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CPT-even electrodynamics in a multidimensional torus: Casimir effect at finite temperature

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Abstract The energy–momentum tensor for the electromagnetism with Lorentz breaking even term of the Standard Model Extended (SME) photon sector confined in a hyper torus is determined. A generalized partition function method is used, following in parallel the thermofield dynamics formalism written in N-dimensional toroidal manifold. After considering general aspects of the SME photon sector in a toroidal manifold, the influence of the isotropic CPT-even electromagnetic sector of the SME is analysed. The approach is then applied to the Casimir effect at finite temperature, corresponding to a topology $(S^1)^r \times R^{N-r}$, where N is the dimension of the Minkowski space-time, and r is the number of compactified dimensions.influence of the isotropic CPT-even electromagnetic sector of the SME is analysed.

1 Introduction

The standard model of particle physics is so far the best theory describing in a unified way the fundamental interactions of nature. However, it fails to address the gravitational interaction. Despite the experimental success, the last being the Higgs boson in 2013, a more complete theory is necessary to predict, for instance, the Higgs mass. Beyond that, remains to be properly explained the origin of the electron's electric dipole moment, d_e, and its experimental upper bounds [1]. Theories beyond the standard model predict a small, but potentially measurable $d_e \leq 10^{-29}\, e \cdot cm$ [2], which presents an asymmetric charge distribution along the spin axis. By these motivations it is mandatory to investigate the physics beyond the standard model.

In 1989, Kostelecký and Samuel [3] proposed a spontaneous Lorentz symmetry violation through nonzero vacuum expectation values of nonscalar fields (vacuum expectation values of tensor fields) based on a string field theory. Taking this violation proposal into a field theory context, Kostelecký and colleagues investigate a possible extension of the standard model [4]. This proposal has been known as Standard Model Extension (SME) [5,6].

The presence of terms that violate the Lorentz symmetry imposes anisotropies in the space-time [7,8]. Relativistic quantum effects [9–15] with Lorentz symmetry breaking and non minimal coupling [16–28] have opened the possibility of investigating implications in quantum mechanics, that this background violating Lorentz invariance can promote.

The first studies addressing the consequences of violating Lorentz symmetries came about in the early 1990s. This subject matter had as its starting point the work carried out by Carrol, Field and Jackiw [29], proposing a modified Maxwell electrodynamics. This modification introduced, in the Lagrangian density, a Chern–Simons-like term, $\varepsilon_{\mu\nu\kappa\lambda} V^{\mu} A^{\nu} F^{\kappa\lambda}$, in (1+3) spacetime dimensions. This stands for a coupling of the gauge field with a field violating the Lorentz symmetry (V^{μ}). Still in the 1990s, Colladay and Kostelecky [5,6,30], developed a theoretical model that would correspond to an extension of the well-known SM of fundamental interactions.

In recent years, efforts have been made as an attempt to test CPT symmetry, as, for example, mechanisms to test Lorentz and CPT symmetries using antimatter experiments [31]. In particular, such symmetries are explored within the scope of chiral perturbation theory [32]. Furthermore, there are tests involving neutrinos from the gamma ray bursts [33–36] and a phenomenological discussion using neutral mesons [37]. Following the same perspective, studies involving the break-

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ing of the CPT symmetry from gravitational effects in a system of self-interacting particles are presented in [38]. The assessment of the values for the Lorentz and CPT coefficients has also been discussed [39–41]. In this realm, the important effect of compactification, including the Casimir effect, in breaking of the CPT symmetry has been only partially addressed in the literature [42].

The Casimir effect is a remarkable manifestations of vacuum fluctuations. For the electromagnetic field, this effect emerges due to the difference between the energy density of the modes outside the plates and those inside the plates, which are subject to the boundary conditions. This leads to the attraction between two metallic plates, parallel to each other, embedded into the vacuum [43]. The attraction is due to a fluctuation of the fundamental energy of the field, that by the presence of plates, select the electromagnetic vacuum modes by boundary conditions- [44–47]. The measurements of this effect in great accuracy in the last decades has gained attention of the theoretical and experimental community [48,49]. One practical implication of these achievements is the development of micro-devices [50–53]. Also in the CPT-even photon sector of the Standard-Model Extension is taken into account in [54–65], and with embedded into a gravitational background [66,67],

In the present work, we consider the effect of compactification of the electromagnetism with breaking of Lorentz symmetry [68], with the preservation of the CPT symmetry [69]. in order to analyze the influence on such breaking of symmetry in the Casimir effect with thermal field treatment [68–77]. In terms of thermofield dynamics, the algebraic formalism of finite-temperature quantum field theory, a Bogoliubov transformation has then been generalized to describe thermal and space-compactification effects, of a field in toroidal topology $\Gamma_N^r = (S^1)^r \times R^{N-r}$ [78]. In Γ_N^r, N stands for the dimension of the Minkowski space-time, whilst r is the number of compactified dimensions. Here, following a different way, we consider a generalization of the partition function approach, in order to handle with the space compactification and temperature. One advantage in this procedure is that we avoid the duplication of the degree of freedom in the Bogoliubov formalism. Considering then general aspects of the SME photon sector in a toroidal manifold, the influence of the isotropic CPT-even electromagnetic sector of the SME in the Casimir effect at finite temperature is analysed.

This work is organized in the following way. In Sect. 2, the gauge sector of the SME is considered. in Sect. 3, the compactification in Γ_N^r is developed through a partition function procedure. In Sect. 4, the vacuum fluctuation effects is studied for $N = 4$, with the compactified dimensions $r = 1, 2$ submitted to the model, and our Concluding remarks are presented in Sect. 5.

2 The theoretical model and notation

The Lagrangian density of the CPT-even electrodynamics in the photon sector of the SME is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} (k_F)_{\mu\nu\lambda\rho} F^{\mu\nu} F^{\lambda\rho}, \tag{1}$$

where $(k_F)_{\mu\nu\lambda\rho}$ is a tensor dimensionless coupling and renormalizable. This tensor has the same symmetries of the Riemann tensor, composed of 19 elements, i.e.

$$(k_F)_{\mu\nu\lambda\rho} = -(\kappa_F)_{\nu\mu\lambda\rho}\,, (k_F)_{\mu\nu\lambda\rho} = -(k_F)_{\mu\nu\rho\lambda}\,,$$
$$(k_F)_{\mu\nu\lambda\rho} = (k_F)_{\lambda\rho\mu\nu}\,,$$
$$(k_F)_{\mu\nu\lambda\rho} + (k_F)_{\mu\lambda\rho\nu} + (k_F)_{\mu\rho\nu\lambda} = 0.$$

With these symmetry and antisymmetry properties associated with a double null trace, $(k_F)^{\mu\nu}{}_{\mu\nu} = 0$, the original 256 components are reduced to 19 independent components. The tensor trace contracted with the two field strengths generates a term proportional to the Maxwell term. Therefore, as the gauge sector of the Standard Model already has this term, the null trace condition removes this extra Maxwell term. It is used to construct the vacuum Maxwell equations similar as appears in a material medium [79].

The starting point for obtaining the structure of the Hamiltonian of electrodynamics comes from the definition of the canonical conjugate momentum

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = -F^{0\mu} - (k_F)^{0\mu\lambda\rho} F_{\lambda\rho}, \tag{2}$$

with the fundamental Poisson brackets written as

$$\left\{A_\mu(\mathbf{x}), \pi^\nu(\mathbf{y})\right\} = \delta_\mu^\nu (\mathbf{x} - \mathbf{y}). \tag{3}$$

We can express the conjugate momentum as,

$$\pi^k = -D^{kj} F^0{}_j - (k_F)^{0kjl} F_{jl}, \tag{4}$$

and the D^{kj} matrix is,

$$D^{kj} = \delta^{kj} + 2 (k_F)^{0k}{}_0{}^j, \tag{5}$$

From the Eq. (2), when taking $\mu = 0$ will imply $\pi^0 = 0$, which corresponds the constraint in the theory. The presence of the constraint suggests that one should study the structure of the Hamiltonian according to the Dirac method. This analysis leads to [80]

$$\mathcal{H}_c = \frac{1}{2}\left[\pi^k + (k_F)^{0kmn} F_{mn}\right] \left(D^{-1}\right)_{kj} \left[\pi^j + (k_F)^{0jmn} F_{mn}\right] + \pi^k \partial_k A_0 + \frac{1}{4}\left(F_{jk}\right)^2 + \frac{1}{4} (k_F)^{kjlm} F_{kj} F_{lm}. \tag{6}$$

The above expression is the canonical Hamiltonian density, and it is kept positive definite for sufficiently small values of $(k_F)_{\mu\nu\lambda\rho}$, providing a stable theory. With this Hamiltonian, in the following section, the partition function in Γ_N^r is introduced as a generalized Matsubara method [81].

3 The partition function in an *r*-dimensional torus

The partition function for the CPT-even sector of the SME in the functional integral representation is defined as

$$\begin{aligned} Z(L_0 \cdots L_r) = &\int DA_\mu D\pi^\mu \delta(\Sigma_a) \\ &\times |\det\{\Sigma_a(x), \Sigma_b(y)\}|^{1/2} \\ &\times \exp\left\{\int_0^{L_0} \cdots \int_0^{L_r} \int d^{D+1}x \left(i\pi^\mu \partial_\tau A_\mu - \mathcal{H}_c\right)\right\}, \end{aligned} \tag{7}$$

where $\Sigma_a = (\phi_1, \phi_2, \psi_1, \psi_2)$ is a second-class set formed by the first-class constraints and the gauge fixing conditions; $M_{ab}(x, y) = \{\Sigma_a(x), \Sigma_b(y)\}$ is the constraint matrix with determinant $\det\{\Sigma_a(x), \Sigma_b(y)\} = \det\left(-D_{kj}\partial_j\partial_k\right)^4$; and $\mathcal{H}_c$ is the canonical Hamiltonian given in Eq. (6). The partition function represents a generalization of the system in $N = D + 1$ dimensions and the topology $\Gamma_N^r = \Gamma_{D+1}^r = \mathbb{S}^{1_0} \times \mathbb{S}^{1_1} \cdots \mathbb{S}^{1_r} \times \mathbb{R}^{D+1-r}$. In this case, the compactification of the CPT-even sector of the SME will occur in $r + 1$ dimensions, with $r \leqslant D$. Therefore, the partition function of the SME for the CPT-even sector reads

$$\begin{aligned} &Z(L_0 \cdots L_r) \\ &= \int DA_0 D\pi^0 DA_k D\pi^k \delta\left(\pi^0\right) \delta\left(\partial_k \pi^k\right) \\ &\times \delta\left(D_{jk}\partial_j A_k\right) \delta\left(D_{jk}\partial_j\partial_k A_0 - (k_F)_{0ijk}\partial_i F_{jk}\right) \\ &\times \det\left(-D_{kj}\partial_j\partial_k\right)^2 \\ &\times \exp\left\{\int_0^{L_0} \cdots \int_0^{L_r} \int d^{D+1}x \left(i\pi^0\partial_\tau A_0 + i\pi^k\partial_\tau A_k\right)\right\} \\ &\times \exp\left\{\int_\beta dx \left[-\frac{1}{2}\left[\pi^k + (k_F)^{0kmn} F_{mn}\right]\right.\right. \\ &\times \left(D^{-1}\right)_{kj}\left[\pi^j + (k_F)^{0jmn} F_{mn}\right] \\ &\left.\left.-\pi^k\partial_k A_0 - \frac{1}{4}\left(F_{jk}\right)^2 - \frac{1}{4}(k_F)^{kjlm} F_{kj} F_{lm}\right]\right\}. \end{aligned} \tag{8}$$

Performing the proper integrations over the canonical conjugate momenta and introducing the following redefinitions,

$$\begin{aligned} F_{0k} &= F_{\tau k}, \\ (k_F)^{0kmn} &= i(k_F)^{\tau kmn}, \end{aligned}$$

it leads to

$$\begin{aligned} Z(L_0 \cdots L_r) = &\int DA_k DA_\tau \delta\left(D_{jk}\partial_j A_k\right) \det\left(-D_{kj}\partial_j\partial_k\right) \\ &\times \exp\left\{\int_0^{L_0} \cdots \int_0^{L_r} \int d^{D+1}x \right. \\ &\times \left[-\frac{1}{2} F_{\tau k} D_{kj} F_{\tau j} - \frac{1}{4}\left(F_{jk}\right)^2\right. \\ &\left.\left.-\frac{1}{4}(k_F)^{kjlm} F_{kj} F_{lm} - i(k_F)^{\tau kmn} F_{\tau k} F_{mn}\right]\right\}. \end{aligned} \tag{9}$$

This expression is rewritten as

$$\begin{aligned} Z(L_0 \cdots L_r) = &N \det\left(-D_{jk}\partial_j\partial_k\right) \int DA_a \delta\left(D_{jk}\partial_j A_k\right) \\ &\times \exp\left\{\int_0^{L_0} \cdots \int_0^{L_r} \int d^{D+1}x \right. \\ &\left.\times \left[-\frac{1}{4} F_{ab} F_{ab} - \frac{1}{4}(k_F)_{abcd} F_{ab} F_{cd}\right]\right\}, \end{aligned} \tag{10}$$

where $a, b, c, d = 0, 1, 2, 3$, considering that x^0 is Euclidian; i.e. $it \to \tau$ such that $0 < \tau < L_0 = \beta$, with $\beta = \frac{1}{T}$, T being the temperature. This partition function is not explicitly Lorentz covariant. To fix that, the Faddeev-Popov ansatz is used. We choose a covariant gauge condition in the form of the Lorenz gauge by

$$G[A_a] = -\frac{1}{\sqrt{\xi}}\partial_a A_a + f, \tag{11}$$

where f is an arbitrary scalar function and ξ is an arbitrary real parameter. The partition function results in

$$\begin{aligned} Z(L_0 \cdots L_r) = &N \int DA_a \det\left|\frac{-\Box}{\sqrt{\xi}}\right| \\ &\times \exp\left\{\int_0^{L_0} \cdots \int_0^{L_r} \int d^{D+1}x \left[-\frac{1}{4} F_{ab} F_{ab}\right.\right. \\ &\left.\left.-\frac{1}{4}(k_F)_{abcd} F_{ab} F_{cd} - \frac{1}{2\xi}\left(\partial_a A_a\right)^2\right]\right\}. \end{aligned} \tag{12}$$

Using the Feynman gauge $\xi = 1$, the partition function for the CPT-even photonic sector of the SME, compactified in Γ_{D+1}^r, reads

$$Z(L_0 \cdots L_r) = \det(-\Box)\left[\det(-\Box\delta_{ab} + S_{ab})\right]^{-1/2}, \tag{13}$$

where $\Box = \partial_a\partial_a$ and $S_{ab} = 2(k_F)_{abcd}\partial_c\partial_d$ corresponds to the symmetric operator containing the parameters responsible for breaking the Lorentz symmetry.

As mentioned earlier, the CPT-even term of the SME is described by the tensor $(k_F)_{\mu\nu\lambda\rho}$, which is constituted by 19 independent coefficients, among which there are 10 birefringent and 9 non-birefringent coefficients. The exact calculation of the partition function of the CPT-even electrodynamics of the SME can become a challenging task. A route to such a condition is to decompose this tensor in terms of four 3×3 matrices, $\kappa_{DE}, \kappa_{HB}, \kappa_{DB}, \kappa_{HE}$ [82–85], i.e.,

$$(\kappa_{DE})^{j\kappa} = -2\,(k_F)^{0j0\kappa}\,, \quad (\kappa_{HB})^{j\kappa} = \frac{1}{2}\epsilon^{jpq}\epsilon^{\kappa lm}\,(k_F)^{pqlm}\,,$$
$$(\kappa_{DB})^{j\kappa} = -\,(\kappa_{HE})^{\kappa j} = \epsilon^{\kappa pq}\,(k_F)^{0jpq}\,. \tag{14}$$

The matrices κ_{DB} and κ_{HE} stand for the parity-odd sector described by 8 components, while κ_{DE} and κ_{HB} represent the parity-even sector and possess together 11 independent components. These four matrices together make up the 19 independent elements of the tensor $(k_F)_{\mu\nu\lambda\rho}$. Hence, from the perspective of parity, such matrices are rearranged into two groups: one group with even parity components $(\widetilde{\kappa}_e)$ and the other with odd parity $(\widetilde{\kappa}_o)$. The even parity components are written as

$$(\widetilde{\kappa}_{e+})^{j\kappa} = \frac{1}{2}(\kappa_{DE} + \kappa_{HB})^{j\kappa},$$
$$(\widetilde{\kappa}_{e-})^{j\kappa} = \frac{1}{2}(\kappa_{DE} - \kappa_{HB})^{j\kappa} - \frac{1}{3}\delta^{j\kappa}(\kappa_{DE})^{ii},$$
$$\widetilde{\kappa}_{tr} = \frac{1}{3}tr(\kappa_{DE}), \tag{15}$$

while the odd parity components are written in terms of an symmetric $(\widetilde{\kappa}_{o-})$ and a antisymmetric matrix $(\widetilde{\kappa}_{o+})$, given in terms of

$$(\widetilde{\kappa}_{o-})^{j\kappa} = \frac{1}{2}(\kappa_{DB} - \kappa_{HE})^{j\kappa},$$
$$(\widetilde{\kappa}_{o+})^{j\kappa} = \frac{1}{2}(\kappa_{DB} + \kappa_{HE})^{j\kappa}. \tag{16}$$

Regarding the compactification, the matrices given in Eq. (14) are written as

$$(\kappa_{DE})^{j\kappa} = 2\,(k_F)^{\tau\kappa\tau j}\,, \quad (\kappa_{HB})^{j\kappa} = \frac{1}{2}\epsilon^{jpq}\epsilon^{\kappa lm}\,(k_F)^{pqlm}\,,$$
$$(\kappa_{DB})^{j\kappa} = -\,(\kappa_{HE})^{\kappa j} = \epsilon^{\kappa pq}\,(k_F)^{\tau jpq}\,. \tag{17}$$

Now, we seek to constrain the tensor $(k_F)_{\mu\nu\lambda\rho}$ for the parity-even and isotropic component configuration. One should start by considering $\widetilde{\kappa}_{e+} = 0$, what leads to $\kappa_{DE} = -\kappa_{HB}$. Then the non-birefringent components are given by

$$(\kappa_{DE})_{jk} = \ (\widetilde{\kappa}_{e-})_{jk} + \widetilde{\kappa}_{tr}\delta_{jk}. \tag{18}$$

The isotropic component $\widetilde{\kappa}_{tr}$ is obtained by imposing $\widetilde{\kappa}_{e-} = 0$, such that

$$(\kappa_{DE})_{jk} = \widetilde{\kappa}_{tr}\delta_{jk}$$
$$(\kappa_{HB})_{jk} = -\widetilde{\kappa}_{tr}\delta_{jk}. \tag{19}$$

In order to calculate the partition function on the influence of the isotropic contribution, using these definitions, the following operators are such that $p^2\delta_{ab} - \widetilde{S}_{ab}$, with $\widetilde{S}_{ab} = 2\,(K_F)_{abcd}\,p_c p_d$. In this way, the matrix $\widetilde{S}_{ab}$ becomes

$$\widetilde{S}_{\tau\tau} = \widetilde{\kappa}_{tr}\mathbf{p}^2, \quad \widetilde{S}_{\tau k} = \widetilde{\kappa}_{tr}p_\tau p_k,$$
$$\widetilde{S}_{jk} = -\widetilde{\kappa}_{tr}\delta_{jk}p^2 + 2\widetilde{\kappa}_{tr}\delta_{jk}\mathbf{p}^2 - \widetilde{\kappa}_{tr}p_j p_k. \tag{20}$$

As a result, it follows,

$$\det\left(-\Box\delta_{ab} + S_{ab}\right) = \det\left[(\widetilde{\kappa}_{tr}+1)^3\,(-\Box)^2\right] \times \det\left[-\Box + \frac{2\widetilde{\kappa}_{tr}}{\widetilde{\kappa}_{tr}+1}\nabla^2\right]^2. \tag{21}$$

Using this results in Eq. (13), the partition function is rewritten as

$$\ln Z\,(L_0\cdots L_r) = -Tr\ln\left[-\Box + \frac{2\widetilde{\kappa}_{tr}}{\widetilde{\kappa}_{tr}+1}\nabla^2\right]. \tag{22}$$

The trace of this equation is calculated by writing the gauge field in terms of a Fourier expansion, i.e.,

$$A_a\,(L_0\cdots L_r,\mathbf{x}) = \left(\frac{L_0}{V}\right)^{1/D-1}\cdots\left(\frac{L_r}{V}\right)^{1/D-1} \times \int\prod_{n=r+1}^{D}V^{n/3}\frac{d^n\mathbf{p}_n}{(2\pi)^n} \times \sum_{n_0\cdots n_r} e^{i\left(\omega_{n_j}x_j+\mathbf{x}\cdot\mathbf{p}\right)}\widetilde{A}_a\,(n_0\cdots n_r,\mathbf{p})\,. \tag{23}$$

In the expression above, j runs over the compactified dimensions, ω_{n_j} are the boson Matsubara-like frequencies, the wave numbers $\omega_{n_j} = \frac{2\pi}{L_j}n_j$ with $n_j = 0, 1, 2, ...$, $\mathbf{p} = (p_{r+1}, \ldots, p_D)$ and V corresponds to the hypervolume of the system. The generalized partition function, given in Eq. (22), of the gauge field is now given as

$$\ln Z\,(L_0\cdots L_r) = -\int\prod_{n=r+1}^{D}V^{n/3}\frac{d^n\mathbf{p}_n}{(2\pi)^n} \times \sum_{n_0\cdots n_r=-\infty}^{+\infty}\ln\prod_{j=0}^{r}L_j^2 V^{\frac{D-3r}{3}} \times \left[\sum_{n_j=0}^{r}\omega_{n_j}^2 + \left(\frac{1-\widetilde{\kappa}_{tr}}{\widetilde{\kappa}_{tr}+1}\right)\mathbf{p}^2\right]. \tag{24}$$

In the next section, this result is explored in applications to the Casimir effect.

4 Vacuum fluctuation in torus

The previous result, given in Eq. (24), is considered in the Γ^2_{3+1} and Γ^1_{3+1}. The physical result is the Casimir effect at finite temperature, followed by the effect of the breaking of the Lorentz symmetry.

4.1 $S^1 \times \mathbb{R}^3$: Casimir Effect

To study the Casimir effect from Eq. (24), we consider $D = 3$ and $r = 1$. In this perspective, the compactification occurs in a spatial dimension, that is, $\Gamma^1_4 = \mathbb{S}^1 \times \mathbb{R}^3$, where the dimension to be compactified is x_1, having the circumference length of $\mathbb{S}^1$ equal to L_1. For this case, the periodicity condition in x_1 leads to $0 \le x_1 \le L_1$, at the same time as the others components vary in the range $(-\infty, +\infty)$. In such configurations, the field is

$$A(x_0, x_1, x_2, x_3) = A(x_0, x_1 + L_1, x_2, x_3).$$

The expression for the partition function, Eq. (24), in this configuration reads

$$\ln Z\left(L_1\right) = -V \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{m=-\infty}^{+\infty} \ln L_1^2 \times \left[\left(\frac{1-\widetilde{\kappa}_{tr}}{\widetilde{\kappa}_{tr}+1}\right)\omega_m^2 + \left(\frac{1-\widetilde{\kappa}_{tr}}{\widetilde{\kappa}_{tr}+1}\right)\mathbf{p}^2 + p_0^2\right], \tag{25}$$

where the trace is calculated in terms of the Fourier expansion

$$A_a\left(x_1, \mathbf{x}\right) = \left(\frac{L_1}{V}\right)^{1/2} V \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_m e^{i(\omega_m x_1 + \mathbf{x}\cdot\mathbf{p})} \widetilde{A}_a\left(m, \mathbf{p}\right).$$

Here, V designates the system volume; $\mathbf{p} = (p_0, p_2, p_3)$; and ω_m are the boson Matsubara frequencies $\omega_m = \frac{2\pi}{L_1} m$ for $m = 0, \pm1, \pm2, \ldots$ To calculate the integrals, we use the transformation $p_i' = p_i \sqrt{\frac{1+\widetilde{\kappa}_{tr}}{1-\widetilde{\kappa}_{tr}}}$. Also, we use spherical coordinates $\mathbf{p} \to (\omega, \theta, \phi)$, $\omega = |\mathbf{p}|$. By performing the summation over m and doing the respective changes in the variable, we have

$$\ln Z\left(L_1\right) = -2V\left(\frac{1+\widetilde{\kappa}_{tr}}{1-\widetilde{\kappa}_{tr}}\right) \int d\Omega \int_0^\infty d\omega \omega^2 \ln\left(1 - e^{-L_1\omega}\right), \tag{26}$$

where $d\Omega = \sin\theta d\theta d\phi$ is the solid-angle element, and the solved integral can be found in [86]. Calculating the derivative with regard to L_1 of Eq. (26), it provides the energy density, u; in other words, the pressure along the direction x^1, that is given by

$$u = \left(\frac{1+\widetilde{\kappa}_{tr}}{1-\widetilde{\kappa}_{tr}}\right) \frac{-\pi^2}{15\left(L_1\right)^4}. \tag{27}$$

This result corresponds to the Casimir effect for the periodic boundary condition. The derivatives along the other directions provides the diagonal terms of the energy momentum tensor. In particular for the compactification in time, it leads to the Stefan–Boltzmann Law. Before to derive this result, it is important to emphasize that the Casimir effect has been measured with precision of few percents of error only over the last two decades (see the references cited about that in the Introduction). Refinements on those measurements would reveal the breaking in the Lorentz symmetry. In addition, the Casimir effect derived here can be considered on a cosmic scale, in the sense of Birell and Ford [70], who have studied a boson field in a toroidal universe model, in order to estimate the origin of matter. In both cases, the effect of temperature has to be taken into consideration. This is the subject matter in the next applications.

4.2 $S^1 \times \mathbb{R}^3$: Stefan–Boltzmann law

Considering, in Eq. (24), $D = 3$ and $r = 1$, where the compactification is the time dimension, i.e., along x_0, $\Gamma^1_4 = \mathbb{S}^1 \times \mathbb{R}^3$, having the circumference length $\mathbb{S}^1$ equal to $L_0 = \beta = 1/T$, where T is the temperature (the Boltzmann constant is $k_B = 1$). The periodicity condition in x_0 leads to $0 \le x_0 \le L_0$, and the trace is calculated in terms of the Fourier expansion, and the field is

$$A_a\left(x_0, \mathbf{x}\right) = \left(\frac{L_0}{V}\right)^{1/2} V \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_n e^{i(\omega_n x_0 + \mathbf{x}\cdot\mathbf{p})} \widetilde{A}_a\left(n, \mathbf{p}\right). \tag{28}$$

where V designates the system volume, $\mathbf{p} = (p_1, p_2, p_3)$, ω_n are the bosonic Matsubara frequencies $\omega_n = \frac{2\pi}{L_0} n$ for $n = 0, \pm1, \pm2, \ldots$ The partition function, Eq. (24), is rewritten as

$$\ln Z\left(L_0\right) = -V \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{n=-\infty}^{+\infty} \ln L_0^2 \left[\omega_n^2 + \left(\frac{1-\widetilde{\kappa}_{tr}}{\widetilde{\kappa}_{tr}+1}\right)\mathbf{p}^2\right]. \tag{29}$$

Again, to calculate the integrals, the transformation $p_i' = p_i \sqrt{\frac{1+\widetilde{\kappa}_{tr}}{1-\widetilde{\kappa}_{tr}}}$ is used. Also, the spherical coordinates $\mathbf{p} \to (\omega, \theta, \phi)$, $\omega = |\mathbf{p}|$are introduced. By performing the summation in n and using the respective changes in the variable, the partition function, given in Eq. (29), takes the form

$$\ln Z\left(\beta\right) = -\frac{2V}{(2\pi)^3}\left(\frac{1+\widetilde{\kappa}_{tr}}{1-\widetilde{\kappa}_{tr}}\right)^{3/2} \int d\Omega \int_0^\infty d\omega \omega^2 \ln\left(1 - e^{-\beta\omega}\right). \tag{30}$$

Calculating the derivative with regard to L_0 of Eq. (30), it provides the energy density, u, that is the free energy. This leads to

$$u = \left(\frac{1+\widetilde{\kappa}_{tr}}{1-\widetilde{\kappa}_{tr}}\right)^{3/2} \frac{\pi^2}{15L_0^4} = \left(\frac{1+\widetilde{\kappa}_{tr}}{1-\widetilde{\kappa}_{tr}}\right)^{3/2} \frac{\pi^2}{15\beta^4}, \qquad (31)$$

which corresponds to the energy density of the cavity or the Stefan–Boltzmann law modified by the isotropic term.

4.3 $S^1 \times S^1 \times \mathbb{R}^2$: Casimir effect at finite temperature

Here, we must consider $D = 2$ and $r = 2$ in Eq. (24). In which compactification will occur in the time dimension and in a spatial dimension, that is, $\Gamma_4^2 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}^2$. The trace is calculated in terms of the Fourier expansion, i.e.,

$$\begin{aligned} A_a(x_0, x_1, \mathbf{x}) = \left(\frac{L_0}{V}\right)^{1/2} \left(\frac{L_1}{V}\right)^{1/2} V^{2/3} \\ \times \int \frac{d^2\mathbf{p}}{(2\pi)^2} \\ \times \sum_{n_0, n_1} e^{i(\omega_{n_0} x_0 + \omega_{n_1} x_1 + \mathbf{x}\cdot\mathbf{p})} \widetilde{A}_a(n_0, n_1, \mathbf{p}), \end{aligned} \qquad (32)$$

where $\mathbf{p} = (p_2, p_3)$, $\omega_{n_0} = \frac{2\pi}{L_0} n_0$ and $\omega_{n_1} = \frac{2\pi}{L_1} n_1$ are the bosonic Matsubara frequencies. This leads the partition function, in Eq. (24), to be written as

$$\begin{aligned} \ln Z(L_0, L_1) = -V^{2/3} \int \frac{d^2\mathbf{p}}{(2\pi)^2} \sum_{n_0, n_1 = -\infty}^{+\infty} \ln L_0{}^2 L_1^2 V^{-4/3} \\ \times \left[\omega_{n_0}^2 + \left(\frac{1-\widetilde{\kappa}_{tr}}{\widetilde{\kappa}_{tr}+1}\right)\omega_{n_1}^2 + \left(\frac{1-\widetilde{\kappa}_{tr}}{\widetilde{\kappa}_{tr}+1}\right)\mathbf{p}^2\right]. \end{aligned} \qquad (33)$$

In order to calculate the integrals, the use of the transformation $p_i' = p_i \sqrt{\frac{1+\widetilde{\kappa}_{tr}}{1-\widetilde{\kappa}_{tr}}}$ leads to

$$\begin{aligned} \ln Z(L_0, L_1) = -\left(\frac{1+\widetilde{\kappa}_{tr}}{1-\widetilde{\kappa}_{tr}}\right) V^{2/3} \int \frac{d^2\mathbf{p}}{(2\pi)^2} \\ \times \left[\sum_{n_0=-\infty}^{+\infty} \ln\left[(2\pi n_0)^2 + (L_0\omega)^2\right]\right. \\ + \sum_{n_1=-\infty}^{+\infty} \ln\left[(2\pi n_1)^2 + (L_1\omega)^2\right] \\ + 4 \sum_{n_0, n_1 = 1}^{+\infty} \ln V^{-4/3} \\ \times \left[(2\pi n_0 L_1)^2 + (2\pi n_1 L_0)^2 + (L_0 L_1 \omega)^2\right] \\ \left. + \ln V^{-4/3} L_0^2 + \ln V^{-4/3} L_1^2\right]. \end{aligned} \qquad (34)$$

In the thermodynamic limit, $V \to \infty$, the last two terms generate a pure divergence. In this way, we will perform a renormalization in Eq. (34). Using the polar coordinates $\mathbf{p} \to (\omega, \theta)$, $\omega = |\mathbf{p}|$, and performing the summation in n, the partition function in Eq. (34) takes the form

$$\begin{aligned} \ln Z(L_0, L_1) = -2\left(\frac{1+\widetilde{\kappa}_{tr}}{1-\widetilde{\kappa}_{tr}}\right) V^{2/3} \\ \times \int \frac{d\phi d\omega\omega}{(2\pi)^2} \Big\{ \ln\left(1 - e^{-L_0\omega}\right) + \ln\left(1 - e^{-L_1\omega}\right) \\ + 2 \sum_{n_0, n_1 = 1}^{+\infty} \ln V^{-4/3} \left[(2\pi n_0 L_1)^2\right. \\ \left. + (2\pi n_1 L_0)^2 + (L_0 L_1 \omega)^2\right] \Big\}. \end{aligned} \qquad (35)$$

Performing the integration over the first two terms, the partition function given in Eq. (35) reads

$$\ln Z(L_0, L_1) = \frac{1}{\pi} \left(\frac{1+\widetilde{\kappa}_{tr}}{1-\widetilde{\kappa}_{tr}}\right) V^{2/3} \left[\frac{\zeta(3)}{L_0^2} + \frac{\zeta(3)}{L_1^2}\right] + \mathcal{O}, \qquad (36)$$

such that

$$\begin{aligned} \mathcal{O} = -\frac{2}{\pi} \left(\frac{1+\widetilde{\kappa}_{tr}}{1-\widetilde{\kappa}_{tr}}\right) V^{2/3} \int d\omega\omega \left(\sum_{n_0=1}^{+\infty} \sum_{n_1=1}^{+\infty} \ln V^{-4/3} \left[(2\pi n_0 L_1)^2\right.\right. \\ \left.\left. + (2\pi n_1 L_0)^2 + (L_0 L_1 \omega)^2\right]\right). \end{aligned} \qquad (37)$$

It is important to realize that in the limit $\beta \to \infty$, the Casimir effect given in Eq. (27) is recovered. For for $L_1 \to \infty$, the Stefan–Boltzmann law, Eq. (27), is obtained again.

5 Conclusions and perspectives

We have investigated the influence of Lorentz symmetry violation by background fields on the vacuum fluctuation in the *SME* of the electromagnetic field in D+1 dimensions with r dimensions compactified in a torus. Based on the results of the generalized real-time thermofield dynamics formalism for the quantum field theory at finite temperatures in a toroidal manifold, we have introduced a generalized partition function to account for the field theory in a space-like torus at finite temperature. This procedure has prevented the use of the duplication in the thermofield dynamics approach. The partition function associated with the CPT-even sector of the SME of the electrodynamics in D+1 dimensions is analyzed.

With compactification in the x_1 direction, the energy density (the pressure) of the system with the Lorentz violation contribution is calculated. This leads to a correction of the standard result of the Casimir effect, due to the background field. The periodicity condition on x_0 leads to the

Stefan–Boltzmann law with the Lorentz violation contribution. Finally, the periodicity condition on x_0 and on x_1 directions provide the modification of the Casimir effect at finite temperature based on the contribution of the Lorentz symmetry violation.

The Casimir effect has been measured only recently with a precision of the order of $\lesssim 5\%$ in error. Therefore, this is a potentially theoretical and experimental realm for testing a Lorentz violation. This is an aspect that deserves more investigation.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: This article is about a theoretical consistency study on the influence of Spontaneous Violation of Lorentz Symmetry on the Casimir effect. At this stage, we still do not relate to possible experimental measurements.]

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