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Thermodynamics of Einstein-Aether black holes

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Abstract We analyze several spherically symmetric exterior vacuum solutions allowed by the Einstein-Aether (EA) theory with a non-static aether and study the thermodynamics of their Killing and universal horizons. We show that there are five classes of solutions corresponding to different values of a combination of the free parameters, c_2 , $c_{13} = c_1 + c_3$ and $c_{14} = c_1 + c_4$, which are: (A) $c_2 \neq 0$ and $c_{13} \neq 0$ and $c_{14} = 0$, (B) $c_2 \neq 0$ and $c_{13} = 0$ and $c_{14} = 0$, (C) $c_2 = 0$ and $c_{13} \neq 0$ and $c_{14} = 0$, (D) $c_2 = 0$, $c_{13} = 0$ and $c_{14} \neq 0$, and (E) $c_2 = -c_{13} \neq 0$ and $c_{14} \neq 0$. We present explicit analytical solutions for these five cases. All these cases have singularities at r = 0 and are asymptotically flat spacetimes and possess both Killing and universal horizons with the universal horizons always being inside the Killing horizons. Finally, we compute the surface gravity, the temperature, the entropy and the first law of thermodynamics for the universal horizons.

1 Introduction

The two recurring themes in our quest for a theory of quantum gravity have been to either introduce a new fundamental symmetry such as supersymmetry or break a fundamental symmetry such as Lorentz invariance (LI). The breaking of LI is found to make the construction of quantum gravity a possible task, at least on paper, see [1] for an example. Currently, one of the principal guiding lights in quantum gravity research is the study of black hole thermodynamics. Thus, it is interesting to investigate the thermodynamics of black holes in a gravitational theory that explicitly breaks LI. This is precisely what we intend to do in this work.

The LI is an exact symmetry in special relativity, quantum field theories, and the standard model of particle physics, while in General Relativity (GR) it is only a local symmetry in freely falling inertial frames [2]. The violation of LI in the gravitational sector is not as well explored as in matter interactions where it is highly constrained by several precision experiments [3]. Jacobson and his collaborators introduced and analyzed a general class of vector-tensor theories called the Einstein-Aether (EA) theory [4-8] to study the effects of violation of LI in gravity. A brief review of the vector-tensor theories of gravity can be found in [9]. The first spherical static vacuum solutions in the EA theory were obtained by Eling and Jacobson in 2006 [10]. Since then several more solutions have been found including our recent analytical solutions for static aether [12]. Most of the literature on black holes in EA theory can be found in the papers [13-35].

The subject of black hole thermodynamics was born in 1974 when Stephen Hawking [36] mathematically showed that a black hole radiates as though it had a temperature proportional to its surface gravity, and he asserted that the similarities between the laws of black-hole mechanics [37] and the laws of thermodynamics were more than a coincidence [38]. Hawking also obtained a precise relationship between the entropy of the black hole and its surface area [39]. The reviews [40,41] give a good outline of the subject.

The paper is organized as follows. Section 2 briefly presents the EA theory, whose field equations are solved for a general spherically symmetric metric in Sect. 3. In Sect. 4, we present the basics of black hole thermodynamics. In Sects. 5-12 we present the explicit analytical solutions and study their thermodynamics. We summarize our results in Sect. 13. The field equations are quite long, so we have relegated them to the Appendix.

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2 Field equations in the EA theory

The general action of the EA theory is given by

$$S = \int \sqrt{-g} \left(L_{\text{Einstein}} + L_{\text{aether}} + L_{\text{matter}} \right) d^4 x, \qquad (1)$$

where, the first term is the usual Einstein–Hilbert Lagrangian, defined by R, the Ricci scalar, and G, the EA gravitational constant, as

$$L_{\text{Einstein}} = \frac{1}{16\pi G} R.$$
 (2)

The second term, the aether Lagrangian is given by

$$L_{\text{aether}} = \frac{1}{16\pi G} [-K^{ab}{}_{mn} \nabla_a u^m \nabla_b u^n + \lambda (g_{ab} u^a u^b + 1)],$$
(3)

where the tensor K^{ab}_{mn} is defined as

$$K^{ab}{}_{mn} = c_1 g^{ab} g_{mn} + c_2 \delta^a_m \delta^b_n + c_3 \delta^a_n \delta^b_m - c_4 u^a u^b g_{mn},$$
(4)

being the c_i dimensionless coupling constants, and λ a Lagrange multiplier enforcing the unit timelike constraint on the aether, and

$$\delta^a_m \delta^b_n = g^{a\alpha} g_{\alpha m} g^{b\beta} g_{\beta n}.$$
⁽⁵⁾

Finally, the last term, L_{matter} is the matter Lagrangian, which depends on the metric tensor and the matter field.

In the weak-field, slow-motion limit EA theory reduces to Newtonian gravity with a value of Newton's constant G_N related to the parameter G in the action (1) by [15],

$$G = G_N \left(1 - \frac{c_{14}}{2} \right). \tag{6}$$

Here, the constant c_{14} is defined as

$$c_{14} = c_1 + c_4. (7)$$

The field equations are obtained by extremizing the action with respect to independent variables of the system. The variation with respect to the Lagrange multiplier λ imposes the condition that u^a is a unit timelike vector, thus

$$g_{ab}u^a u^b = -1, (8)$$

while the variation of the action with respect u^a , leads to [15]

$$\nabla_a J_b^a + c_4 a_a \nabla_b u^a + \lambda u_b = 0, \tag{9}$$

where,

$$J_m^a = K_{mn}^{ab} \nabla_b u^n, \tag{10}$$

and

$$a_a = u^b \nabla_b u_a. \tag{11}$$

The variation of the action with respect to the metric g_{mn} gives the dynamical equations,

$$G_{ab}^{Einstein} = T_{ab}^{aether} + 8\pi G T_{ab}^{matter},$$
(12)

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where

$$G_{ab}^{Einstein} = R_{ab} - \frac{1}{2}g_{ab}R,$$

$$T_{ab}^{aether} = \nabla_c [J^c \ _{(a}u_{b)} + u^c J_{(ab)} - J_{(a} \ ^c u_{b)}]$$

$$- \frac{1}{2}g_{ab}J_d^c \nabla_c u^d + \lambda u_a u_b$$

$$+ c_1 [\nabla_a u_c \nabla_b u^c - \nabla^c u_a \nabla_c u_b] + c_4 a_a a_b,$$

$$T_{ab}^{matter} = \frac{-2}{\sqrt{-g}} \frac{\delta \left(\sqrt{-g}L_{matter}\right)}{\delta g_{ab}}.$$
(13)

Later, when we solve the field equations (12), we do take into consideration the Eqs. (8)–(11) in the process of simplification. Thus, in this paper (as in the Eqs. (126)–(131) below) we seem to solve only the dynamical equations, but in fact, we are also solving the equations arising from the variations of the action with respect λ and u^a .

In a more general situation, the Lagrangian of GR is recovered, if and only if, the coupling constants are identically zero, e.g., $c_1 = c_2 = c_3 = c_4 = 0$, considering the Eqs. (4) and (8).

3 Spherical solutions of EA field equations

We start with the most general spherically symmetric static metric

$$ds^{2} = -A(r)dt^{2} + B(r)dr^{2} + r^{2}d\theta^{2} + r^{2}\sin^{2}\theta d\phi^{2}.$$
 (14)

In accordance with Eq. (8), the aether field is assumed to be unitary and timelike, chosen as

$$u^{a} = \left[\frac{b(r)}{\sqrt{-a(r)^{2}B(r) + b(r)^{2}A(r)}}, \frac{a(r)}{\sqrt{-a(r)^{2}B(r) + b(r)^{2}A(r)}}, 0, 0\right],$$
(15)

where $x^{\mu} = (t, r, \theta, \phi)$ are the directions of the aether vector. Since the aether vector is unitary we have $a(r)^2 B(r) - b(r)^2 A(r) = -1$. This kind of aether vector with time and radial components we have called *non-static aether*, based on the nomenclature created by Eling and Jacobson [11]. They have referred as *static aether* when the aether is parallel to the Killing vector, i.e., the aether vector has only time component.

Substituting

$$b(r) = \frac{\epsilon}{A(r)} \sqrt{A(r)[a(r)^2 B(r) + 1]}$$
(16)

from this last condition we obtain the aether vector depending only of a(r), where $\epsilon = \pm 1$.

$$u^{a} = \left[\epsilon \frac{\sqrt{A(r)[a(r)^{2}B(r) + 1]}}{A(r)}, a(r), 0, 0 \right],$$
(17)

The timelike Killing vector of the metric (14) is giving by

$$\chi^{\alpha} = (-1, 0, 0, 0). \tag{18}$$

The Killing and the universal horizon [42,43] are obtained finding the largest root of

$$\chi^{\alpha}\chi_{\alpha} = 0, \tag{19}$$

and

$$\chi^{\alpha} u_{\alpha} = 0, \tag{20}$$

respectively, where χ^{α} is the timelike Killing vector. In our case,

$$\chi^{\alpha}\chi_{\alpha} = -A(r), \tag{21}$$

$$\chi^{\alpha} u_{\alpha} = \frac{A(r)b(r)}{\sqrt{-a(r)^2 B(r) + b(r)^2 A(r)}}.$$
(22)

In order to identify eventual singularities in the solutions, it is useful to calculate the Kretschmann scalar invariant K. For the metric (14), it is given by

$$K = \frac{1}{4r^{4}B^{4}A^{4}} \left(16B^{4}A^{4} - 32B^{3}A^{4} + 16B^{2}A^{4} + 8A^{\prime 2}r^{2}B^{2}A^{2} + 8B^{\prime 2}r^{2}A^{4} + 4r^{4}A^{\prime \prime 2}B^{2}A^{2} - 4r^{4}A^{\prime \prime}BA^{2}B^{\prime}A^{\prime} - 4r^{4}A^{\prime \prime}B^{2}AA^{\prime 2} + r^{4}B^{\prime 2}A^{\prime 2}A^{2} + 2r^{4}B^{\prime}A^{\prime 3}AB + r^{4}A^{\prime 4}B^{2} \right).$$
(23)

The field equations are given in full detail in the Appendices A (collecting the terms c_2 , c_{13} and c_{14}) and B (collecting the terms c_{123} , c_{14} and c_2). Notice that assuming $c_{123} = 0$ does not give the same field equations that assuming $c_2 + c_{13} = 0$ or $c_2 = 0$ with $c_{13} = 0$. This characteristic of the field equations imposes different solutions for each case that will be clearer below.

Solving simultaneously Eqs. (126)–(131) using Maple 16 we get five particular families of analytic solutions: (A) $c_2 \neq 0$ and $c_{13} \neq 0$ and $c_{14} = 0$, (B) $c_2 \neq 0$ and $c_{13} = 0$ and $c_{14} = 0$, (C) $c_2 = 0$ and $c_{13} \neq 0$ and $c_{14} = 0$, (D) $c_2 = 0$, $c_{13} = 0$ and $c_{14} \neq 0$, and (E) $c_2 = -c_{13} \neq 0$ and $c_{14} \neq 0$. Let us now analyze these five possible cases in detail. The solutions for only $c_2 = 0$ or only $c_{13} = 0$ were not shown because they are a special case of static aether, imposing a(r) = 0 (see our previous work [12] for the static aether case).

It was shown [20] that Smarr formula and corresponding first law of black hole mechanics exists for ranges of the c_i 's, $0 \le c_{14} < 2, c_{13} < 1$ and $2+c_{13}+3c_2 > 0$. All our solutions fall within this interval. Imposing these limits on our choices, we get $c_2 > -2/3, -2 < c_{13} < 1$ and $0 < c_{14} < 2$.

4 Black hole thermodynamics

We briefly discuss the thermodynamics of black holes for the sake of completeness. The essential ingredient for this is the existence of horizons, the usual event horizon in the relativistic case and universal horizon in the Lorentz-violating theories like the one we are considering here. For more discussion on black hole thermodynamics in EA theory, we refer the readers to references [14,22,45–47]. Recently, the thermodynamics of black holes in Einstein-Aether–Maxwell theory was investigated by the phase-space solution method in [47].

Besides the horizons, the other important quantity in black hole thermodynamics is the surface gravity κ . Once we have these two quantities, we can interpret the $\kappa/2\pi$ as the temperature of the black hole and A/4 as the entropy of the black hole where A is the surface area of the horizon. The surface gravity for Killing and universal horizons is defined respectively as (see the equations (37) and (39) of the reference [47]),

$$\kappa_{kh} = \frac{1}{2} \sqrt{-\chi^{a;b} \chi_{b;a}} |_{r=r_{kh}},$$

$$\kappa_{uh} = \frac{1}{2} u^{b} (u^{a} \chi_{a})_{;b} |_{r=r_{uh}},$$
(24)

where the symbol semicolon means covariant derivative. Calculating these quantities we get

$$\kappa_{kh} = \frac{1}{2} \sqrt{\frac{A'^2}{BA}} |_{r=r_{kh}},$$

$$\kappa_{uh} = -\frac{\epsilon a \left(A'a^2 B + A' + 2AaBA' + Aa^2 B'\right)}{4\sqrt{A \left(a^2 B + 1\right)}} |_{r=r_{uh}}.$$
(25)

We would like to mention that since any constant multiple of a Killing vector is also another Killing vector, it does not uniquely specify the scaling of the surface gravity; it can be changed by a constant factor by rescaling the Killing vector. If the spacetime is asymptotically flat, then one can normalize the Killing vector at the spatial infinity thereby obtaining a unique value for the surface gravity – which is what we have done here.

For the universal horizon, we can directly write down (see equation (63) of [47]) the surface gravity, temperature, entropy and the first law for the universal horizon respectively as,

$$T_{uh} = \frac{\kappa_{uh}}{2\pi},$$

$$S_{uh} = \frac{\pi r_{uh}^2}{G} = \frac{A_{uh}}{4G},$$

$$\delta S_{uh} = \frac{\delta M}{T_{uh}},$$
(26)

where A_{uh} is the the area of universal horizon. It is clear from the equations above that they are exactly the same as in GR. However, for a non-static aether, one may have to redefine surface gravity because of the contribution of aether vector field [47] as we have in Eq. (24).

In order to compare our results with the GR black we calculate the thermodynamical quantities for $r_{kh} = r_{uh} = r_{GR} = 2M$ or the Schwarzschild radius $r_s = \frac{2G_NM}{c^2}$ giving

$$\kappa_{GR} = \frac{G_N M}{r_s^2} = \frac{c^4}{4G_N M} = \frac{1}{4M},$$

$$T_{GR} = \frac{\hbar \kappa_{GR}}{2\pi ck_B} = \frac{\hbar c^3}{8\pi k_B G_N M} = \frac{1}{8\pi M},$$

$$S_{GR} = \frac{k_B c^3 A_{GR}}{4\hbar G_N} = \frac{4\pi k_B G_N M^2}{\hbar c} = 4\pi M^2,$$

$$A_{GR} = 4\pi r_s^2 = \frac{16\pi G_N^2 M^2}{c^4} = 16\pi M^2,$$

$$\delta S_{GR} = \frac{8\pi k_B G_N M \,\delta M}{\hbar c} = 8\pi M \,\delta M.$$
(27)

We are explicitly showing all the constants because the gravitational coupling constant is different in GR and EA. But, from now on, we work with $G_N = c = \hbar = k_B = 1$.

5 Solutions for case (A): $c_2 \neq 0$ and $c_{13} \neq 0$ and $c_{14} = 0$

The solution of the field equations (126)–(131) for $c_{14} = 0$ we get,

$$A = G_1 + \frac{G_2}{r} + G_3 r^2 + \frac{G_4}{r^4},$$

$$B = \frac{r^4}{r^4 + G_2 r^3 + G_4},$$
(28)

where G_1 , G_2 , G_3 and G_4 are arbitrary integration constants we have chosen $G_1 = 1$ and $G_3 = 0$ in order to have a flat spacetime at infinity and $G_2 = -|G_2|$ in order to have a resemblance with the Schwarzschild solution as in the GR and $|G_2| = 2M$, where *M* is the Schwarzschild mass. Note again as in the Case C, the term r^{-4} it could be compared in GR to the Hartle and Thorne [48] solution of a slowly rotating deformed relativistic star, assuming the quadrupole moment is zero. Thus, A(r) and B(r) can be rewritten as

$$A = 1 - \frac{2M}{r} + \frac{G_4}{r^4},$$

$$B = \frac{r^4}{r^4 - 2Mr^3 + G_4}.$$
(29)

The solutions for a(r) and b(r) are

$$a = -\frac{8\zeta}{r^2(24c_2 + 8c_{13})c_{13}}\sqrt{-G_4c_{13}(3c_2 + c_{13})^2},$$

$$b = \frac{\epsilon r^4}{r^4 - 2Mr^3 + G_4} \sqrt{\frac{r^4 c_{13} - 2c_{13}Mr^3 + c_{13}G_4 - G_4}{r^4 c_{13}}},$$
(30)

where $\zeta = \pm 1$, hereinafter, $-G_4c_{13}(3c_2 + c_{13})^2 > 0$ and $(r^4c_{13} - 2c_{13}Mr^3 + c_{13}G_4 - G_4)/c_{13} > 0$, in order to ensure that the components of the aether vector are real. The first condition imposes that G_4 and c_{13} must have opposite signs, while the second one is identical to the Case C, changing E_4 by G_4 . Then, we have the same limit for G_4 in order to ensure that b be real, that is,

$$G_4 \ge \frac{27}{16} \left(\frac{c_{13}}{c_{13} - 1} \right) M^4. \tag{31}$$

Note that the solutions presented in this section depend explicitly on the parameters c_2 and c_{13} .

The Kretschmann scalar is given by

$$K = \frac{12(4M^2r^6 - 20MG_4r^3 + 39G_4^2)}{r^{12}}.$$
(32)

Note that r = 0 is the singularity of the spacetime.

The Killing horizon equation is given by

$$\chi^{\alpha}\chi_{\alpha} = -1 + \frac{2M}{r} - \frac{G_4}{r^4} = 0,$$
(33)

whose roots are

where $\eta_1 = \pm 1$, hereinafter,

$$\delta_{1} = \left(432G_{4}M^{2} + 12\sqrt{-768G_{4}^{3} + 1296G_{4}^{2}M^{4}}\right)^{\frac{1}{3}},$$

$$\delta_{2} = \sqrt{\frac{12M^{2}\delta_{1} + 2\delta_{1}^{2} + 96G_{4}}{\delta_{1}}}.$$
 (35)

The universal horizon equation is

$$\chi^{\alpha} u_{\alpha} = \epsilon \sqrt{\frac{r^4 c_{13} - 2c_{13} M r^3 + c_{13} G_4 - G_4}{r^4 c_{13}}},$$
 (36)

whose four solutions are

$$r_{uh1,2,3,4} = \frac{1}{2}M + \eta_1 \frac{\sqrt{3}}{12} \lambda_3 + \zeta \frac{1}{12} \\ \left[-(-72M^2 c_{13} \lambda_2 \lambda_3 + 6 \right] \\ \times 12^{\frac{1}{3}} \lambda_3 \lambda_2^2 - 24 \times 12^{\frac{2}{3}} \lambda_3 G_4 c_{13} \\ + 24 \times 12^{\frac{2}{3}} \lambda_3 G_4 c_{13}^2 \\ - 144\sqrt{3}M^3 c_{13} \lambda_2)(\lambda_2 \lambda_3)^{-1} \right]^{\frac{1}{2}},$$
(37)

where

$$\lambda_{1} = 36M^{2} + \sqrt{3}\sqrt{\frac{-256c_{13}G_{4} + 256G_{4} + 432M^{4}c_{13}}{c_{13}}},$$

$$\lambda_{2} = \left[(-1 + c_{13})G_{4}\lambda_{1}c_{13}^{2}\right]^{\frac{1}{3}},$$

$$\lambda_{3} = \sqrt{\frac{12M^{2}c_{13}\lambda_{2} + 212^{\frac{1}{3}}\lambda_{2}^{2} + 8G_{4}12^{\frac{2}{3}}c_{13}^{2} - 8G_{4}12^{\frac{2}{3}}c_{13}}{c_{13}\lambda_{2}}}.$$
(38)

In order to have a real aether vector we must impose that

$$c_{13} < 1,$$
 (39)

$$c_{13} < -2(1+c_2), \tag{40}$$

and

$$c_{13}G_4 \le 0.$$
 (41)

In order to have real Killing horizons we must impose that

$$G_4 \le \frac{27}{16} M^4, \tag{42}$$

and real universal horizons we must have

$$G_4 \ge \frac{27}{16} \left(\frac{c_{13}}{c_{13} - 1}\right) M^4.$$
(43)

From these two conditions we have that

$$G_4 = \frac{27}{16} \left(\frac{c_{13}}{c_{13} - 1} \right) M^4.$$
(44)

We can observe that the condition (44) satisfies simultaneously the conditions for a real aether vector.

Thus, using Eq. (44), the aether vector can be written as

$$a = \frac{-3\sqrt{3}\zeta M^2}{4r^2\sqrt{1-c_{13}}} \tag{45}$$

$$b = \frac{4\epsilon (c_{13} - 1) (-2r + 3M) r^2 \sqrt{3M^2 + 4Mr + 4r^2}}{(c_{13} - 1) 16r^3 (r - 2M) + 27c_{13}M^4}$$
(46)

Again, using Eq. (44), the universal horizons are given by

$$r_{uh1,2} = \frac{3M}{2},\tag{47}$$

$$r_{uh3,4} = -\frac{M}{2} + \eta_1 \frac{\sqrt{2}}{2} \sqrt{-M^2}.$$
(48)

Substituting the Eq. (44) into the Killing horizons, we get that they depend on c_{13} and the mass M. Thus, we plot the real Killing and universal horizons shown in the Fig. 1, for two different values of M = 1 and M = 2.

Since the outermost universal horizon is $r_{uh1} = r_{uh2}$, the surface gravity, temperature, entropy and the first law and using Eq. (26) we have

$$\kappa_{uh1} = -\frac{\epsilon \zeta \sqrt{6}}{9M} \frac{|(c_{13} + 3c_2) c_{13}|}{(c_{13} + 3c_2) c_{13}} \frac{1}{\sqrt{1 - c_{13}}},$$
(49)

$$T_{uh1} = -\frac{\epsilon \zeta \sqrt{6}}{18\pi M} \frac{|(c_{13} + 3c_2) c_{13}|}{(c_{13} + 3c_2) c_{13}} \frac{1}{\sqrt{1 - c_{13}}},$$
(50)

$$S_{uh1} = \frac{9\pi M^2}{4G},\tag{51}$$

$$\delta S_{uh1} = -\frac{3\sqrt{6\pi}}{\epsilon\zeta} \,\delta M \,M \frac{(c_{13} + 3c_2) \,c_{13}}{|(c_{13} + 3c_2) \,c_{13}|} \sqrt{1 - c_{13}}.$$
 (52)

Since the surface gravity must be positive, we have to choose $-\epsilon\zeta/c_{13}(c_{13}+3c_2) > 0$. The first law of the thermodynamics is calculated using Eq. (26), i.e., $\delta S_{uh1} = \delta M/T_{uh1}$.

6 Solutions for case (B): $c_2 \neq 0$ and $c_{13} = 0$ and $c_{14} = 0$

For the field equations (126)–(131) for $c_{13} = 0$ and $c_{14} = 0$ are given by

$$A = D_2 + D_3 r^2 + \frac{D_4}{r}, B = \frac{r}{r + D_4},$$
(53)

where the two solutions for a(r) and b(r) can be written as

$$a = \frac{D_1}{r^2},$$

$$b = \epsilon \sqrt{\frac{D_1^2 + r^4 + r^3 D_4}{r^2 (r + D_4)^2}},$$
(54)

where D_1 , D_2 , D_3 and D_4 are arbitrary integration constants and we have chosen $D_2 = 1$ and $D_3 = 0$ in order to have a flat spacetime at infinity. The choice $D_4 = -|D_4|$ is in order to have a resemblance with the Schwarzschild solution as in the GR and $|D_4| = 2M$, where *M* is the Schwarzschild mass. Thus, A(r), B(r) and b(r) can be rewritten as

$$A = 1 - \frac{2M}{r},$$

$$B = \frac{r}{r - 2M},$$

$$b = \epsilon \sqrt{\frac{D_1^2 + r^4 - 2r^3M}{r^2(r - 2M)^2}},$$
(55)

where $D_1^2 + r^4 - 2r^3M > 0$. Note that the solution presented in this section does not depend explicitly on the parameter c_2 , although $c_2 \neq 0$.

The Kretschmann scalar is given by

$$K = \frac{48M^2}{r^6}.$$
 (56)

We can note that r = 0 is the only singularity of this spacetime.

The Killing horizon equation is obtained from

$$\chi^{\alpha}\chi_{\alpha} = -1 + \frac{2M}{r} = 0, \qquad (57)$$

Fig. 1 These figures show the Killing and universal radii for the Case B where we have: r_{kh1} (blue dot-dashed line), r_{kh2} (green dashed line), r_{uh1} (violet long-dashed). The Killing and universal horizons that are not displayed in these figures are imaginaries

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whose solution is

$$r_{kh} = 2M. \tag{58}$$

Note that the Killing horizon coincides just as in GR.

The universal horizon equation is

$$\chi^{\alpha} u_{\alpha} = \epsilon \sqrt{\left(1 - \frac{2M}{r}\right) \left[\frac{D_1^2}{r^3(r - 2M)} + 1\right]} = 0,$$
 (59)

whose four solutions are

$$r_{uh1,2,3,4} = \frac{1}{2}M + \eta_1 \frac{\sqrt{3}}{12}\alpha_2 + \eta_2 \frac{\sqrt{6}}{12}$$

$$\sqrt{\frac{12M^2\alpha_1\alpha_2 - \alpha_2\alpha_1^2 - 48\alpha_2D_1^2 + 24\sqrt{3}M^3\alpha_1}{\alpha_1\alpha_2}},$$
(60)

where $\eta_2 = \pm 1$, hereinafter, and

$$\alpha_{1} = \left(432D_{1}^{2}M^{2} + 12\sqrt{-768D_{1}^{6} + 1296D_{1}^{4}M^{4}}\right)^{\frac{1}{3}},$$

$$\alpha_{2} = \sqrt{\frac{12M^{2}\alpha_{1} + 2\alpha_{1}^{2} + 96D_{1}^{2}}{\alpha_{1}}}.$$
(61)

Since α_1 must be real we have that $D_1 \le \sqrt{1296/768}M^2$. Besides, we must have that $D_1^2 + r^4 - 2r^3M \ge 0$, in order to *b* be real, as pointed out after (55). These two conditions simultaneously applied impose that

$$D_1 = \sqrt{\frac{27}{16}}M^2.$$
 (62)

Thus, the aether vector is given by

$$a = \frac{3\sqrt{3}M^2}{4r^2}$$
(63)

$$b = \frac{\epsilon \left|-2r + 3M\right| \sqrt{3M^2 + 4Mr + 4r^2}}{4r \left|-r + 2M\right|}$$
(64)

The Killing and universal horizons are given by

$$r_{uh1,2} = \frac{3M}{2},$$
(65)

$$r_{uh3,4} = -\frac{M}{2} + \eta_1 \frac{\sqrt{2}}{2} \sqrt{-M^2}.$$
(66)

The only real universal horizons are r_{uh1} and r_{uh2} . Since the outermost horizon is $r_{uh1} = 3M/2$, we get for the surface gravity, temperature, entropy and the first law, using Eq. (26), thus

$$\kappa_{uh1} = \frac{\sqrt{6}\epsilon}{9M},\tag{67}$$

$$T_{uh1} = \frac{\sqrt{6\epsilon}}{18M\pi},\tag{68}$$

$$S_{uh1} = \frac{9\pi \ M^2}{4G},\tag{69}$$

$$\delta S_{uh1} = \frac{3\sqrt{6}\,\delta M\,M\pi}{\epsilon}.\tag{70}$$

Since the surface gravity must be positive, we have to choose $\epsilon = 1$. The first law of the thermodynamics is calculated using Eq. (26), i.e., $\delta S_{uh1} = \delta M / T_{uh1}$.

7 Solutions for case (C): $c_2 = 0$ and $c_{13} \neq 0$ and $c_{14} = 0$

The solution of the field equations (126)–(131) for $c_2 = 0$ and $c_{14} = 0$ we get,

$$A = E_1 + \frac{E_2}{r^4} + E_3 r^2 + \frac{E_4}{r},$$

$$B = \frac{1}{1 + \frac{E_2}{r^4} + \frac{E_4}{r}},$$
(71)

where the solutions of a(r) and b(r) are given by

$$a = \zeta \sqrt{-\frac{E_2}{c_{13} r^4}},$$

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$$b = \epsilon \sqrt{\frac{r^4 \left(-E_2 + c_{13}r^4 + c_{13}E_2 - 2c_{13}Mr^3\right)}{c_{13} \left(r^4 + E_2 - 2Mr^3\right)^2}},$$
(72)

where E_1 , E_2 , E_3 and E_4 are arbitrary integration constants we have chosen $E_1 = 1$ and $E_3 = 0$ in order to have a flat spacetime at infinity and $E_4 = -|E_4|$ in order to have a resemblance with the Schwarzschild solution as in the GR and $|E_4| = 2M$, where *M* is the Schwarzschild mass. Thus, A(r), B(r) and b(r) can be rewritten as

$$A = 1 + \frac{E_2}{r^4} - \frac{2M}{r},$$

$$B = \frac{1}{1 + \frac{E_2}{r^4} - \frac{2M}{r}},$$

$$b = \epsilon \sqrt{\frac{r^4 \left(-E_2 + c_{13}r^4 + c_{13}E_2 - 2c_{13}Mr^3\right)}{c_{13} \left(r^4 + E_2 - 2Mr^3\right)^2}},$$
(73)

where $-E_2/c_{13} > 0$ and $(-E_2 + c_{13}r^4 + c_{13}E_2 - 2c_{13}Mr^3)/c_{13} > 0$. The first condition is imposed in order to have *a* real, while the second one ensures that *b* be real. The last one is not obvious but it can be found if we analyze the first and second derivatives of the expression inside the parenthesis, beyond it behavior in the limits for $r \to \pm \infty$ and the position of the minimum of the function.

Note that the solutions presented in this section depend explicitly on the parameter c_{13} . Besides, the term r^{-4} is interesting because it could be compared in GR to the Hartle and Thorne [48] solution of a slowly rotating deformed relativistic star, assuming the quadrupole moment is zero.

The Kretschmann scalar is given by

$$K = \frac{12(39E_2^2 - 20E_2Mr^3 + 4M^2r^6)}{r^{12}}.$$
(74)

Notice again that r = 0 is the only singularity.

The Killing horizon equation is given by

$$\chi^{\alpha}\chi_{\alpha} = -1 - \frac{E_2}{r^4} + \frac{2M}{r} = 0,$$
(75)

whose four roots are

$$r_{kh1,2,3,4} = \frac{1}{2}M + \eta_1 \frac{\sqrt{3}}{12}\beta_2 + \eta_2 \frac{1}{12}$$
$$\sqrt{-\frac{-72M^2\beta_1\beta_2 + 6\beta_2\beta_1^2 + 288\beta_2E_2 - 144\sqrt{3}M^3\beta_1}{\beta_1\beta_2}},$$
(76)

where

$$\beta_1 = \left(432E_2M^2 + 12\sqrt{-768E_2^3 + 1296E_2^2M^4}\right)^{\frac{1}{3}},$$

$$\beta_2 = \sqrt{\frac{12M^2\beta_1 + 2\beta_1^2 + 96E_2}{\beta_1}}.$$
 (77)

The universal horizon is

$$\chi^{\alpha}u_{\alpha} = \epsilon \sqrt{-\frac{E_2 - c_{13}r^4 - c_{13}E_2 + 2c_{13}Mr^3}{c_{13}r^4}} = 0,$$
(78)

whose four solutions are

$$\begin{aligned} \dot{x}_{uh1,2,3,4} &= \frac{1}{2}M + \eta_1 \frac{\sqrt{3}}{12} \gamma_3 \\ &+ \eta_2 \frac{\sqrt{6}}{12\sqrt{\gamma_2\gamma_3}} \left[12M^2 c_{13}\gamma_2\gamma_3 - 12^{\frac{1}{3}}\gamma_3\gamma_2^2 \right. \\ &+ 4 \times 12^{\frac{2}{3}}\gamma_3 E_2 c_{13} - 4 \times 12^{\frac{2}{3}}\gamma_3 E_2 c_{13}^2 \\ &+ 24\sqrt{3}M^3 c_{13}\gamma_2 \right]^{\frac{1}{2}}, \end{aligned}$$
(79)

where

r

$$\begin{aligned} \gamma_1 &= 36M^2 + \sqrt{3}\sqrt{\frac{256E_2 - 256c_{13}E_2 + 432M^4c_{13}}{c_{13}}}, \\ \gamma_2 &= \left[(-1 + c_{13})E_2\gamma_1c_{13}^2\right]^{\frac{1}{3}}, \\ \gamma_3 &= \sqrt{\frac{12M^2c_{13}\gamma_2 + 2 \times 12^{\frac{1}{3}}\gamma_2^2 - 8 \times 12^{\frac{2}{3}}E_2c_{13} + 8 \times 12^{\frac{2}{3}}E_2c_{13}^2}{c_{13}\gamma_2}}. \end{aligned}$$

$$\end{aligned}$$

$$(80)$$

In order to have real Killing horizons we must impose that

$$E_2 \le \frac{27}{16}M^4,\tag{81}$$

and real universal horizons we must have

$$E_2 \ge \frac{27}{16} \left(\frac{c_{13}}{c_{13} - 1} \right) M^4.$$
(82)

From these two conditions we have that

$$E_2 = \frac{27}{16} \left(\frac{c_{13}}{c_{13} - 1} \right) M^4.$$
(83)

Thus, using Eq. (83), the aether vector can be written as

$$a = \frac{3\sqrt{3}\zeta M^2}{4r^2\sqrt{1-c_{13}}}$$
(84)

$$b = \frac{4\epsilon (c_{13} - 1) (-2r + 3M) r^2 \sqrt{3M^2 + 4Mr + 4r^2}}{(c_{13} - 1) 16r^3 (r - 2m) + 27c_{13}M^4}$$
(85)

Again, using Eq. (83), the universal horizons are given by

$$r_{uh1,2} = \frac{3M}{2},\tag{86}$$

$$r_{uh3,4} = -\frac{M}{2} + \eta_1 \frac{\sqrt{2}}{2} \sqrt{-M^2}.$$
(87)

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Substituting the Eq. (83) into the Killing horizons, we get that they depend on c_{13} and the mass *M*. Thus, we plot the

We can notice again that r = 0 is the only singularity. The solutions for a(r) and b(r) are

$$a = \frac{\zeta}{c_{14}r} \sqrt{c_{14}(2Mc_{14}r - c_{14}r^2 + F_3r^2F_4^2c_{14}^2 + 2\sqrt{2}F_3F_4c_{14}r - F_3c_{14} + 2F_3)},$$

$$b = \frac{\epsilon r^2}{(r^2 - 2Mr + F_3)} \sqrt{\frac{F_3(r^2F_4^2c_{14}^2 + 2\sqrt{2}F_4c_{14}r + 2)}{c_{14}r^2}},$$
(95)

real Killing and universal horizons shown in the Fig. 2, for two different values of M = 1 and M = 2.

Since from Fig. 2 the outermost universal horizon is r_{uh1} , the surface gravity, temperature, entropy and the first law and using Eq. (26) we have

$$\kappa_{uh1} = \frac{\sqrt{6}}{9M\sqrt{1 - c_{13}}},\tag{88}$$

$$T_{uh1} = \frac{\sqrt{6}}{18\pi M \sqrt{1 - c_{13}}},\tag{89}$$

$$S_{uh1} = \frac{9\pi M^2}{4G},$$
(90)

$$\delta S_{uh1} = 3\sqrt{6}\,\pi\delta M\,M\sqrt{1-c_{13}}.$$
(91)

Since the surface gravity must be positive and real, we have to choose $\epsilon = \eta = 1$ and $c_{13} < 1$. The first law of the thermodynamics is calculated using Eq. (26), i.e., $\delta S_{uh1} = \delta M/T_{uh1}$.

8 Solutions for case (D): $c_2 = 0$ and $c_{13} = 0$ and $c_{14} \neq 0$

The solution of the field equations (126)–(131) for $c_2 = 0$ and $c_{13} = 0$ we get,

$$A = F_1 + \frac{F_2}{r} + \frac{F_3}{r^2}, \quad B = \frac{r^2}{r^2 + F_2 r + F_3},$$
(92)

where F_1 , F_2 , F_3 and F_4 are arbitrary integration constants we have chosen $F_1 = 1$ in order to have a flat spacetime at infinity and $F_2 = -|F_2|$ in order to have a resemblance with the Schwarzschild solution as in the GR and $|F_2| = 2M$, where *M* is the Schwarzschild mass. Thus, A(r) and B(r)can be rewritten as

$$A = 1 - \frac{2M}{r} + \frac{F_3}{r^2}, \quad B = \frac{r^2}{r^2 - 2Mr + F_3}.$$
 (93)

The metric of this case can be associated to the Reissner– Nordström spacetime in GR, identifying F_3 with the electric charge.

The Kretschmann scalar is given by

$$K = \frac{4(12M^2r^2 - 24MF_3r + 14F_3^2)}{r^8}.$$
(94)

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where $(2Mr - c_{14}r^2 + F_3r^2F_4^2c_{14}^2 + 2\sqrt{2}F_3F_4c_{14}r - F_3c_{14} + 2F_3) > 0$ and $F_3(r^2F_4^2c_{14}^2 + 2\sqrt{2}F_4c_{14}r + 2) > 0$, in order to have *a* and *b* real.

Looking at the term under the square root at *b*, we can see that it vanishes only at $r = -\frac{\sqrt{2}}{F_4C_{14}}$, which coincides with the minimum or maximum of the function, depending on whether the sign of F_4 is positive or negative, respectively. If $F_4 > 0$, the minimum occurs for r < 0, while if $F_4 < 0$, the maximum occurs for r > 0. Thus, to ensure that *b* is real for every value of $r \ge 0$, we must choose $F_3 > 0$ and $F_4 > 0$. Doing a similar analysis of the term under the square root in *a* we see that it has a minimum, that occurs for some r < 0 or a maximum, that occurs for some r > 0, depending on whether $F_3 > \frac{1}{F_4^2c_{14}}$ or if $F_3 < \frac{1}{F_4^2c_{14}}$, respectively. Imposing, as for *b*, that *a* is real for all $r \ge 0$, the only possible option is $F_3 > \frac{1}{F_4^2c_{14}} \cdot S_0$, the existence of the aether vector field imposes $F_4 > 0$ and $F_3 > \frac{1}{F_4^2c_{14}}$.

We can factorize these aether component equations giving

$$a = \frac{\epsilon}{r} \sqrt{\frac{F_3(c_{14}F_4r + \sqrt{2})^2 - c_{14}(r^2 - 2Mr + F_3)}{c_{14}}},$$

$$b = \frac{\epsilon r(c_{14}F_4r + \sqrt{2})}{(r^2 - 2Mr + F_3)} \sqrt{\frac{F_3}{c_{14}}}.$$
 (96)

Note that the solutions presented in this section depend explicitly on the parameter c_{14} .

The Killing horizon equation is

$$\chi^{\alpha}\chi_{\alpha} = -1 + \frac{2M}{r} - \frac{F_3}{r^2} = 0, \qquad (97)$$

whose two roots are

$$r_{kh1,2} = M + \eta_1 \sqrt{M^2 - F_3},\tag{98}$$

where $F_3 \leq M^2$, establishing an upper limit for the constant F_3 , to insure that exist Killing horizons. Then, the solution for this case is restricted to values of F_3 and F_4 such that $F_4 > 0$ and

$$\frac{1}{F_4{}^2c_{14}} < F_3 \le M^2. \tag{99}$$

The universal horizon equation is given by

Fig. 2 These figures show the Killing and universal radii for the Case C, where we have: r_{kh1} (blue dot-dashed line), r_{kh2} (green dashed line), r_{uh1} (violet long-dashed). The Killing and universal horizons that are not displayed in these figures are imaginaries



metric. See Figs. 3, 4 and 5. From the Figs. 3, 4 and 5, as

$$\chi^{\alpha} u_{\alpha} = \epsilon \sqrt{1 - \frac{2M}{r} + \frac{F_3}{r^2}} \times \sqrt{\frac{2c_{14}Mr - c_{14}r^2 + F_3r^2F_4^2c_{14}^2 + 2\sqrt{2}F_3F_4c_{14}r - F_3c_{14} + 2F_3}{c_{14}(r^2 - 2Mr + F_3)}} + 1 = 0,$$
(100)

whose three solutions are

$$r_{uh1} = -\frac{\sqrt{2}}{F_4 c_{14}},$$

$$r_{uh2,3} = M + \eta_1 \sqrt{M^2 - F_3}.$$
 (101)

As we saw earlier, F_4 must be positive leading to $r_{uh1} < 0$ and therefore this does not correspond to a real horizon. Thus, we have that $r_{uh2,3}$ are the outermost universal horizon, if $F_3 \le M^2$, and we have for the surface gravity, temperature, entropy and the first law, using Eq. (26)

$$\kappa_{uh2} = \frac{F_3 \left[2 + \sqrt{2}c_{14}F_4 \left(M + \sqrt{M^2 - F_3} \right) \right]}{2c_{14} \left(M + \sqrt{M^2 - F_3} \right)^3} \tag{102}$$

$$T_{uh2} = \frac{F_3 \left[2 + \sqrt{2}c_{14}F_4 \left(M + \sqrt{M^2 - F_3} \right) \right]}{4\pi c_{14} \left(M + \sqrt{M^2 - F_3} \right)^3}$$
(103)

$$S_{uh2} = \frac{\pi \left(M + \sqrt{M^2 - F_3}\right)^2}{G},$$
 (104)

$$\delta S_{uh2} = \frac{4\pi c_{14} \left(M + \sqrt{M^2 - F_3} \right)^3 \delta M}{F_3 \left[2 + \sqrt{2} c_{14} F_4 \left(M + \sqrt{M^2 - F_3} \right) \right]}.$$
 (105)

The first law of the thermodynamics is calculated using Eq. (26), i.e., $\delta S_{uh2} = \delta M / T_{uh2}$.

Then, when we establish the condition $F_3 < M^2$, this is in agreement with the solution of the Reissner–Nordström already pointed out, we can see that for values of $F_3 > M^2$ we do not have any horizon, thus, we have naked singularities.

9 Solutions for case (E): $c_2 = -c_{13} \neq 0$ and $c_{14} \neq 0$

The solution of the field equations (126)–(131) for $c_2 = -c_{13}$ we get,

$$A = H_2 + \frac{H_4}{r} + \frac{H_3}{r^2},$$

$$B = \frac{H_1}{H_2 + \frac{H_4}{r} + \frac{H_3}{r^2}},$$
(106)

where H_1 , H_2 , H_3 and H_4 are arbitrary integration constants we have chosen $H_1 = H_2 = 1$ in order to have a flat spacetime at infinity and $H_4 = -2M$ in order to have a resemblance with the Schwarzschild solution, where *M* is the Schwarzschild mass.

Thus, A(r) and B(r) can be rewritten as

$$A = 1 - \frac{2M}{r} + \frac{H_3}{r^2},$$

$$B = \frac{1}{1 - \frac{2M}{r} + \frac{H_3}{r^2}}.$$
(107)

The solutions for a(r) and b(r) are

$$a = \frac{\epsilon}{r} \sqrt{\frac{2\sqrt{2}\zeta r \Delta + 2Mr \left(2c_{13} - c_{14}\right) + H_3 \left(c_{14} - 2\right)}{\left(2c_{13} - c_{14}\right)}}$$

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Fig. 3 These figures show the Killing and universal radii for the Case D where we have: r_{kh1} or r_{uh1} (blue dot-dashed line), r_{kh2} or r_{uh2} (green dashed line), r_{kh3} or r_{uh3} (black dotted line). The violet long-dashed straight lines represent the inferior and superior limits of F_3 . The horizons that are not displayed in these figures are imaginaries. The gray areas are the regions where the condition (99) is valid



$$b = \frac{\epsilon r \left(\sqrt{2H_3(-1-c_{13})} + \zeta r \sqrt{(2c_{13}-c_{14})}\right)}{\left(2Mr - r^2 - H_3\right)\sqrt{(2c_{13}-c_{14})}},$$
 (108)

where $\Delta = \sqrt{H_3 (-1 + c_{13}) (2c_{13} - c_{14})}$ and $H_3 (-1 + c_{13})$

 $(2c_{13} - c_{14}) > 0$, in order to ensure that the components of

the aether vector are real. Note that the solutions presented in this section depend explicitly on the parameters c_{13} and The Kretschmann scalar is given by

$$K = \frac{8(7H_3^2 - 12H_3Mr + 6M^2r^2)}{r^8}.$$
 (109)

Note that r = 0 is the singularity of the spacetime.

The Killing horizon equation is given by

$$\chi^{\alpha}\chi_{\alpha} = -1 + \frac{2M}{r} - \frac{H_3}{r^2} = 0, \qquad (110)$$

 $c_{14}.$

Fig. 4 These figures show the Killing and universal radii for the Case D where we have: r_{kh1} or r_{uh1} (blue dot-dashed line), r_{kh2} or r_{uh2} (green dashed line), r_{kh3} or r_{uh3} (black dotted line). The violet long-dashed straight lines represent the inferior and superior limits of F_3 . The horizons that are not displayed in these figures are imaginaries. The gray areas are the regions where the condition (99) is valid



where $H_3 \leq M^2$.

whose roots are

The universal horizon equation is

Fig. 5 These figures show the universal horizon temperatures, for the Case D, where we have: T_{uh1} (blue dot-dashed line) and T_{GR} (green dashed line). The violet long-dashed straight lines represent the inferior and superior limits of F_3 . The gray areas are the regions where the condition (99) is valid



whose four solutions are

$$r_{uh1,2} = \eta_1 \sqrt{\frac{2H_3\left(-1+c_{13}\right)}{2c_{13}-c_{14}}},$$
(113)

Since the outermost universal horizon is r_{uh1} , the surface gravity, temperature, entropy and the first law and using Eq. (26) we have

$$\begin{aligned} \kappa_{uh1} &= \frac{\sqrt{2}}{8} \sqrt{\frac{\left(-2c_{13}H_3 + 2c_{13}M\sqrt{2}\Delta_1 + c_{14}H_{3c_{13}} + 4|c_{13}(-1+c_{13})H_3|\right)c_{13}}{(-1+c_{13})H_3}} \\ &\times \sqrt{c_{13}^2H_3 - c_{13}H_3 + |c_{13}(-1+c_{13})H_3|} \left|\frac{2c_{13} - c_{14}}{c_{13}(-1+c_{13})H_3}\right| \\ &\times \frac{1}{\sqrt{c_{13}(2c_{13} - c_{14})}}, \end{aligned}$$
(114)
$$\begin{aligned} T_{uh1} &= \frac{\sqrt{2}}{16\pi} \\ &\times \sqrt{\frac{\left(-2c_{13}H_3 + 2c_{13}M\sqrt{2}\Delta_1 + c_{14}H_3c_{13} + 4|c_{13}(-1+c_{13})H_3|\right)c_{13}}{(-1+c_{13})H_3}} \\ &\times \sqrt{c_{13}^2H_3 - c_{13}H_3 + |c_{13}(-1+c_{13})H_3|} \left|\frac{2c_{13} - c_{14}}{c_{13}(-1+c_{13})H_3}\right| \\ &\times \frac{1}{\sqrt{c_{13}(2c_{13} - c_{14})}}, \end{aligned}$$
(115)
$$\begin{aligned} S_{uh1} &= \frac{2\pi H_3 (-1+c_{13})}{G (2c_{13} - c_{14})}, \\ \delta S_{uh1} &= 8\pi \,\delta M \,\sqrt{2} \left|\frac{c_{13}(-1+c_{13})H_3}{2c_{13} - c_{14}}\right| \sqrt{c_{13}(2c_{13} - c_{14})} \\ &\times \frac{1}{\sqrt{\left(-2c_{13}H_3 + 2c_{13}M\sqrt{2}\Delta_1 + c_{14}H_{3c_{13}} + 4|c_{13}(-1+c_{13})H_3|\right)}c_{13}} \\ &\times \frac{1}{\sqrt{c_{12}^2H_3 - c_{13}H_3 + |c_{13}(-1+c_{13})H_3|}}, \end{aligned}$$
(115)

where $\epsilon = \zeta = 1$. The first law of the thermodynamics is calculated using Eq. (26), i.e., $\delta S_{uh1} = \delta M / T_{uh1}$.

10 Solutions for case (F): $c_2 = 0$ and $c_{13} \neq 0$ and $c_{14} \neq 0$

The solution of the field equations (126)–(131) for $c_2 = 0$ is given by a(r) = 0, i.e., the radial component of the aether vector is null, thus, we have a static aether. Therefore, we have not considered this case.

11 Solutions for case (G): $c_2 \neq 0$ and $c_{13} = 0$ and $c_{14} \neq 0$

The solution of the field equations (126)–(131) for $c_{13} = 0$ it is also given by a(r) = 0, i.e., the radial component of the aether vector is null, thus, again we have a static aether. Therefore, we have not considered this case.

12 Solutions for case (H): $c_2 = -c_{13} \neq 0$ and $c_{14} = 0$

The solution of the field equations (126)–(131) for Thus, A(r) and B(r) can be rewritten as

$$A = \frac{J_2}{r},$$

$$B = J_1 r.$$
(118)

The solutions for a(r) and b(r) are

$$a = \frac{\sqrt{c_{13} r (r + J_3 c_{13})}}{c_{13} r},$$

$$b = \sqrt{\frac{r (J_1 r + J_1 J_3 c_{13} + c_{13})}{J_2 c_{13}}},$$
(119)

where J_1 , J_2 and J_3 are arbitrary integration constants. This solution does not have a flat spacetime at infinity. Therefore, we have also not considered this case.

13 Conclusions

In the present work, we analyze several spherically symmetric exterior vacuum solutions allowed by the Einstein-Aether

Table 1 Summary of the Aether conditions						
Solution	<i>c</i> ₂	c_{13}	c_{14}	c _i		
A			0	$c_{13} < 1$ and $2 + c_{13} + 2c_2 >$		
В		0	0	$c_2 > -2/3$		
С	0		0	$-2 < c_{13} < 1$		
D	0	0		$0 < c_{14} < 2$		
F				$a_{2} = a_{12} \neq 0$		

Ŀ $-c_{13} \neq 0$

Here c_i (i=1..4) are the parameters of the aether field. In order to have a theory that obeys the solar system tests, we must have $0 \le c_{14} < 2$, $c_{13} < 1$ and $2 + c_{13} + 2c_2 > 0$ [8,49]

(EA) theory with a non-static aether. We show that there are five classes of solutions corresponding to different values of a combination of the free parameters, c_2 , $c_{13} = c_1 + c_3$

Table 2 Summary of the solutions

and $c_{14} = c_1 + c_4$, which are: (A) $c_{14} = 0$ (B) $c_{13} = 0$ and $c_{14} = 0$, (C) $c_2 = 0$ and $c_{14} = 0$, (D) $c_2 = 0$ and $c_{13} = 0$, and (E) $c_2 = -c_{13} \neq 0$. We present explicit analytical solutions for these five cases. The cases where only $c_2 = 0$ or only $c_{13} = 0$ are not analytic solutions. All these cases present singularities at r = 0 and are asymptotically flat spacetimes, and posses both Killing, and universal horizons. We call attention to the fact that in all the cases presented here, we have several solutions of the aether vector field for the same spacetime. This means that the geometry of the spacetime, defined by the metric, is not sensitive to different aether fields of the same spacetime. Also, it should be noted that all the solutions presented in this paper depend explicitly on some of the aether parameters except the solution (B).

Case	Metric functions	Aether components	Horizons	Ob
(<i>A</i>)	$A = 1 - \frac{2M}{r} + \frac{G_4}{r^4}$	$a = \frac{-3\sqrt{3}\zeta M^2}{4r^2\sqrt{1-c_{13}}}$		(1)
	$B = \left(1 - \frac{2M}{r} + \frac{G_4}{r^4}\right)^{-1}$		$r_{uh} = \frac{3M}{2}$	
	$G_4 = \frac{27}{16} \left(\frac{c_{13}}{c_{13} - 1} \right) M^4$	$b = \frac{4\epsilon(c_{13}-1)(-2r+3M)r^2\sqrt{3M^2+4Mr+4r^2}}{(c_{13}-1)16r^3(r-2m)+27c_{13}M^4}$		
(<i>B</i>)	$A = 1 - \frac{2M}{r}$	$a = \frac{3\sqrt{3}M^2}{4r^2}$	$r_{uh} = \frac{3M}{2}$	(2)
	$B = \left(1 - \frac{2M}{r}\right)^{-1}$	$b = \frac{\epsilon(-2r+3M)\sqrt{3M^2+4Mr+4r^2}}{4r(-r+2M)}$		
(<i>C</i>)	$A = 1 + \frac{E_2}{r^4} - \frac{2M}{r}$	$a = \frac{3\sqrt{3}\zeta M^2}{4r^2\sqrt{1-c_{13}}}$		
	$B = \left(1 + \frac{E_2}{r^4} - \frac{2M}{r}\right)^{-1}$		$r_{uh} = \frac{3M}{2}$	
	$E_2 = \frac{27}{16} \left(\frac{c_{13}}{c_{13} - 1} \right) M^4$	$b = \frac{4\epsilon(c_{13}-1)(-2r+3M)r^2\sqrt{3M^2+4Mr+4r^2}}{(c_{13}-1)16r^3(r-2m)+27c_{13}M^4}$		
(<i>D</i>)	$A = 1 - \frac{2M}{r} + \frac{F_3}{r^2}$	$a = \frac{\zeta}{r} \sqrt{\frac{F_3(c_{14}F_4r + \sqrt{2})^2 - c_{14}(r^2 - 2Mr + F_3)}{c_{14}}}$	$r_{uh} = M +$	
	$B = \left(1 - \frac{2M}{r} + \frac{F_3}{r^2}\right)^{-1}$	$b = \frac{\epsilon r (c_{14}F_4r + \sqrt{2})}{(r^2 - 2Mr + F_3)} \sqrt{\frac{F_3}{c_{14}}}$	$\sqrt{M^2-F_3}$	
	$rac{1}{F_4{}^2c_{14}} < F_3 \le M^2$	$F_4 > 0$		
	$A = 1 - \frac{2M}{r} + \frac{M^2}{r^2}$	$a = \frac{\zeta}{r} \sqrt{\frac{M^2(c_{14}F_4r + \sqrt{2})^2 - c_{14}(r^2 - 2Mr + M^2)}{c_{14}}}$	$r_{uh} = M$	
	$B = \left(1 - \frac{2M}{r} + \frac{M^2}{r^2}\right)^{-1}$	$b = \frac{\epsilon r (c_{14}F_4r + \sqrt{2})}{(r^2 - 2Mr + M^2)} \sqrt{\frac{M^2}{c_{14}}}$		
	$F_3 = M^2$			
(<i>E</i>)	$A = 1 - \frac{2M}{r} + \frac{H_3}{r^2}$	$a = \frac{\epsilon}{r} \sqrt{\frac{2\sqrt{2}\zeta r \Delta + 2Mr(2c_{13} - c_{14}) + H_3(c_{14} - 2)}{(2c_{13} - c_{14})}}$	$r_{uh} =$	
	$B = \left(1 - \frac{2M}{r} + \frac{H_3}{r^2}\right)^{-1}$		$\sqrt{\frac{2H_3(-1+c_{13})}{(2c_{13}-c_{14})}}$	
	$H_3(-1+c_{13}) \times$	$b = \frac{\epsilon r \left(\sqrt{2H_3(-1-c_{13})} + \zeta r \sqrt{(2c_{13}-c_{14})}\right)}{(2Mr-r^2-H_3)\sqrt{(2c_{13}-c_{14})}}$		
	$(2c_{13} - c_{14}) > 0$			
	& $H_3 \leq M^2$	$\Delta = \sqrt{H_3 \left(-1 + c_{13}\right) \left(2c_{13} - c_{14}\right)}$		
	$A = 1 - \frac{2M}{r} + \frac{M^2}{r^2}$	$a = \frac{\epsilon}{r} \sqrt{\frac{(\sqrt{2M^2} - r\sqrt{c_{14}})^2 - c_{14}(r^2 - 2Mr + M^2)}{c_{14}}}$	$r_{uh} = M \sqrt{\frac{2}{c_{14}}}$	
	$B = \left(1 - \frac{2M}{r} + \frac{M^2}{r^2}\right)^{-1}$	$b = \frac{\epsilon r}{2Mr - r^2 - M^2} \left(\sqrt{\frac{2M^2}{c_{14}}} + r \right)$		
	$c_{13} = 0 \& H_3 = M^2$			

0

(1) Notice that $G_4 \equiv E_2$ of the Case (C). (2) $c_2 \neq 0$, yet black hole thermodynamics is not exactly the same as in GR, where $r_{uh} = 2M$

We have shown that the universal horizons are always situated inside than the Killing horizons. We have also computed the surface gravity, the temperature, the entropy, and the first law of thermodynamics for the outermost universal horizons. The temperature of a black hole in EA theory can be higher or lower than in GR. In Case (B), the temperature is always lower than GR. However, in the Cases (A), (C), (D) and (E) depend on the values of c_{13} , c_{14} or c_2 . Besides, the Case (D) also depends on two arbitrary constants (F_3 and F_4) [see Table 3]. We also notice that the temperature tends to $+\infty$ when $c_{13} \rightarrow 1$ in Cases (C) and (E), and when $F_3 \rightarrow M^2$ in Case (D) and also when $c_{13} \rightarrow -3c_2$ in the Case (E). See Fig. 5 for the details.

As with temperature, the entropy of the EA black hole can also be higher or lower than in GR. In Cases (A), (B), and (C) the entropy is lower than the GR. However, in Case (D) this quantity depends on the values of c_{14} and of the arbitrary constant F_3 , with, c_{14} causing entropy to increase and F_3 causing it to decrease, in comparison with the GR (see Table 3). The lowest value of the entropy is for the values $c_{14} = 0$ and $F_3 = M^2$, giving $S = \pi M^2$. We can also notice that the entropy tends to $+\infty$ when $c_{14} \rightarrow 2$ in this case.

Finally, we want compare our results in Schwarzschild coordinates with those of the references [20,34,43,44] in Eddington–Finkelstein coordinates, and show that our results are new. In order to compare, we first transform their results in Eddington–Finkelstein coordinates into Schwarzschild coordinates. From the Eddington–Finkelstein metric, we have

$$ds^{2} = -e(r)dv^{2} + 2f(r)drdv + r^{2}d\theta^{2} + r^{2}\sin^{2}\theta d\phi^{2},$$
(120)

with the aether vector given by

$$u^{a} = [\alpha_{1}(r), \beta_{1}(r), 0, 0], \qquad (121)$$

Table 3 Summary of the temperature and entropy

Case	Temperature	Entropy	Notes
(A)	$\frac{\sqrt{6}}{18\pi M\sqrt{1-c_{13}}}$	$\frac{9\pi M^2}{4}$	
(<i>B</i>)	$\frac{\sqrt{6}}{18M\pi}$	$\frac{9\pi M^2}{4}$	
(<i>C</i>)	$\frac{\sqrt{6}}{18\pi M\sqrt{1-c_{13}}}$	$\frac{9\pi M^2}{4}$	
(<i>D</i>)	$\frac{F_3 \Big[2 + \sqrt{2}c_{14}F_4 \Big(M + \sqrt{M^2 - F_3} \Big) \Big]}{4\pi c_{14} \Big(M + \sqrt{M^2 - F_3} \Big)^3}$	$\frac{2\pi \left(M + \sqrt{M^2 - F_3}\right)^2}{2 - c_{14}}$	(1)
	$\frac{2+\sqrt{2}F_{4}c_{14}M}{4\pi c_{14}M}$	$\frac{2\pi M^2}{2-c_{1A}}$	(2)
(<i>E</i>)	$\frac{1}{8\pi} \left \frac{\sqrt{2c_{13}-c_{14}}}{(-1+c_{13})H_3} \right \times$	$\frac{2\pi H_3(-1+c_{13})}{G(2c_{13}-c_{14})}$	(3)
	$\sqrt{2\sqrt{2}M\Delta + H_3(c_{14}-2) + 4H_3(-1+c_{13})}$		
	$\frac{\sqrt{c_{14}(6-c_{14}-2\sqrt{2c_{14}})}}{8\pi}$	$\frac{2\pi M^2}{c_{14}G}$	(4)

(1) See also the Fig. 5. Notice that G for the Cases (A), (B) and (C) is equal to $G_N = 1$, since $c_{14} = 0$. (2) Solution (D) assuming $F_3 = M^2$. (3) $\Delta = \sqrt{(2c_{13} - c_{14})(-1 + c_{13})H_3}$ (4) assuming $H_3 = M^2$ and $c_{13} = 0$

Table 4 Summary of the results of Eddington–Finkelstein coordinates transformed into Schwarzschild coordinates [20,34,43,44]

Cases	$I(c_{14} = 0)$	II $(c_{123} = 0)$
Metric functions	$A = 1 - \frac{2M}{r} - \frac{E_2}{r^4}$	$A = 1 - \frac{2M}{r} + \frac{r_u(2M + r_u)}{r^2}$
	$B = \frac{1}{A}$	$B = \frac{1}{A}$
Aether components	$a = -\frac{\sqrt{E_2}}{r^2\sqrt{c_{13}}}$	$a = -\frac{M+r_u}{r}$
	$b = \frac{\sqrt{r^4[c_{13}(r^4 - 2Mr^3) + E_2(1 - c_{13})]}}{\sqrt{c_{13}(r^4 - 2r^3M - E_2)}}$	$b = \frac{(r-M)r}{r^2 + 2Mr - 2r_u M - r_u^2}$
Universal Horizon	$r_{uh} = \frac{3M}{2}$	$r_{uh} = M$
Temperature	$T = \frac{\sqrt{6}}{18M\pi\sqrt{1-c_{13}}}$	$T = \frac{\sqrt{2}\sqrt{2-c_{14}}}{8M\pi}$
Entropy	$S = \frac{9\pi M^2}{4}$	$S = \frac{2\pi M^2}{2-c_{14}}$

Note that, using the notation of the references [43,44], we have the following relations: $e(r) \equiv A$; $B(r) \equiv e(r)^{-1}$; $r_0 \equiv 2M$; $r_{a}^4 = \frac{27}{16} \left(\frac{c_{13}}{1 - c_{13}} \right) M^4 \equiv r_{a}$

 $\frac{E_2}{c_{13}}$; $r_u = M\sqrt{1 - \frac{c_{14}}{2}}$, that help to make the comparison. In reference [20], the authors assume null charge. The universal horizons are reobtained from the transformed metrics and aether vectors

when normalized we get

$$u^{a} = \left[\alpha_{1}(r), \frac{e(r)\alpha_{1}(r)^{2} - 1}{2f(r)\alpha_{1}(r)}, 0, 0\right].$$
 (122)

We can make a coordinate transformation (dv = dt + dr/e(r)) and assuming the same radial coordinate) in order to transform them into Schwarzschild coordinates (see more details in [29]), thus we get

$$ds^{2} = -e(r)dt^{2} + e(r)^{-1}dr^{2} + r^{2}d\theta^{2} + r^{2}\sin^{2}\theta d\phi^{2},$$
(123)

with the aether vector given by

$$u^{a} = \left[\alpha_{1}(r) - \frac{\beta_{1}(r)}{e(r)}, \beta_{1}(r), 0, 0\right],$$
(124)

when normalized we obtain

$$u^{a} = \left[\frac{\alpha_{1}(r)e(r) - \beta_{1}(r)}{e(r)\sqrt{\alpha_{1}(r)[\alpha_{1}(r)e(r) - 2\beta_{1}(r)]}}, \frac{\beta_{1}(r)}{\sqrt{\alpha_{1}(r)[\alpha_{1}(r)e(r) - 2\beta_{1}(r)]}}, 0, 0\right],$$
(125)

and the timelike Killing vector is also given by Eq. (18).

In these previous papers, they have presented only two analytical solutions for $c_{14} = 0$ (Case I) and $c_{123} = 0$ (Case II). Using these coordinate transformations we can get their results in our coordinates (Table 1). See Table 4 for the details.

Comparing the Tables 2 and 3 with the correspondent cases in Table 4, we observe clearly from the metric functions, the aether vectors and the universal horizons that they are different from each other. However, the universal horizons of the Cases (A), (B) and (C) coincide with the Case I.

We notice that the thermodynamical quantities such as temperature and entropy, of the Case I coincide with ours ones of the Case A. In the rest of the cases the thermodynamical quantities are different to each other.

Thus, we conclude that our results are completely different from the previously published papers (albeit in a different coordinate system), except our Case (C) coincides with Case I. The reason that the universal horizons and their thermodynamical properties are different is because the surface gravity depends explicitly on the aether vector.

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14 Appendix A

The aether field equations, collecting the terms c_2 , c_{13} and c_{14} , are given by

$$\begin{split} G_{tt}^{aether} &= -\frac{1}{8A(a^2B+1)B^2r^2} \\ \times [c_{13}\left(-4a'^2r^2A^2B^2+3a^2B'^2r^2A^2+3a^2A'^2r^2B^2+3a^4A'^2r^2B^3-2r^2AB^2A'a'-2a^2r^2AB^2A'B'A'-8ar^2AB^2A'a'-2a^2r^2ABA'B'-8ar^2A^2Ba'B'-8ar^2A^2Ba'B'-8ar^2A^2B^2A''-4a^2r^2A^2BB''-8ar^2A^2B^2a''-4a^2r^2A^2BB''-8ar^2A^2B^2a''-4a^2r^2A^2BB''-8ar^2A^2B^2B'-6ar^2A^2BB''-16arA^2B^2a'-8a^4AB^3A'r-8B^2ArA'a^2+8B^2A^2a^2+8A^2B^3a^4)-c_{14}\left(8a'^2r^2A^2B^2-2a^2B'^2r^2A^2-6a^2A'^2r^2B^2-3a^4A'^2r^2B^3-2B'A'r^2A^2BB'-6a^2A'^2r^2B^2-3a^4A'^2r^2B^3-2B'A'r^2A^2B^2A'r^2A^2+8ArA'B+4A''Br^2A+2r^2AB^2a'a'B+8r^2A^2B^2A'a'+12ar^2A^2Ba'B'+8r^2AB^3A'a'^3+8a^2r^2AB^2A''+4a^2r^2A^2BB''+4a^2r^2A^2BB''+8ar^2A^2B^2a''+4a^2r^2A^2BB''+8ar^2A^2B^2a''+4a^2r^2A^2BB''+8ar^2A^2B^2a''+4a^2r^2A^2BB''+8ar^2A^2B^2A''+4a^2r^2A^2BB''+8ar^2A^2BB''+8a^2r^2B''+8a^2r^2B''+8a^2r^2B''+8a^2r^2B''+8a^2r^2B''+8a^2r^2B''+8a^2r^2B''+8a^2r^2B''+8a^2r^2B''+8a^2r^2B''+8a^2r^2B''+8a^2r^2B''+8a^2r^2B''+8a^2r^2B''+8a^2r^2A'^2B''+8a^2r^2B'$$

$$\begin{aligned} &-4a^{2}r^{2}A^{2}B^{3}a'^{2} - 8ar^{2}AB^{2}A'a' \\ &-32rA^{2}B^{3}a^{3}a' - 2a^{2}r^{2}ABA'B' \\ &-8r^{2}AB^{3}a'A'a^{3} - 8r^{2}A^{2}B^{2}a'a^{3}B' \\ &-8r^{2}AB^{2}a^{4}B'A' - 8a^{2}rA^{2}BB' \\ &-8B^{2}ArA'a^{2} - 8a^{4}AB^{3}A'r \\ &+3a^{4}r^{2}A^{2}BB'^{2} + 3a^{2}A'^{2}r^{2}B^{2} \\ &-4a'^{2}r^{2}A^{2}B^{2} + 3a^{2}B'^{2}r^{2}A^{2} \\ &+3a^{4}A'^{2}r^{2}B^{3} - 8ar^{2}A^{2}Ba'B' \\ &-8ar^{2}A^{2}B^{2}a'' - 4a^{2}r^{2}A^{2}BB'' \\ &-4a^{2}r^{2}AB^{2}A'' - 4r^{2}A^{2}B^{2}a^{4}B'' \\ &-4r^{2}AB^{3}A''a^{4} \\ &-8r^{2}B^{3}A^{2}a''a^{3}) \\ &-(-8a^{2}rA^{2}BB' + 8B^{2}A^{2}a^{2} \\ &-8B^{2}A^{2} + 8BA^{2} - 8B^{3}A^{2}a^{2} - 8rB'A^{2})] = 0, \end{aligned}$$
(126)

$$G_{tr}^{aether} = \frac{\epsilon}{\sqrt{R^2 r^2 \Lambda(a^2 R + 1)} \sqrt{\Lambda(a^2 R + 1)}}$$

$$\begin{array}{l} 4B^{2}r^{2}A(a^{2}B+1)\sqrt{A(a^{2}B+1)}\\ \times [c_{13}(4A^{\prime 2}B^{3}r^{2}a^{3}+2r^{2}B^{4}a^{5}A^{\prime 2}\\ +2B^{\prime 2}r^{2}A^{2}a-4B^{2}r^{2}A^{2}a^{\prime \prime}\\ +2A^{\prime 2}B^{2}r^{2}a-8B^{2}rA^{2}a^{\prime \prime}-4r^{2}B^{4}A^{2}a^{\prime \prime}a^{4}\\ -2AB^{4}r^{2}a^{5}A^{\prime \prime}\\ -2B^{3}r^{2}A^{2}a^{5}B^{\prime \prime}-4Ar^{2}B^{3}a^{3}A^{\prime \prime}\\ -2B^{2}r^{2}AaA^{\prime \prime}-2Br^{2}A^{2}aB^{\prime \prime}\\ -4B^{2}r^{2}A^{2}a^{3}B^{\prime \prime}-8B^{3}r^{2}A^{2}a^{2}a^{\prime \prime}\\ -8B^{2}rA^{2}a^{3}B^{\prime \prime}-8B^{3}r^{2}A^{2}a^{2}a^{\prime \prime}\\ -4BrA^{2}aB^{\prime }-8B^{4}A^{2}ra^{4}a^{\prime }\\ -4B^{3}rA^{2}a^{5}B^{\prime }-4AB^{4}a^{5}A^{\prime r}\\ +2B^{2}r^{2}A^{2}a^{5}B^{\prime \prime}-8B^{4}A^{2}a^{3}B^{\prime \prime}\\ -4B^{2}r^{2}A^{2}a^{5}B^{\prime \prime}-8B^{3}a^{3}A^{\prime }\\ -4rAB^{2}aA^{\prime }-2B^{2}r^{2}AA^{\prime}a^{\prime }\\ -2Br^{2}A^{2}a^{\prime}B^{\prime }+4Br^{2}A^{2}a^{3}B^{\prime 2}\\ -4B^{2}r^{2}A^{2}a^{\prime}B^{\prime }+4Br^{2}A^{2}a^{3}B^{\prime 2}\\ -4B^{2}r^{2}A^{2}a^{\prime}B^{\prime }+4Br^{2}A^{2}a^{3}B^{\prime 2}\\ -4B^{2}r^{2}A^{2}a^{\prime}B^{\prime }+4Br^{2}A^{2}a^{3}B^{\prime 2}\\ +8B^{2}A^{2}a+16B^{3}A^{2}a^{3}\\ +8B^{4}A^{2}a^{5})\\ +c_{14}\left(-4A^{\prime 2}B^{3}r^{2}a^{3}-2r^{2}B^{4}a^{5}A^{\prime 2}\\ -2A^{\prime 2}B^{2}r^{2}a+4B^{3}r^{2}A^{2}aa^{\prime 2}\\ +4r^{2}B^{4}A^{2}a^{\prime }a^{4}+2AB^{4}r^{2}a^{5}A^{\prime }\\ +2B^{3}r^{2}A^{2}a^{5}B^{\prime \prime}+4Ar^{2}B^{3}a^{3}A^{\prime \prime}\\ +2B^{3}r^{2}A^{2}a^{2}a^{\prime }+8B^{4}A^{2}a^{3}B^{\prime }\\ +4B^{3}rA^{2}a^{2}a^{\prime }+8B^{4}A^{2}a^{4}a^{\prime }\\ +4B^{3}rA^{2}a^{5}B^{\prime }+4AB^{4}a^{5}A^{\prime }r\\ -2B^{2}r^{2}Aa^{2}a^{5}B^{\prime }+4AB^{4}a^{5}A^{\prime }r\\ -2B^{2}r^{2}A^{2}a^{5}B^{\prime }+4AB^{4}a^{5}A^{\prime }r\\ +2B^{2}r^{2}A^{2}a^{5}B^{\prime }+4AB^{4}a^{5}A^{\prime }r\\ +2B^{2}r^{2}A^{2}a^{5}B^{\prime }+4AB^{4}a^{5}A^{\prime }r\\ +4B^{3}rA^{2}a^{5}B^{\prime }+4AB^{4}a^{5}A^{\prime }r\\ +2B^{2}r^{2}A^{2}a^{5}B^{\prime }+4AB^{4}a^{5}A^{\prime }r\\ +4B^{3}rA^{2}a^{5}$$

$$\begin{aligned} +4rAB^{2}aA' - Br^{2}A^{2}a^{3}B'^{2} \\ +6B^{2}r^{2}A^{2}a'B'a^{2} + 2AB^{4}r^{2}A'a^{4}a' \\ +2B^{3}r^{2}A^{2}a^{4}a'B' \\ +2A'B^{3}r^{2}Aa'a^{2} - Br^{2}AaA'B' - B^{2}r^{2}AA'a^{3}B') \\ +c_{2}(4A'^{2}B^{3}r^{2}a^{3} + 2r^{2}B^{4}a^{5}A'^{2} \\ +2B'^{2}r^{2}A^{2}a - 4B^{2}r^{2}A^{2}a'' \\ +2A'^{2}B^{2}r^{2}a - 8B^{2}rA^{2}a' \\ -4r^{2}B^{4}A^{2}a''a^{4} - 2AB^{4}r^{2}a^{5}A'' \\ -2B^{3}r^{2}A^{2}a^{5}B'' - 4Ar^{2}B^{3}a^{3}A'' \\ -2B^{2}r^{2}AaA'' - 2Br^{2}A^{2}aB'' \\ -4B^{2}r^{2}A^{2}a^{3}B'' - 8B^{3}r^{2}A^{2}a^{2}a'' \\ -16B^{3}rA^{2}a^{2}a' - 8B^{4}A^{2}ra^{4}a' \\ +2B^{2}r^{2}A^{2}a^{5}B'^{2} - 2B^{2}r^{2}AA'a' \\ -2Br^{2}A^{2}a'B' + 4Br^{2}A^{2}a^{3}B'^{2} \\ -4B^{2}r^{2}A^{2}a'B'a' - 2AB^{4}r^{2}A'a^{4}a' \\ +2B^{2}r^{2}A^{2}a'B'a' - 2AB^{4}r^{2}A'a^{4}a' \\ -2Br^{2}A^{2}a'B'a'A' - 2AB^{4}r^{2}A'a^{4}a' \\ -2B^{3}r^{2}A^{2}a'A'B' + 4Br^{2}A^{2}a^{3}B'^{2} \\ -4B^{2}r^{2}A^{2}a'A'B'A'A'B^{3}r^{2}Aa'a^{2} \\ +8B^{2}A^{2}a + 16B^{3}A^{2}a^{3} + 8B^{4}A^{2}a^{5}) = 0, \quad (127) \end{aligned}$$

$$\begin{split} G_{rr}^{aether} &= 0, \\ G_{rr}^{aether} &= -\frac{1}{8A^2Br^2(a^2B+1)} \\ \times [\ c_{13}\ (a^4r^2A^2BB'^2+4a^2r^2A^2B^3a'^2 \\ &+4ar^2A^2Ba'B' + 4r^2AB^3a'A'a^3 \\ &+4r^2A^2B^2a'a^3B' + 2r^2AB^2a^4B'A' \\ &+4ar^2AB^2A'a' + 2a^2r^2ABA'B' \\ &+8B^2A^2a^2 + 8A^2B^3a^4 + 4a'^2r^2A^2B^2 \\ &+a^2B'^2r^2A^2 + a^2A'^2r^2B^2 \\ &+a^4A'^2r^2B^3) \\ &-c_{14}\ (-4a^2r^2A^2B^3a'^2 - a^4r^2A^2BB'^2 \\ &-2a^2r^2ABA'B' - 4r^2AB^3a'A'a^3 \\ &-2r^2AB^2a^4B'A' - 4r^2A^2B^2a'a^3B' \\ &-4ar^2AB^2A'a' - A'^2Br^2 \\ &-a^4A'^2r^2B^3 - 2a^2A'^2r^2B^2) \\ &-c_2\ (16arA^2B^2a' + 8a^2rA^2BB' \\ &+8B^2ArA'a^2 + 8a^4AB^3A'r \\ &+a^4r^2A^2BB'^2 + 16rA^2B^3a^3a' \\ &+8a^4rA^2B^2B' + 4a^2r^2A^2Ba'B' \\ &+2a^2r^2ABA'B' + 4r^2AB^3a'A'a^3 \\ &+4r^2A^2B^2a'a^3B' + 2r^2AB^2a^4B'A' \\ &+16B^2A^2a^2 + 16A^2B^3a^4 \\ &+a^2A'^2r^2B^2 + 4a'^2r^2A^2B^2 \\ &+a^2B'^2r^2A^2 + a^4A'^2r^2B^3) \\ &-(8B^2A^2 - 8BA^2 + 8A^2B^3a^2 - 8ArA'B \end{split}$$

$$G_{\theta\theta}^{aether} = -\frac{r}{8B^2Ar^2(a^2B+1)}$$
(129)

$$S_{\theta\theta}^{aether} = -\frac{r}{8B^2Ar^2(a^2B+1)}$$
(129)

$$\times [c_{13}(-A^2rBa^4B'^2 - 4A^2rB^3a^2a'^2 - 4arA^2Ba'B' - 2B'A'rAa^2B - 2ArB^2a^4B'A' - 4ArB^2aa'A' + 4A'B^2Aa^2 + 4B^3AA'a^4 - rB^3A'^2a^4 - A'^2B^2ra^2 + 16aA^2B^2a' + 4B'A^2a^2B + 16A^2B^3a^3a' + 4A^2B^2a^4B' - 4a'^2rA^2B^2 - a^2B'^2rA^2) - c_{14}(A'^2Br + 4A^2rB^3a^3a'A' + 4ArB^2aa'A' + 4ArB^3a^3a'A' + 4ArB^2aa'A' + 4A^2rB^2a^3B' + 2ArB^2a^4B'A' + 2B'A'rAa^2B + rB^3A'^2a^4 + 2A'^2B^2a^2) - c_2(8arA^2B^2a'' + 4a^2rA^2BB'' + 4A^2rB^2a^3A'B' + 2ArB^2a^4B'A' + 8A^2rB^3a'a'^2 + 4A^2rB^2a^4B'' + 4ArB^3a^4A'' + 8A^2rB^3a'a'^2 + 8ArB^2aa'A' + 8ArB^3a^3a'A' + 8ArB^2aa'A' + 8ArB^2a'A' + 8ArB^3a^3a'A' + 8ArB^2a'A' + 8ArB^2a'A' + 8ArB^2a'A' + 8ArB^3a^3a'A' + 8A^2rB^2a^3a'B' + 2ArB^2a^4B'A' + 32aA^2B^2a'A' + 8A'A^2B^2a^2B + 8A'B^2Aa^2 + 8B'AA'a^2B^2ra^2 + 4a'^2rA^2B^2ra^2 + 4A'B^2Aa^2 + 8B^3AA'a^4 + 32A^2B^3a'a' + 8A^2B^2a^4B' - 3A'^2B^2ra^2 + 4a'^2rA^2B^2ra^2 + 4a'^2rA^2B^2ra^2 + 4A'B^2Aa^2 + 2B'A'rAa^2B - 4A'BA - 4A''B^2rAa^2 + 2A'^2BrA^2 + 2A'^2B^2ra^2 + 4a'^2rA^2B^2ra^2 + 4A'B^2rA^2B - 4A'BA - 4A''B^2rAa^2 + 2A'^2B^2rA^2 - 3rB^3A'2a^4) - (4B'A^2a^2B - 4A'B^2Aa^2 + 2A'^2B^2B - 4A'B^2Aa^2 + 2A''B^2A^2B - 4A'B^2Aa^2 + 2A''B^2rA^2 + 2A''B^2rA^2 + 2A''B^2A^2B - 4A'B^2Aa^2 + 2A''B^2rA^2 + 2A''B^2A^2B - 4A'B^2A - 4A''B^2A^2 + 2A''B^2A^2B - 4A'B^2A - 4A''B^2A^2 + 2A''B^2B - 4A'B^2A - 4A''B^2rAa^2 + 2A''B^2B - 4A'B^2A - 4A''B^2A^2 + 2A''B^2A^2B - 4A'B^2A - 4A''B^2A^2 + 2A''B^2B^2A^2B + 4A'B^2A^2B - 4A''B^2A^2B + 4A''B^2A^2$$

where $G_{\mu\nu}^{aether} = T_{\mu\nu}^{aether}$ and the symbol prime denotes the differentiation with respect to *r*. We can notice here that when $c_{13} = 0$, $c_{14} = 0$ and $c_2 = 0$ we obtain the same field equations of the GR.

15 Appendix B

The aether field equations, collecting the terms c_{123} , c_{14} and c_2 , are given by

$$G_{tt}^{aether} = \frac{1}{8A(a^2B + 1)B^2r^2}$$

$$\begin{split} & \times \left[c_{14} \left(-8a^{3}r^{2}AB^{3}A'a' \right. \\ & -8a^{3}r^{2}A^{2}B^{2}a'B' - 12ar^{2}A^{2}Ba'B' \right. \\ & -2a^{4}r^{2}AB^{2}A'B' - 8a^{2}r^{2}AB^{2}A'a' \\ & -8a^{2}rA^{2}BB' - 4a^{2}r^{2}A^{2}B^{3}a'^{2} \\ & -4A^{2}r^{2}Ba^{2}B'' - 8a^{2}r^{2}AB^{2}A'' \\ & -8B^{2}r^{2}A^{2}aa'' - 4a^{4}r^{2}A^{2}B^{2}B'' \\ & -4a^{4}r^{2}AB^{3}A'' - 8a^{3}r^{2}A^{2}B^{3}a'' \\ & +3a^{4}r^{2}A^{2}BB'^{2} - 8a^{4}rA^{2}B^{2}B' \\ & -8a^{4}AB^{3}A'r - 16rA^{2}B^{3}a^{3}a' \\ & -16arA^{2}B^{2}a' - 16B^{2}ArA'a^{2} \\ & +2B'A'r^{2}A - 8ArA'B + 3r^{2}B^{3}a^{4}A'^{2} \\ & +6a^{2}A'^{2}r^{2}B^{2} - 8a'^{2}r^{2}A^{2}B^{2} \\ & +2a^{2}B'^{2}r^{2}A^{2} - 4A''Br^{2}A + 3A'^{2}Br^{2} \right) \\ & +c_{2}\left(16arA^{2}B^{2}a' + 16rA^{2}B^{3}a^{3}a' \\ & +8A^{2}B^{3}a^{4} + 8B^{2}A^{2}a^{2} \right) \\ & +c_{123}\left(8a^{3}r^{2}AB^{3}A'a' + 8a^{3}r^{2}A^{2}B^{2}a'B' \\ & +2a^{4}r^{2}AB^{2}A'B' + 8ar^{2}AB^{2}A'a' \\ & +2a^{2}r^{2}ABA'B' + 8a^{2}rA^{2}BB' \\ & +4a^{2}r^{2}A^{2}B^{3}a'^{2} + 4A^{2}r^{2}Ba^{2}B' \\ & +4a^{2}r^{2}A^{2}B^{3}a' - 3a^{4}r^{2}A^{2}BB'^{2} \\ & +8a^{4}rA^{2}B^{2}B' + 8a^{4}AB^{3}A'r \\ & +16rA^{2}B^{3}a^{3}a' + 16arA^{2}B^{2}a' \\ & -3a^{2}A'^{2}r^{2}B^{2} + 4a'^{2}r^{2}A^{2}B^{2} \\ & -3a^{2}B'^{2}r^{2}A^{2} - 8A^{2}B^{3}a^{4} \\ & +(8a^{2}rA^{2}BB' - 8B^{2}A^{2}a^{2} + 8r'A^{2})\right] = 0 \\ G_{tr}^{aether} = \frac{\epsilon}{AB^{2}r^{2}A(a^{2}B + 1)\sqrt{A(a^{2}B + 1)}} \end{split}$$

$$= \frac{\epsilon}{4B^2r^2A(a^2B+1)\sqrt{A(a^2B+1)}}$$

$$\times [c_{14} (2B^2r^2A^2a^3B''+4B^3r^2A^2a^2a'' + 2B^4r^2Aa^5A'' + 2B^3r^2A^2a^5B'' + 4B^4r^2A^2a''a^4 + 4Ar^2B^3a^3A'' + 2B^2r^2AaA'' + 4B^3A^2ra^5B' + 4B^3r^2A^2aa'^2 + 4B^2rA^2a^3B' - Br^2A^2a^3B'^2 + 8B^3A^2ra^2a' + 8B^4A^2ra^4a' - 2B^2r^2A^2a^5B'^2 + 8rAB^3a^3A' + 4rAB^2aA' + 4a^5AB^4rA' + 6B^2r^2A^2a'B'a^2 + 2B^4r^2AA'a^4a' + 2B^3r^2A^2a^4a'B' + 2A'B^3r^2Aa'a^2 - Br^2AaA'B'$$

$$-B^{2}r^{2}AA'a^{3}B' - 4A'^{2}B^{3}r^{2}a^{3}$$

$$-2A'^{2}B^{2}r^{2}a - 2B^{4}r^{2}a^{5}A'^{2})$$

$$+c_{2}(4B^{3}A^{2}ra^{5}B' + 4BrA^{2}aB' + 8B^{2}rA^{2}a^{3}B' + 8rAB^{3}a^{3}A' + 4rAB^{2}aA' + 4a^{5}AB^{4}rA')$$

$$+c_{123}(-4B^{2}r^{2}A^{2}a^{3}B'' - 2B^{4}r^{2}Aa^{5}A'' - 2B^{3}r^{2}A^{2}a^{5}B'' - 4B^{4}r^{2}A^{2}a''a^{4} - 4Ar^{2}B^{3}a^{3}A'' - 2B^{2}r^{2}AaA'' - 4B^{3}A^{2}ra^{5}B' - 8B^{2}rA^{2}a^{3}B' + 4Br^{2}A^{2}a^{3}B'^{2} - 16B^{3}A^{2}ra^{2}a' - 8B^{4}A^{2}ra^{4}a' + 2B^{2}r^{2}A^{2}a^{5}B'^{2} - 8rAB^{3}a^{3}A' - 4rAB^{2}aA' - 4a^{5}AB^{4}rA' - 4BrA^{2}aB' - 4B^{2}r^{2}A^{2}a'B'a^{2} - 2B^{4}r^{2}AA'a^{4}a' - 2B^{3}r^{2}A^{2}a^{4}a'B' - 2B^{3}r^{2}A^{2}a^{4}a'B' - 4A'B^{3}r^{2}Aa'a^{2} - 8B^{2}rA^{2}a' + 2B'r^{2}A^{2}a'B'a^{2} - 2B^{4}r^{2}AA'a^{4}a' - 2B^{3}r^{2}A^{2}a^{4}a'B' - 4A'B^{2}r^{2}A^{2}a'A'a^{2} - 8B^{2}rA^{2}a' + 2B'r^{2}A^{2}a'B'a^{2} - 2B^{2}r^{2}A^{2}a'A'a^{4}a' - 2B^{3}r^{2}A^{2}a^{3} + 2A'^{2}B^{2}r^{2}a + 16B^{3}A^{2}a^{3} + 8B^{4}A^{2}a^{5} - 2B^{2}r^{2}AA'a' - 2Br^{2}A^{2}a'B' - 2Br^{2}A^{2}aB''] = 0,$$
(133)

$$G_{rt}^{aether} = 0,$$
(134)

$$G_{rr}^{aether} = -\frac{r}{8A^2Br^2(a^2B+1)} \times [c_{14}(-4a^2r^2A^2B^3a'^2 - a^4r^2A^2BB'^2 - 2a^4r^2AB^2A'B' - 4a^3r^2A^2BB'^2 - 2a^4r^2AB^2A'B' - 4ar^2AB^2A'a' - 2a^2r^2ABA'B' - 4ar^2AB^2A'a' - 2a^2r^2ABA'B' - A'^2Br^2 - r^2B^3a^4A'^2 - 2a^2A'^2r^2B^2) - c_2(16arA^2B^2a' + 8a^2rA^2BB' + 8B^2ArA'a^2 + 8a^4AB^3A'r + 16rA^2B^3a^3a' + 8a^4rA^2B^2B' + 8B^2A^2a^2 + 8A^2B^3a^4) - c_{123}(4a^2r^2A^2B^3a'^2 + 4a^2r^2AB^2A'a' + 2a^2r^2ABA'B' + 4ar^2A^2Ba'B' + 4a^3r^2AB^3A'a' + 4a^3r^2A^2Ba'B' + 2a^4r^2AB^2A'a' + 2a^2r^2ABA'B' + 8B^2A^2a^2 + 8A^2B^3a^4 + 4a'^2r^2A^2B^2 + a^2B'^2r^2A^2 + a^2A'^2r^2B^2 + a^2B'^2r^2A^2 + a^2A'^2r^2B^2 + a^2B'^2r^2A^2 + a^2A'^2r^2B^2 + a^2B'^2r^2A^2 + a^2A'^2r^2B^2 + r^2B^3a^4A'^2) - (-8ArA'B - 8B^2A^2a^2 - 8BA^2 + 8B^2A^2 - 8B^2ArA'a^2 + 8A^2B^3a^2)] = 0, \quad (135)$$

$$\begin{split} G^{aether}_{\theta\theta} &= -\frac{r}{8B^2A^2(a^2B+1)} \\ \times [c_{14}(A'^2Br + 4rA^2B^2a'a^3B' \\ + 2ArB^2a^4B'A' + 2B'A'rAa^2B \\ + 4ArB^3a^3a'A' + 4ArB^2aa'A' \\ + 4rA^2B^3a^2a'^2 + rA^2Ba^4B'^2 \\ + rB^3A'^2a^4 + 2A'^2B^2ra^2) \\ - c_2(12ArB^2aa'A' + 12arA^2Ba'B' \\ + 4B'A'rAa^2B + 12ArB^3a^3a'A' \\ + 12rA^2B^2a'a^3B' + 4ArB^2a^4B'A' \\ + 8arA^2B^2a'' + 4ArB^3a^4A'' \\ + 4rA^2B^2a^4B'' + 8rA^2B^3a'a^3 \\ - 2rA^2Ba^4B'^2 + 8rA^2B^3a'a^2 \\ + 4a^2rA^2BB'' + 4A''B^2rAa^2 + 16aA^2B^2a' \\ - 2A'^2B^2ra^2 + 8a'^2rA^2B^2 \\ - 2a^2B'^2rA^2 - 2rB^3A'^2a^4 + 4B'A^2a^2B \\ + 4A'B^2Aa^2 + 4B^3AA'a^4 \\ + 16A^2B^3a^3a' + 4A^2B^2a^4B') \\ - c_{123}(-2B'A'rAa^2B - 2ArB^2a^4B'A' \\ - 4ArB^2aa'A' - 4arA^2Ba'B' \\ - 4ArB^3a^3a'A' - 4rA^2B^3a^2a'^2 \\ + 4A'B^2Aa^2 + 4B^3AA'a^4 - rB^3A'^2a^4 \\ - A'^2B^2ra^2 + 16A^2B^3a^3a' \\ + 4A^2B^2a^4B' + 16aA^2B^2a' + 4B'A^2a^2B \\ - 4a'^2R^2B^2 - a^2B'^2rA^2) \\ - (4B'A^2a^2B - 4A'B^2Aa^2 \\ + 2A'^2B^2ra^2 - 4A''BrA + 2B'A'rAa^2B \\ + 2A'^2Br - 4A'BA + 2B'A'rAa^2B \\ + 4B'A^2 - 4A''B^2rAa^2] = 0, \quad (136) \\ G^{aether}_{\theta\theta} = G^{aether}_{\theta\theta}\sin^2\theta. \quad (137) \end{split}$$

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