central extensions of the algebra or equivalently the elements of the second real cohomology of the algebra.

The central term in the commutator $[J_m, J_n]$, which we denoted as $u(m, n)$, is an arbitrary antisymmetric function. The Jacobi identity

$$[J_m, [J_n, J_l]] + [J_n, [J_l, J_m]] + [J_l, [J_m, J_n]] = 0, \qquad \text{(A.1)}$$

leads to the relation

$$(n-l)u(m, n+l)+(l-m)u(n, m+l)+(m-n)u(l, n+m) = 0, \qquad \text{(A.2)}$$

which has the nontrivial solution $u(m, n) = C^{(1)}_{jj} (m^3 - m)\, \delta_{m+n,0}$. This, as expected, is in the form of the usual Virasoro central charge. Other Jacobi identities do not put any new constraint on $u(m, n)$. A redefinition of $J_m \to J_m + A\delta_{m,0}$ with an appropriate choice of A can be used to absorb the linear term in m.

One can fix the central term $v(m, n)$ of the $[J_m, P_n]$ commutator in the following way. The Jacobi identity between J_m, J_n and P_l leads to

$$\begin{aligned}&-(a + bn + l)v(m, n + l) + (a + bm + l)v(n, m + l)\\ &\quad-(m - n)v(n + m, l) = 0.\end{aligned} \qquad \text{(A.3)}$$

Specific values of a and b yield even more nontrivial solutions. We systematically tabulate all the cases below

1. $a = b = 0$ where, $v(m, n) = (C^{(1)}_{jp} m^2 + C^{(2)}_{jp} m)\delta_{m+n,0}$,
2. $a = 0, b = 1$ where, $v(m, n) = (C^{(3)}_{jp} m + C^{(4)}_{jp})\delta_{m+n,0}$,
3. $a = 0$, $b = -1$ where, $v(m, n) = (C^{(5)}_{jp} m^3 + C^{(6)}_{jp} m)\delta_{m+n,0}$,
4. $a = 0, b \neq 0, 1, -1$ where, $v(m, n) = C^{(7)}_{jp} m\delta_{m+n,0}$,
5. $a \neq 0$ and b is arbitrary where, $v(m, n) = C^{(8)}_{jp} \left(1 + \frac{b-1}{a} m\right) \delta_{m+n,0}$.

Here the subscript jp denotes the central extension in the $[J, P]$ commutator. Out of the eight central terms appearing in the above five scenarios, only $C^{(1)}_{jp}$, $C^{(3)}_{jp}$, $C^{(4)}_{jp}$ and $C^{(5)}_{jp}$ are the nontrivial ones. Other central terms can be absorbed by a simple redefinition of $P_m \to P_m + B\delta_{m,0}$ and subsequently choosing the constant B in an appropriate manner. Thus we drop the remaining central terms $C^{(2)}_{jp}$, $C^{(6)}_{jp}$, $C^{(7)}_{jp}$ and $C^{(8)}_{jp}$ for the remaining analysis.

The above analysis demonstrates that there may be certain values for the parameters a and b for which certain central terms will be allowed in the algebra. This opens up a host of possibilities in the central extension. We will focus on the most general extension that is admissible for arbitrary values of a and b.

The commutator $[J_m, Q^1_r]$ may admit a central term given by

$$[J_m, Q^1_r] = -\left(\frac{bm + a}{2} + r\right) Q^1_{m+r} + x_1(m, r), \qquad \text{(A.4)}$$

where $x_1(m, r)$ is an arbitrary function. The Jacobi identity between J_m, J_n and Q^1_r gives us

$$\begin{aligned}&-\left(\frac{bn + a}{2} + r\right) x_1(m, n + r)\\ &+\left(\frac{bm + a}{2} + r\right) x_1(n, m + r)\\ &-(m - n)x_1(n + m, r) = 0.\end{aligned} \qquad \text{(A.5)}$$

The x_1 central term appearing in the $[J_m, , Q^1_r]$ commutator is identically zero since a central term proportional to $\delta_{m+r,0}$ is identically zero, as m is an integer and r is a half-integer. An identically similar argument is true for $x_2(m, r)$ which is the central extension in the $[J_m, Q^2_r]$ commutator.

The central term in the $[J_m, S_n]$ commutator is denoted by $z(m, n)$, and the full commutator is written as

$$[J_m, \mathcal{S}_n] = -(a + n + (b + 1)m)\mathcal{S}_{n+m} + z(m, n). \qquad \text{(A.6)}$$

The Jacobi identity of J_m, J_n and $\mathcal{S}_l$ yields

$$\begin{aligned}&-(a + (b + 1)n + l)z(m, n + l)\\ &\quad+(a + (b + 1)m + l)z(n, m + l) - (m - n)z(n + m, l) = 0,\end{aligned} \qquad \text{(A.7)}$$

which admits the following nontrivial solutions

1. $a \neq 0$ and b are arbitrary, where $z(m,n) = C^{(0)}_{js}\left(1+\frac{b}{a}m\right)\delta_{m+n,0}$,
2. $a = 0$ and $b \neq 0,-1,-2$, where $z(m,n) = C^{(1)}_{js} m\delta_{m+n,0}$,
3. $a = 0$ and $b = -1$, where $z(m,n) = (C^{(2)}_{js} m + C^{(3)}_{js} m^2)\delta_{m+n,0}$,
4. $a = 0$ and $b = -2$, where $z(m,n) = (C^{(4)}_{js} m + C^{(5)}_{js} m^3)\delta_{m+n,0}$,
5. $a = b = 0$, where $z(m,n) = (C^{(6)}_{js} + C^{(7)}_{js} m)\delta_{m+n,0}$.

Again, performing the shift $\mathcal{S}_m \to \mathcal{S}_m + S\delta_{m,0}$ will remove some of the constants appearing above with an appropriate choice of S. A detailed analysis reveals that we can drop $C^{(0)}_{js}, C^{(1)}_{js}, C^{(2)}_{js}$ and $C^{(4)}_{js}$.

The central term $y(m,n)$

$$[J_m, \mathcal{R}_n] = -n\mathcal{R}_{n+m} + y(m,n), \tag{A.8}$$

can be determined from the Jacobi identity of J_m, J_n and $\mathcal{R}_l$, which gives

$$-ly(m,n+l)+ly(n,m+l)-(m-n)y(n+m,l) = 0. \tag{A.9}$$

This has the nontrivial solution $y(m,n) = C^{(0)}_{jr} m^2\delta_{m+n,0}$. This was discussed in earlier works [37,68].

The anticommutator $\{Q^1_r, Q^2_s\}$ may have the possible central term $f(r,s)$ and is given by

$$\{Q^1_r, Q^2_s\} = P_{r+s} + (r-s)\mathcal{S}_{r+s} + f(r,s). \tag{A.10}$$

The Jacobi identity of Q^1_r, Q^2_s and J_m gives

$$\left(\frac{bm+a}{2}+s\right) f(r,s+m) + \left(\frac{bm+a}{2}+r\right) f(r+m,s) = -v(m,r+s) - (r-s)z(m,r+s). \tag{A.11}$$

The above equation needs to be dealt with on a case-by-case basis. We tabulate all possible solutions for various values of the deformation parameter a and b.

1. When $a = b = 0$, we obtain the solution as $f(r,s) = C^{(0)}_{qq} r\delta_{r+s,0}$. Also, for consistency of the above equation, we must have $C^{(1)}_{jp} = C^{(6)}_{js} = C^{(7)}_{js} = 0$. This in turn ensures that for $a = b = 0$, $v(m,n) = z(m,n) = 0$. Note that the linear term appearing in $f(r,s)$ cannot be absorbed in the shift of generators. A possible absorbing of the central term can be performed by shifting the supertranslation generators P_n. This was already performed earlier to ensure that $C^{(2)}_{jp}$ drops out in the expression of the central charge. Thus, there is no more freedom to absorb this piece in the generators.
2. When $a = 0$ and $b = 1$, we obtain the solution $f(r,s) = C^{(1)}_{qq}\delta_{r+s,0}$. Again, consistency demands that we set $C^{(3)}_{jp} = C^{(4)}_{jp} = 0$, again ensuring that $v(m,n)$ vanishes for this case.
3. When $a = 0$ and $b = -1$, we obtain the solution $f(r,s) = 2C^{(5)}_{jp} r^2\delta_{r+s,0}$ along with the constraint that $C^{(3)}_{js} = 0$. This implies that for $a = 0$ and $b = -1$, we have $z(m,n) = 0$.
4. When $a = 0$ and $b = -2$, we obtain the solution as $f(r,s) = C^{(5)}_{js} r^3\delta_{r+s,0}$.
5. When $a = 0$ and $b = 2$, we obtain the solution $f(r,s) = C^{(2)}_{qq} r^2\delta_{r+s,0}$, while for $a = 0$ and $b \neq -2,-1,0,1,2$, $f(r,s)$ must vanish identically.
6. When $a \neq 0$ and b are arbitrary, we recover $f(r,s) = 0$ identically.

$g_1(m,n)$ is an arbitrary symmetric function which denotes the central term in the $[\mathcal{R}_m, Q^1_r]$ commutator and is given by

$$[\mathcal{R}_m, Q^1_r] = Q^1_{m+r} + g_1(m,r). \tag{A.12}$$

For $g_1(m,r) \propto \delta_{m+r,0}$, we can easily conclude that this will be zero identically, since m is an integer and r is a half-integer.

The commutator of $[P_m, \mathcal{R}_n]$ may admit a central term $h(m,n)$, which appears as follows:

$$[P_m, \mathcal{R}_n] = -2n\mathcal{S}_{m+n} + h(m,n). \tag{A.13}$$

The Jacobi identity of J_m, $\mathcal{R}_n$ and P_l leads to

$$(bm+a+l)h(l+m,n) + nh(l,m+n) = 2nz(m,n+l). \tag{A.14}$$

Clearly, the RHS of the above equation depends crucially on the values of a and b. However, dealing case-by-case, it turns out that $h(m,n) = 0$ for all values of a and b along with the constraint $C^{(5)}_{js} = 0$. This in turn implies that $z(m,n)$ vanishes for $a = 0$ and $b = -2$.

The commutator of $[\mathcal{R}_m, \mathcal{S}_n]$ can admit a central extension given by

$$[\mathcal{R}_m, \mathcal{S}_n] = k(m,n). \tag{A.15}$$

The Jacobi identity of Q^1_r, Q^2_s and $\mathcal{R}_m$ gives

$$(r-s)k(m,r+s) + f(r,m+s) - f(m+r,s) = 0. \tag{A.16}$$

Depending on the form of $f(r,s)$, we will have different solutions for $k(m,n)$. We list the possible solutions as follows:

1. For $a = b = 0$, the above equation simplifies to

$$(r - s)k(m, r + s) = C_{qq}^{(0)} m\delta_{m+r+s,0}. \tag{A.17}$$

For the above equation to be consistent, we must have $C_{qq}^{(0)} = 0$, which further implies for this case that $f(r, s) = k(m, n) = 0$.
2. For $a = 0, b = 1$, we simply recover $(r-s)k(m, r+s) = 0$, which immediately implies that $k(m, n) = 0$.
3. For $a = 0, b = -1$, we see that the equation for $k(m, n)$ is satisfied provided that $k(m, n) = 2C_{jp}^{(5)} m\delta_{m+n,0}$.
4. For $a = 0, b = -2$, consistency demands us to set $C_{js}^{(5)} = 0$, which in turn ensures that $z(m, n) = f(r, s) = k(m, n) = 0$.
5. For $a = 0, b = 2$, we see the solution of $k(m, n) = C_{qq}^{(2)} m\delta_{r+s,0}$.
6. For $a \neq 0$ and arbitrary b, we must have $k(r, s) = 0$.

A quick glance at (2.44) tells us that the central extension to the commutator $[\mathcal{R}_m, \mathcal{R}_n]$ will affect the Jacobi identity between J_m, $\mathcal{R}_n$ and $\mathcal{R}_p$. Denoting the central extension in this case as

$$[\mathcal{R}_m, \mathcal{R}_n] = w(m, n), \tag{A.18}$$

the $J_m, \mathcal{R}_n, \mathcal{R}_p$ Jacobi identity leads to

$$pw(n, p + m) = nw(p, m + n). \tag{A.19}$$

A little algebra shows that the solution to the above functional equation is given by $w(m, n) = C_{rr} m\delta_{m+n,0}$.

The central term in the $[S_m, S_n]$ commutator is denoted as $s(m, n)$ and can be explicitly written as

$$[S_m, \mathcal{S}_n] = s(m, n). \tag{A.20}$$

The Jacobi identity between P_m, $\mathcal{R}_n$ and $\mathcal{S}_l$ gives

$$ns(l, m + n) = 0, \tag{A.21}$$

which naturally implies $s(m, n) = 0$ identically. The reader can easily verify that the Jacobi identities of $(\mathcal{R}_m, \mathcal{S}_n, Q_r^i)$, $(\mathcal{R}_m, P_n, Q_r^i)$ and (Q_r^1, Q_s^2, P_m) imply $f^i(m, n) = h^i(m, n) = t_2(m, n) = 0$. The supertranslation commutator $[P_m, P_n]$ also does not admit any central term, as discussed in further detail in an earlier work [36] by one of the authors.

Finally, we consider the anticommutator $\{Q_r^1, Q_s^1\}$ which may admit a central term as

$$\{Q_r^1, Q_s^1\} = w_1(r, s). \tag{A.22}$$

It is clear that $w_1(r, s)$ must be symmetric in its arguments. The Jacobi identity between Q_r^1, Q_s^1 and R_m gives

$$w_1(r, s + m) + w_1(s, m + r) = 0. \tag{A.23}$$

For $m = 0$, we see that $w_1(m, n)$ should be antisymmetric, which is clearly a contradiction. Thus, $w_1(m, n) = 0$ identically, and a similar argument involving Q_r^2 yields $w_2(m, n) = 0$. This completes a full description of the central extension for the $W(a, b)$ algebra, which clearly depends on the values of the parameters a and b.

Appendix B: Solving ansatz for supersymmetric $W(a, b; \bar{a}, \bar{b})$

In this section, we solve for the indices and structure constants proposed in the ansatz (4.2). As mentioned earlier, the linearity of the indices implies

$$\begin{aligned} f(r, s) &= f_0 + f_1 r + f_2 s, \\ g(r, s) &= g_0 + g_1 r + g_2 s. \end{aligned} \tag{B.1}$$

where f_i and g_i are constants. Using (4.1), along with (3.6)–(3.6), we must have

$$\begin{aligned} \{Q_1^i, \bar{Q}_1^j\} &= -2(P_0 - P_3)\delta^{ij} = 4T_{1,0}\delta^{ij}, \\ \{Q_1^i, \bar{Q}_2^j\} &= 2(P_1 - iP_2)\delta^{ij} = -4iT_{1,1}\delta^{ij}, \\ \{Q_2^i, \bar{Q}_1^j\} &= 2(P_1 + iP_2)\delta^{ij} = -4iT_{0,0}\delta^{ij}, \\ \{Q_2^i, \bar{Q}_2^j\} &= -2(P_0 + P_3)\delta^{ij} = -4T_{0,1}\delta^{ij}. \end{aligned} \tag{B.2}$$

Thus, the mapping (4.3) requires the functions $f(r, s)$ and $g(r, s)$ to satisfy

$$\begin{aligned} &f\left(+\frac{1}{2}, +\frac{1}{2}\right) = 1, \quad f\left(+\frac{1}{2}, -\frac{1}{2}\right) = 1, \\ &f\left(-\frac{1}{2}, +\frac{1}{2}\right) = 0, \quad f\left(-\frac{1}{2}, -\frac{1}{2}\right) = 0, \\ &g\left(+\frac{1}{2}, +\frac{1}{2}\right) = 0, \quad g\left(+\frac{1}{2}, -\frac{1}{2}\right) = 1, \\ &g\left(-\frac{1}{2}, +\frac{1}{2}\right) = 0, \quad g\left(-\frac{1}{2}, -\frac{1}{2}\right) = 1. \end{aligned} \tag{B.3}$$

Thus, we need to solve for the six unknowns $f_i, g_i (i = 0, 1, 2)$ appearing in (B.1) from the above eight equations. There does exist a consistent solution to the above system given by

$$f_0 = g_0 = +\frac{1}{2}, \quad f_1 = -g_2 = 1, \quad f_2 = g_1 = 0. \tag{B.4}$$

Thus, we eventually recover

$$\{Q^i_r, \bar{Q}^j_s\} = \delta^{ij} T_{r+\frac{1}{2},-s+\frac{1}{2}}, \quad \{Q^i_r, Q^j_s\} = \{\bar{Q}^i_r, \bar{Q}^j_s\} = 0 \tag{B.5}$$

The exact map (4.3) can be seen to be

$$\mathcal{Q}^i_1 = 2Q^i_{+\frac{1}{2}}, \quad \bar{\mathcal{Q}}^i_1 = 2\bar{Q}^i_{+\frac{1}{2}}, \quad \mathcal{Q}^i_2 = -2iQ^i_{-\frac{1}{2}},$$
$$\bar{\mathcal{Q}}^i_2 = -2i\bar{Q}^i_{-\frac{1}{2}}. \tag{B.6}$$

Using the above map, along with the map described in Sect. 3, we obtain

$$\begin{aligned}
&[\mathcal{L}_{-1}, Q^i_{+\frac{1}{2}}] = -Q^i_{-\frac{1}{2}}, \quad [\mathcal{L}_0, Q^i_{+\frac{1}{2}}] = -\frac{1}{2}Q^i_{+\frac{1}{2}},\\
&\quad [\mathcal{L}_{+1}, Q^i_{+\frac{1}{2}}] = 0,\\
&[\mathcal{L}_{-1}, Q^i_{-\frac{1}{2}}] = 0, \quad [\mathcal{L}_0, Q^i_{-\frac{1}{2}}] = \frac{1}{2}Q^i_{-\frac{1}{2}},\\
&\quad [\mathcal{L}_{+1}, Q^i_{-\frac{1}{2}}] = Q^i_{+\frac{1}{2}}.
\end{aligned} \tag{B.7}$$

Further assuming linearity of the structure constant $\alpha(m, r)$ and $h(m, r)$ appearing in (4.2), we see that the above global sector is consistent provided one has

$$[\mathcal{L}_m, Q^i_r] = \left(\frac{m}{2} - r\right) Q^i_{m+r}. \tag{B.8}$$

An identical exercise on the “barred” sector first leads us to the relations

$$\begin{aligned}
&[\bar{\mathcal{L}}_{-1}, \bar{Q}^i_{+\frac{1}{2}}] = 0, \quad [\bar{\mathcal{L}}_0, \bar{Q}^i_{+\frac{1}{2}}] = \frac{1}{2}\bar{Q}^i_{+\frac{1}{2}},\\
&\quad [\bar{\mathcal{L}}_{+1}, \bar{Q}^i_{+\frac{1}{2}}] = \bar{Q}^i_{-\frac{1}{2}},\\
&[\bar{\mathcal{L}}_{-1}, \bar{Q}^i_{-\frac{1}{2}}] = -\bar{Q}^i_{+\frac{1}{2}}, \quad [\bar{\mathcal{L}}_0, \bar{Q}^i_{-\frac{1}{2}}] = -\frac{1}{2}\bar{Q}^i_{-\frac{1}{2}},\\
&\quad [\bar{\mathcal{L}}_{+1}, \bar{Q}^i_{-\frac{1}{2}}] = 0.
\end{aligned} \tag{B.9}$$

This shows that we must have

$$[\bar{\mathcal{L}}_m, \bar{Q}^j_s] = \left(\frac{m}{2} + s\right) \bar{Q}^j_{-m+s}. \tag{B.10}$$

Appendix C: Jacobi identities for supersymmetric $W(a, b; \bar{a}, \bar{b})$

1. **Jacobi identity for $\mathcal{L}_m, Q^i_r, R_n$ and $\mathcal{L}_m, \bar{Q}^j_s, R_n$**
   We start with the Jacobi identity for $\mathcal{L}_m, Q^i_r$ and R_n, which is given by

$$[[\mathcal{L}_m, Q^i_r], R_n] + [[Q^i_r, R_n], \mathcal{L}_m] + [[R_n, \mathcal{L}_m], Q^i_r] = 0. \tag{C.1}$$

Using (4.5) and (5.6), we can simplify the above equation to obtain

$$[[R_n, \mathcal{L}_m], Q^i_r] + [\alpha(m, r)\beta(m + r, n)$$
$$-\beta(r, n)\alpha(m, cn + r)]Q^i_{cn+m+r} = 0. \tag{C.2}$$

Clearly, looking at the above, on very general grounds, one can schematically write

$$[R_n, \mathcal{L}_m] = w_1(n, m)\mathcal{L}_{t_1(n,m)}$$
$$+h_1(n, m)R_{u_1(n,m)} + \bar{h}_1(n, m)\bar{R}_{v_1(n,m)}. \tag{C.3}$$

The Jacobi identity for $\mathcal{L}_m, \bar{Q}^j_s$ and R_n along with (4.5) and (5.3) leads to

$$[[R_n, \mathcal{L}_m], \bar{Q}^j_s] = 0. \tag{C.4}$$

Now, if we consider the $[R_n, \mathcal{L}_m]$ to be of the form as (C.3), we easily see that the part $[R_{u(n,m)}, \bar{Q}^j_s]$ and $[\bar{R}_{v(n,m)}, \bar{Q}^j_s]$ will be generically nonzero *individually* for arbitrary values of n, m and s; however, a linear combination with specific forms of h_1 and $\bar{h}_1$ might presumably ensure that the expression vanishes. Note that (C.3) has certain features which put it in stark contrast to its three-dimensional $W(a, b)$ analog. Firstly, the first term appearing on the RHS in the above equation has no analog for the $W(a, b)$ algebra as stated explicitly in (2.43). We can also see that commutators similar to the $W(a, b)$ algebra, i.e.,

$$[R_n, \mathcal{L}_m] \sim R_{u(n,m)} \text{ and } [\bar{R}_n, \mathcal{L}_m] \sim \bar{R}_{v(n,m)} \tag{C.5}$$

are clearly inconsistent with (C.4) for arbitrary values of the indices m and n.

2. **Jacobi identity for $\bar{\mathcal{L}}_m, \bar{Q}^j_s, R_n$ and $\bar{\mathcal{L}}_m, Q^i_r$ and R_n**
   The Jacobi identity of $\bar{\mathcal{L}}_m, \bar{Q}^j_s$ and R_n leads us to

$$\begin{aligned}
&[[R_n, \bar{\mathcal{L}}_m], \bar{Q}^j_s]\\
&\quad + \left(\bar{\beta}(s, n)\left(\bar{a} + \bar{b}m - \bar{c}n - s + \frac{1}{2}\right)\right.\\
&\quad \left. -\bar{\beta}(-m + s, n)\left(\bar{a} + \bar{b}m - s + \frac{1}{2}\right)\right) \bar{Q}^j_{\bar{c}n+s-m} = 0.
\end{aligned} \tag{C.6}$$

The above, along with the Jacobi identity for $\bar{\mathcal{L}}_m, Q^i_r$, which simplifies to

$$[[R_n, \bar{\mathcal{L}}_m], Q^i_r] = 0, \tag{C.7}$$

suggests a relation of the form

$$[R_n, \bar{\mathcal{L}}_m] = w_2(n, m)\bar{\mathcal{L}}_{t_2(n,m)}$$

$$+h_2(n,m)R_{u_2(n,m)}+\bar{h}_2(n,m)\bar{R}_{v_2(n,m)}. \qquad \text{(C.8)}$$

3. **Jacobi identity for $\mathcal{L}_m, Q^i_r, \bar{R}_n$ and $\mathcal{L}_m, \bar{Q}^j_s, \bar{R}_n$**
The Jacobi identity for $\mathcal{L}_m$, Q^i_r and $\bar{R}_n$ leads to

$$\begin{aligned}&[[\bar{R}_m,\mathcal{L}_m],Q^i_r]\\&+\left(\kappa(r,n)\left(a+bm+kn+r+\frac{1}{2}\right)\kappa(m+r,n)\right.\\&\left.-\left(a+bm+r+\frac{1}{2}\right)\right)Q^i_{kn+m+r}=0,\end{aligned} \qquad \text{(C.9)}$$

while the Jacobi identity for $\mathcal{L}_m$, $\bar{Q}^j_s$ and $\bar{R}_n$ gives

$$[[\bar{R}_n,\mathcal{L}_m],\bar{Q}^j_s]=0. \qquad \text{(C.10)}$$

This leads us to propose

$$\begin{aligned}&[\bar{R}_n,\mathcal{L}_m]=w_3(n,m)\mathcal{L}_{t_3(n,m)}+h_3(n,m)R_{u_3(n,m)}\\&+\bar{h}_3(n,m)\bar{R}_{v_3(n,m)}.\end{aligned} \qquad \text{(C.11)}$$

4. **Jacobi identity for $\bar{\mathcal{L}}_m, Q^i_r, \bar{R}_n$ and $\bar{\mathcal{L}}_m, \bar{Q}^j_s, \bar{R}_n$**
The Jacobi identity for $\bar{\mathcal{L}}_m$, Q^i_r, $\bar{R}_n$ simplifies to

$$[[\bar{R}_n,\bar{\mathcal{L}}_m],Q^i_r]=0 \qquad \text{(C.12)}$$

and the $\bar{\mathcal{L}}_m$, $\bar{Q}^j_s$, $\bar{R}_n$ Jacobi identity leads to

$$\begin{aligned}&[[\bar{R}_n,\bar{\mathcal{L}}_m],\bar{Q}^j_s]\\&+\left(\bar{\kappa}(s,n)\left(\bar{a}+\bar{b}m-\bar{k}n-s+\frac{1}{2}\right)\right.\\&\left.-\bar{\kappa}(-m+s,n)\left(\bar{a}+\bar{b}m-s+\frac{1}{2}\right)\right)\bar{Q}^j_{\bar{k}n+s-m}=0.\end{aligned} \qquad \text{(C.13)}$$

The above two equations lead us to the ansatz

$$\begin{aligned}&[\bar{R}_n,\bar{\mathcal{L}}_m]=w_4(n,m)\bar{\mathcal{L}}_{t_4(n,m)}\\&+h_4(n,m)R_{u_4(n,m)}+\bar{h}_4(n,m)\bar{R}_{v_4(n,m)}.\end{aligned} \qquad \text{(C.14)}$$

5. **Jacobi identity for $\mathcal{L}_m, T_{p,q}, R_n$ and $\bar{\mathcal{L}}_m, T_{p,q}, R_n$**
The Jacobi identity for the operators $\mathcal{L}_m$, $T_{p,q}$ and R_n is given by

$$[[\mathcal{L}_m,T_{p,q}],R_n]+[[T_{p,q},R_n],\mathcal{L}_m]+[[R_n,\mathcal{L}_m],T_{p,q}]=0, \qquad \text{(C.15)}$$

which upon using (4.5) and (5.6) leads us to the relation

$$\begin{aligned}&[[R_n,\mathcal{L}_m],T_{p,q}]\\&+\left((a+bm+cn+p)\beta\left(p-\frac{1}{2},n\right)\right.\\&\left.-(a+bm+p)\beta\left(m+p-\frac{1}{2},n\right)\right)T_{cn+m+p,q}=0,\end{aligned} \qquad \text{(C.16)}$$

which is consistent with the ansatz (C.3). Further, the Jacobi identity for $\bar{\mathcal{L}}_m$, $T_{p,q}$, R_n gives

$$\begin{aligned}&[[R_n,\bar{\mathcal{L}}_m],T_{p,q}]\\&+\left(\bar{\beta}\left(-q+\frac{1}{2},n\right)(\bar{a}+\bar{b}m-\bar{c}n+q)\right.\\&\left.-\bar{\beta}\left(-m-q+\frac{1}{2},n\right)(\bar{a}+\bar{b}m+q)\right)T_{p,-\bar{c}n+m+q}=0,\end{aligned} \qquad \text{(C.17)}$$

which is consistent with the ansatz (C.8).

6. **Jacobi identity for $\mathcal{L}_m, T_{p,q}, \bar{R}_n$ and $\bar{\mathcal{L}}_m, T_{p,q}, \bar{R}_n$**
The Jacobi identity for $\mathcal{L}_m$, $T_{p,q}$, $\bar{R}_n$ leads to

$$\begin{aligned}&[[\bar{R}_n,\mathcal{L}_m],T_{p,q}]\\&+\left(\kappa\left(p-\frac{1}{2},n\right)(a+bm+kn+p)\right.\\&\left.-\kappa\left(m+p-\frac{1}{2},n\right)(a+bm+p)\right)T_{kn+m+p,q}=0,\end{aligned} \qquad \text{(C.18)}$$

while for the other tuple, namely $\bar{\mathcal{L}}_m$, $T_{p,q}$, $\bar{R}_n$, we have

$$\begin{aligned}&[[\bar{R}_n,\bar{\mathcal{L}}_m],T_{p,q}]\\&+\left(\bar{\kappa}\left(-q+\frac{1}{2},n\right)(\bar{a}+\bar{b}m-\bar{k}n+q)\right.\\&\left.-\bar{\kappa}\left(-m-q+\frac{1}{2},n\right)(\bar{a}+\bar{b}m+q)\right)T_{p,-\bar{k}n+m+q}=0.\end{aligned} \qquad \text{(C.19)}$$

One can easily see that the above two equations are indeed consistent with the proposed ansatze (C.8) and (C.14), respectively.

Armed with a series of ansatze and a number of Jacobi identities, we enumerate below the index structure and which equations they follow from.

- Plugging ansatz (C.3) in (C.2), we get

$$-w_1(n,m)\alpha(t_1,r)Q^i_{t_1+r}-h_1(n,m)\beta(r,u_1)Q^i_{cu_1+r}$$

$$-\bar{h}_1(n,m)\kappa(r,v_1)Q^i_{kv_1+r} \\ +[\alpha(m,r)\beta(m+r,n)-\beta(r,n)\alpha(m,cn+r)]Q^i_{cn+m+r}=0. \tag{C.20}$$

The above equation must be satisfied for arbitrary permissible values of m, n and r. Also, by definition, β and κ cannot be identically zero. This implies that

$$u_1(n,m)=n+\frac{m}{c} \\ v_1(n,m)=\frac{c}{k}n+\frac{m}{k}.$$

Since, clearly, the indices must be integers, we see that both k and c must divide every integer, which naturally implies that both can take values ± 1. Using (C.4), we further get

$$\frac{\bar{c}}{c}=\frac{\bar{k}}{k}=\xi \text{ (say)}, \tag{C.21}$$

where ξ is some real number. Further looking at (C.20), we can have w_1 be either identically zero or $t_1=cn+m$. Further, (C.16) gives

$$\begin{aligned}
&(-w_1(n,m)(a+bcn+bm+p)\\
&\quad-h_1(n,m)\beta\left(p-\frac{1}{2},n+\frac{m}{c}\right)\\
&\quad-\bar{h}_1(n,m)\kappa\left(p-\frac{1}{2},\frac{c}{k}n+\frac{m}{k}\right)\\
&\quad+(a+bm+cn+p)\beta\left(p-\frac{1}{2},n\right)\\
&\quad-(a+bm+p)\beta\left(m+p-\frac{1}{2},n\right)\Big)T_{cn+m+p,q}\\
&\quad+\left(-h_1(n,m)\bar{\beta}\left(-q+\frac{1}{2},n+\frac{m}{c}\right)-\bar{h}_1(n,m)\bar{\kappa}\right.\\
&\quad\left.\times\left(-q+\frac{1}{2},\frac{c}{k}n+\frac{m}{k}\right)\right)T_{p,-\bar{c}n-\xi m+q}=0.
\end{aligned} \tag{C.22}$$

In the above expression, each of the coefficients of the $T_{cn+m+p,q}$ and $T_{p,-\bar{c}n-\xi m+q}$ has to vanish individually.

- Ansatz (C.8) along with (C.6) implies $t_2(n,m)=-\bar{c}n+m$ provided w_2 does not vanish identically. Along similar arguments as before, we also conclude that

$$u_2=n-\frac{m}{\bar{c}} \\ v_2=\frac{\bar{c}}{\bar{k}}n-\frac{m}{\bar{k}}.$$

Again, since both u_2 and v_2 must be integers for all values of m and n, this leads us to conclude that $\bar{c}$ and $\bar{k}$ can only take values ± 1. Thus, ξ appearing in (C.21) can only be ± 1. Now, since $c^2=k^2=\bar{c}^2=\bar{k}^2=\xi^2=1$, we can rewrite

$$u_1=n+mc,\quad v_1=kcn+km,\quad u_2=n-\xi cm,\quad v_2=kcn-\xi km. \tag{C.23}$$

Further, using (C.17), we get

$$\begin{aligned}
&-w_2(n,m)(\bar{a}-\bar{b}\bar{c}n+\bar{b}m+q)-h_2(n,m)\bar{\beta}\\
&\quad\times\left(-q+\frac{1}{2},n-\xi\frac{m}{c}\right)\\
&-\bar{h}_2(n,m)\bar{\kappa}\left(-q+\frac{1}{2},\frac{c}{k}n-\xi\frac{m}{k}\right)\\
&+\bar{\beta}\left(-q+\frac{1}{2},n\right)(\bar{a}+\bar{b}m-\bar{c}n+q)\\
&\quad-\bar{\beta}\left(-m-q+\frac{1}{2},n\right)(\bar{a}+\bar{b}m+q)=0,
\end{aligned} \tag{C.24}$$

and

$$\begin{aligned}
&h_2(n,m)\beta\left(p-\frac{1}{2},n-\xi\frac{m}{c}\right)+\bar{h}_2(n,m)\kappa\\
&\times\left(p-\frac{1}{2},\frac{c}{k}n-\xi\frac{m}{k}\right)=0.
\end{aligned} \tag{C.25}$$

- Ansatz (C.11) along with (C.9) tells us that we must have

$$u_3=\frac{k}{c}n+\frac{m}{c}\equiv kcn+cm \\ v_3=n+\frac{m}{k}\equiv n+km,$$

while w_3 identically vanishes or $t_3=kn+m$. The Jacobi identity (C.18) implies

$$\begin{aligned}
&-w_3(n,m)(a+bkn+bm+p)-h_3(n,m)\beta\\
&\quad\times\left(p-\frac{1}{2},kcn+cm\right)\\
&\quad-\bar{h}_3(n,m)\kappa\left(p-\frac{1}{2},n+km\right)\\
&\quad+\kappa\left(p-\frac{1}{2},n\right)(a+bm+kn+p)\\
&\quad-\kappa\left(m+p-\frac{1}{2},n\right)(a+bm+p)=0,
\end{aligned} \tag{C.26}$$

and

$$h_3(n,m)\bar\beta\left(-q+\frac{1}{2},kcn+cm\right)+\bar h_3(n,m)\bar\kappa \times\left(-q+\frac{1}{2},n+km\right)=0. \tag{C.27}$$

- Ansatz (C.14) along with (C.13) tells us that

$$u_4=\frac{k}{c}n-\xi\frac{m}{c}\equiv kcn-\xi cm$$
$$v_4=n-\xi\frac{m}{k}\equiv n-\xi km,$$

along with $t_4=-\bar k n+m$ provided if w_4 is nonzero. The Jacobi identity (C.19) gives

$$\begin{aligned}&-w_4(n,m)(\bar a+\bar b m-\xi k\bar b n+q)-h_4(n,m)\bar\beta\\&\quad\times\left(-q+\frac{1}{2},kcn-\xi cm\right)\\&\quad-\bar h_4(n,m)\bar\kappa\left(-q+\frac{1}{2},n-\xi km\right)\\&\quad+\bar\kappa\left(-q+\frac{1}{2},n\right)(\bar a+\bar b m-\bar k n+q)\\&\quad-\bar\kappa\left(-m-q+\frac{1}{2},n\right)(\bar a+\bar b m+q)=0,\end{aligned} \tag{C.28}$$

and

$$h_4(n,m)\beta\left(p-\frac{1}{2},kcn-\xi cm\right)+\bar h_4(n,m)\kappa \left(p-\frac{1}{2},n-\xi km\right)=0\,. \tag{C.29}$$

To finalize the form of w and h we need to consider Jacobi identities involving two R-supercurrent generators and one superrotation generator.

1. **Jacobi identity for R_m, R_n and $\mathcal{L}_p$ leads to**

$$\begin{aligned}&w_1(n,p)w_1(m,cn+p)=w_1(m,p)w_1(n,cm+p),\\&h_1(m,cn+p)w_1(n,p)=h_1(n,cm+p)w_1(m,p),\\&\bar h_1(m,cn+p)w_1(n,p)=\bar h_1(n,cm+p)w_1(m,p).\end{aligned} \tag{C.30}$$

2. **Jacobi identity for R_m, R_n and $\bar{\mathcal{L}}_p$ leads to**

$$\begin{aligned}&w_2(n,p)w_2(m,-\xi cn+p)=w_2(m,p)w_2(n,-\xi cm+p),\\&h_2(m,-\xi cn+p)w_2(n,p)=h_2(n,-\xi cm+p)w_2(m,p),\\&\bar h_2(m,-\xi cn+p)w_2(n,p)=\bar h_2(n,-\xi cm+p)w_2(m,p).\end{aligned} \tag{C.31}$$

3. **Jacobi identity for $\bar R_m$, $\bar R_n$ and $\mathcal{L}_p$ leads to**

$$\begin{aligned}&w_3(n,p)w_3(m,kn+p)=w_3(m,p)w_3(n,km+p),\\&h_3(m,kn+p)w_3(n,p)=h_3(n,km+p)w_3(m,p),\\&\bar h_3(m,kn+p)w_3(n,p)=\bar h_3(n,km+p)w_3(m,p).\end{aligned} \tag{C.32}$$

4. **Jacobi identity for $\bar R_m$, $\bar R_n$ and $\bar{\mathcal{L}}_p$ leads to**

$$\begin{aligned}&w_4(n,p)w_4(m,-\xi kn+p)=w_4(m,p)w_4(n,-\xi km+p),\\&h_4(m,-\xi kn+p)w_4(n,p)=h_4(n,-\xi km+p)w_4(m,p),\\&\bar h_4(m,-\xi kn+p)w_4(n,p)=\bar h_4(n,-\xi km+p)w_4(m,p).\end{aligned} \tag{C.33}$$

5. **Jacobi identity for R_m, $\bar R_n$ and $\mathcal{L}_p$ leads to**

$$\begin{aligned}&w_3(n,p)w_1(m,kn+p)=w_1(m,p)w_3(n,cm+p),\\&h_1(m,kn+p)w_3(n,p)=h_3(n,cm+p)w_1(m,p),\\&\bar h_1(m,kn+p)w_3(n,p)=\bar h_3(n,cm+p)w_1(m,p).\end{aligned} \tag{C.34}$$

6. **Jacobi identity for R_m, $\bar R_n$ and $\bar{\mathcal{L}}_p$ leads to**

$$\begin{aligned}&w_4(n,p)w_2(m,-\xi kn+p)=w_2(m,p)w_4(n,-\xi cm+p),\\&h_2(m,-\xi kn+p)w_4(n,p)=h_4(n,-\xi cm+p)w_2(m,p),\\&\bar h_2(m,-\xi kn+p)w_4(n,p)=\bar h_4(n,-\xi cm+p)w_2(m,p).\end{aligned} \tag{C.35}$$

Finally, we have another family of Jacobi identities involving two superrotation generators and one R-supercurrent generator. The Jacobi identities for $\mathcal{L}_m$, $\bar{\mathcal{L}}_n$, R_n and $\mathcal{L}_m$, $\bar{\mathcal{L}}_n$, $\bar R_n$ are trivially satisfied. We list the equations that follow from the nontrivial Jacobi identities systematically.

1. **Jacobi identity for $\mathcal{L}_m$, $\mathcal{L}_n$ and R_p implies**

$$\begin{aligned}&(m-n)w_1(p,m+n)+w_1(p,m)(n-cp-m)\\&\quad-w_1(p,n)(m-cp-n)\\&\quad+h_1(p,n)w_1(p+cn,m)\\&\quad-h_1(p,m)w_1(p+cm,n)\\&\quad+\bar h_1(p,n)w_3(kcp+kn,m)\\&\quad-\bar h_1(p,m)w_3(kcp+km,n)=0,\end{aligned} \tag{C.36}$$

$$\begin{aligned}&(m-n)h_1(p,m+n)+h_1(p,n)h_1(p+cn,m)\\&\quad-h_1(p,m)h_1(p+cm,n)\\&=\bar h_1(p,m)h_3(kcp+km,n)\end{aligned}$$

$$-\bar{h}_1(p,n)h_3(kcp+kn,m), \quad (C.37)$$

$$(m-n)\bar{h}_1(p,m+n)+h_1(p,n)\bar{h}_1(p+cn,m)$$
$$-h_1(p,m)\bar{h}_1(p+cm,n)=$$
$$\times\bar{h}_1(p,m)\bar{h}_3(kcp+km,n)$$
$$-\bar{h}_1(p,n)\bar{h}_3(kcp+kn,m). \quad (C.38)$$

2. **Jacobi identity for $\mathcal{L}_m$, $\mathcal{L}_n$ and $\bar{R}_p$ implies**

$$(m-n)w_3(p,m+n)+w_3(p,m)(n-kp-m)$$
$$-w_3(p,n)(m-kp-n)$$
$$+h_3(p,n)w_1(kcp+cn,m)$$
$$-h_3(p,m)w_1(kcp+cm,n)$$
$$+\bar{h}_3(p,n)w_3(p+kn,m)$$
$$-\bar{h}_3(p,m)w_3(p+km,n)=0, \quad (C.39)$$

$$(m-n)h_3(p,m+n)+h_3(p,n)h_1(kcp+cn,m)$$
$$-h_3(p,m)h_1(kcp+cm,n)=$$
$$\bar{h}_3(p,m)h_3(p+km,n)$$
$$-\bar{h}_3(p,n)h_3(p+kn,m), \quad (C.40)$$

$$(m-n)\bar{h}_3(p,m+n)+h_3(p,n)\bar{h}_1(kcp+cn,m)$$
$$-h_3(p,m)\bar{h}_1(kcp+cm,n)=$$
$$\bar{h}_3(p,m)\bar{h}_3(p+km,n)$$
$$-\bar{h}_3(p,n)\bar{h}_3(p+kn,m). \quad (C.41)$$

3. **Jacobi identity for $\bar{\mathcal{L}}_m$, $\bar{\mathcal{L}}_n$ and R_p implies**

$$(m-n)w_2(p,m+n)+w_2(p,m)(n+\xi cp-m)$$
$$-w_2(p,n)(m+\xi cp-n)$$
$$+h_2(p,n)w_2(p-\xi cn,m)$$
$$-h_2(p,m)w_2(p-\xi cm,n)$$
$$+\bar{h}_2(p,n)w_4(kcp-\xi kn,m)$$
$$-\bar{h}_2(p,m)w_4(kcp-\xi km,n)=0, \quad (C.42)$$

$$(m-n)h_2(p,m+n)+h_2(p,n)h_2(p-\xi cn,m)$$
$$-h_2(p,m)h_2(p-\xi cm,n)$$
$$=\bar{h}_2(p,m)h_4(kcp-\xi km,n)$$
$$-\bar{h}_2(p,n)h_4(kcp-\xi kn,m), \quad (C.43)$$

$$\times(m-n)\bar{h}_2(p,m+n)+h_2(p,n)\bar{h}_2(p-\xi cn,m)$$
$$-h_2(p,m)\bar{h}_2(p-\xi cm,n)$$
$$=\bar{h}_2(p,m)\bar{h}_4(kcp-\xi km,n)$$
$$-\bar{h}_2(p,n)\bar{h}_4(kcp-\xi kn,m). \quad (C.44)$$

3. **Jacobi identity for $\bar{\mathcal{L}}_m$, $\bar{\mathcal{L}}_n$ and $\bar{R}_p$ implies**

$$(m-n)w_4(p,m+n)+w_4(p,m)(n-\xi kp-m)$$
$$-w_4(p,n)(m-\xi kp-n)$$
$$+h_4(p,n)w_2(kcp-\xi cn,m)$$
$$-h_4(p,m)w_2(kcp-\xi cm,n)$$
$$+\bar{h}_4(p,n)w_4(p-\xi kn,m)$$
$$-\bar{h}_4(p,m)w_4(p-\xi km,n)=0, \quad (C.45)$$

$$(m-n)h_4(p,m+n)+h_4(p,n)h_2(kcp-\xi cn,m)$$
$$-h_4(p,m)h_2(kcp-\xi cm,n)$$
$$=\bar{h}_4(p,m)h_4(p-\xi km,n)$$
$$-\bar{h}_4(p,n)h_4(p-\xi kn,m), \quad (C.46)$$

$$(m-n)\bar{h}_4(p,m+n)+h_4(p,n)\bar{h}_2(kcp-\xi cn,m)$$
$$-h_4(p,m)\bar{h}_2(kcp-\xi cm,n)$$
$$=\bar{h}_4(p,m)\bar{h}_4(p-\xi km,n)$$
$$-\bar{h}_4(p,n)\bar{h}_4(p-\xi kn,m). \quad (C.47)$$

Appendix D: Algebra with non-linear structure constants

We can relax the criteria of the linearity of structure constants and assume further that they can be at best quadratic in the arguments. Continuing to refer to the structure constants w_i, h_i and $\bar{h}_i$ as μ_i, we write down an ansatz of the form

$$\mu_i(n,p)=\omega_{i0}+\omega_{i1}n+\omega_{i2}p+\omega_{i3}np+\omega_{i4}n^2+\omega_{i5}p^2. \quad (D.1)$$

Since $\mu_i(0,p)$ must vanish for $p=0,\pm1$, we can write

$$\mu_i(n,p)=n(\omega_{i1}+\omega_{i3}p+\omega_{i4}n). \quad (D.2)$$

Demanding (5.14), we see that β and κ (and similarly $\bar{\beta}$ and $\bar{\kappa}$) must take the form

$$\beta(r,n)=\beta_0+\beta_2n+\beta_3r^2+\beta_4n^2+\beta_5rn, \quad (D.3)$$
$$\kappa(r,n)=\kappa_0+\kappa_2n+\kappa_3r^2+\kappa_4n^2+\kappa_5rn \quad (D.4)$$

Unlike the linear case, (5.8) and (5.9) are not trivially satisfied now. Focusing specifically on (5.9), we recover the following equations

$$\begin{aligned}
&\beta_0\kappa_3=\beta_3\kappa_0=\beta_2\kappa_3=\beta_3\kappa_2=\beta_3\kappa_4=\beta_4\kappa_3=0,\\
&\beta_5\kappa_2=\beta_2\kappa_5=\beta_5\kappa_4=\beta_4\kappa_5=\beta_3\kappa_3=\beta_5\kappa_5=0,\\
&k\beta_5\kappa_3=c\beta_3\kappa_5,\ k\beta_5\kappa_0=c\beta_0\kappa_5,\\
&k\beta_3\kappa_3+2\beta_3\kappa_5=c\beta_3\kappa_3+2\beta_5\kappa_3=c\beta_5\kappa_3\\
&\quad+2\beta_4\kappa_3=k\beta_3\kappa_5+2\beta_3\kappa_4=0.
\end{aligned} \quad (D.5)$$

We will obtain an analogous set of equations following from (5.9). The above set of equations implies

$$\beta_3=\kappa_3=\beta_5=\kappa_5=0, \quad (D.6)$$

thus making the structure constants $\beta(r,n)$ (and $\bar{\beta}(r,n)$) and $\kappa(r,n)$ (and $\bar{\kappa}(r,n)$) independent of r. As before, (5.14) implies

$$\beta_0 = -\bar{\beta}_0 \quad \text{(and similarly } \kappa_0 = -\bar{\kappa}_0) \tag{D.7}$$

Now, armed with the above results from (C.22), we again recover equations very close to those of (5.18) and (5.19), which again leads us to conclude that $w_i = 0$ identically. Thus, in this case, the two equations following from (C.22) simplify to

$$\begin{aligned} &-h_1(n,m)\beta(n+cm)\\ &-\bar{h}_1(n,m)\kappa(kcn+km)+cn\beta(n)=0, \end{aligned} \tag{D.8}$$

$$\begin{aligned} &-h_1(n,m)\bar{\beta}(n+cm)\\ &-\bar{h}_1(n,m)\bar{\kappa}(kcn+km)=0. \end{aligned} \tag{D.9}$$

Plugging in the forms of the ansatze for $h_1(n,m)$, $\bar{h}_1(n,m)$, $\beta(n)$ and $\kappa(n)$ in (D.8) and equating the various coefficients of the monomials in m and n gives

$$c\beta_0 - \omega_1\beta_0 - \bar{\omega}_1\kappa_0 = 0, \tag{D.10}$$
$$\omega_3\beta_0 + c\omega_1\beta_2 + \bar{\omega}_3\kappa_0 + k\bar{\omega}_1\kappa_2 = 0, \tag{D.11}$$
$$\omega_4\beta_0 - c\beta_2 + \omega_1\beta_2 + \bar{\omega}_4\kappa_0 + ck\bar{\omega}_1\kappa_2 = 0, \tag{D.12}$$
$$\omega_4\beta_2 - c\beta_4 + \omega_1\beta_4 + ck\bar{\omega}_4\kappa_2 + \bar{\omega}_1\kappa_4 = 0, \tag{D.13}$$
$$c\omega_3\beta_2 + \omega_1\beta_4 + k\bar{\omega}_3\kappa_2 + \bar{\omega}_1\kappa_4 = 0, \tag{D.14}$$
$$\begin{aligned} &\omega_3\beta_2 + c\omega_4\beta_2 + 2c\omega_1\beta_4 + ck\bar{\omega}_3\kappa_2\\ &\quad + k\bar{\omega}_4\kappa_2 + 2c\bar{\omega}_1\kappa_4 = 0, \end{aligned} \tag{D.15}$$
$$\omega_4\beta_4 + \bar{\omega}_4\kappa_4 = 0, \tag{D.16}$$
$$\omega_3\beta_4 + \bar{\omega}_3\kappa_4 = 0, \tag{D.17}$$
$$2c\omega_3\beta_4 + \omega_4\beta_4 + 2c\bar{\omega}_3\kappa_4 + \bar{\omega}_4\kappa_4 = 0, \tag{D.18}$$
$$\omega_3\beta_4 + 2c\omega_4\beta_4 + \bar{\omega}_3\kappa_4 + 2c\bar{\omega}_4\kappa_4 = 0\,. \tag{D.19}$$

The same exercise with (D.9) gives the following equations

$$\omega_1\beta_0 + \bar{\omega}_1\kappa_0 = 0, \tag{D.20}$$
$$\omega_4\beta_0 - \omega_1\bar{\beta}_1 + \bar{\omega}_4\kappa_0 - ck\bar{\omega}_1\bar{\kappa}_2 = 0, \tag{D.21}$$
$$\omega_3\beta_0 - c\omega_1\bar{\beta}_1 + \bar{\omega}_3\kappa_0 - k\bar{\omega}_1\bar{\kappa}_2 = 0, \tag{D.22}$$
$$\omega_1\bar{\beta}_4 + \omega_4\bar{\beta}_1 + ck\bar{\omega}_4\bar{\kappa}_2 + \bar{\omega}_1\bar{\kappa}_4 = 0, \tag{D.23}$$
$$\omega_1\bar{\beta}_4 + c\omega_3\bar{\beta}_1 + k\bar{\omega}_3\bar{\kappa}_2 + \bar{\omega}_1\bar{\kappa}_4 = 0, \tag{D.24}$$
$$\begin{aligned} &2c\omega_1\bar{\beta}_4 + \omega_3\bar{\beta}_1 + c\omega_4\bar{\beta}_1\\ &\quad + ck\bar{\omega}_3\bar{\kappa}_2 + k\bar{\omega}_4\bar{\kappa}_2 + 2c\bar{\omega}_1\bar{\kappa}_4 = 0, \end{aligned} \tag{D.25}$$
$$\omega_4\bar{\beta}_4 + \bar{\omega}_4\bar{\kappa}_4 = 0, \tag{D.26}$$
$$\omega_3\bar{\beta}_4 + \bar{\omega}_3\bar{\kappa}_4 = 0, \tag{D.27}$$
$$2c\omega_3\bar{\beta}_4 + \omega_4\bar{\beta}_4 + 2c\bar{\omega}_3\bar{\kappa}_4 + \bar{\omega}_4\bar{\kappa}_4 = 0, \tag{D.28}$$
$$\omega_3\bar{\beta}_4 + 2c\omega_4\bar{\beta}_4 + \bar{\omega}_3\bar{\kappa}_4 + 2c\bar{\omega}_4\bar{\kappa}_4 = 0. \tag{D.29}$$

The above system of equations is highly nontrivial to solve in general. But from (D.10) and (D.20) we easily see that

$$\beta_0 = 0, \tag{D.30}$$

which implies that $\beta(0) = 0$, and similarly we find $\bar{\beta}(0) = \kappa(0) = \bar{\kappa}(0) = 0$. This is in contradiction with the constraint from the global subalgebra (5.14). Thus, akin to the case of linear structure constants, even for quadratic structure constants, an extension by $U(1)_V \times U(1)_A$ realized as infinitely many generators R_n and $\bar{R}_n$ (R-symmetry generators) of $W(a,b,\bar{a},\bar{b})$ algebra turns out to be *impossible*.

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