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New wormhole models with stability analysis via thin-shell in teleparallel gravity

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Abstract This study explores new wormhole solutions in the background of teleparallel gravity. All the energy conditions are investigated for two different new calculated shape functions. The presence of exotic matter is confirmed due to the violation of the energy conditions. Thin-shell around the wormhole geometry is obtained by using the cut and paste approach taking the Schwarzschild black hole as an exterior manifold. The stability of thin-shell is explored by using linearized radial perturbation about equilibrium shell radius for both choices of calculated shape functions. It is concluded that stable regions and the position of the expected event horizon depend on the choice of physical parameters.

1 Introduction

The progressive revolutions in general relativity (GR) referred to the understanding of a stimulating framework incorporating gravity and also, shed light on various astrophysical phenomena in the cosmos. Recent developments in this aspect have led to many remarkable investigations into the evolutionary mechanism of the Universe. Accelerated expansion of our Universe is one of the fundamental facts that have brought many challenges to the present arena of research. Researchers attempted different astronomical probes (type Ia supernova, large-scale structure and cosmic microwave background radiation (CMBR), the integrated Sachs-Wolfe effect, baryon acoustic oscillations, gamma-ray bursts, etc.) and found evidence about the mysterious exotic source that takes part in the expansion of the Universe [1–6]. These observational advances indicate two cosmic phases of accelerated expansion, i.e., the cosmic era before radiation and eventually, the present state following the matter-dominated phase. In the last couple of decades, a consensus about this accelerating behavior exhibits that an unusual anti-gravitational force termed dark energy (DE), causes the current cosmic expansion. The observational data about the presence of DE was firstly proposed by physicists in the early 1980s while exploring the structure formation of galaxies in the cosmos [7].

The investigation of dominant cosmic ingredients in a matter distribution has remained one of the biggest puzzles in Cosmology. Recent observations indicate their contribution in the ratio: 4% baryonic matter, 28% dark matter, and 68% DE of the total budget. Dark energy comprises a significant feature of having large negative pressure (responsible for acceleration) but does not the cluster at large scales. This exotic nature of DE makes it inconsistent with strong energy conditions (SEC), thereby the major part of cosmic contents remains unspecified. Several attempts have been made in the literature to comprehend the ambiguous nature of the DE. Without having any solid argument in favor of dark sources, extensive approaches have been chosen as an alternative. The study of mysterious approaches to these exotic terms have been illustrated in two ways, i.e., using modified matter sources or modifying the gravity by introducing some extra degrees of freedom in the action for the field equations. The first approach refers to the modification of the matter sector of the Einstein-Hilbert Lagrangian density by taking different

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proposals like quintessence energy [8], phantom [9], tachyon field [10], k-essence [11], generalization of barotropic equation of state (EoS) such as Chaplygin gas [12] and its modification [13] etc. that effectively suggest the dynamical behavior of the cosmos. The other approach is an extension of the gravitational part in GR by incorporating a DE source while the matter part remains fixed. Although the first category has interesting implications but could not be endorsed as much promising due to the presence of some ambiguities, whereas modified frameworks of gravity are quite useful due to their effective cosmological executions.

Einstein developed the concept of geometry-matter coupling whose modifications are as old as GR itself. Modified theories of gravity supplemented by extra curvature terms in the action have been extensively studied in the literature. Some well-known modifications of gravity include f(R)gravity [14], Gauss–Bonnet gravity [15], f(R, T) gravity [16], and the scalar-tensor theory [17]. Further, in the context of wormhole study some modified theories of gravity, like f(T) gravity [18,19], $f(T, T_G)$ gravity [20], f(Q) gravity [21–23], and Einstenian cubic gravity [24,25] have been explored. The attractive insights of these alternative gravitational theories have become a center of interest for many researchers in the fields of high energy physics, astrophysics, and modern Cosmology. These modifications are based on the metric tensor g_{ij} being a dynamical variable. It is intriguing to note that an alternative approach to GR has been getting momentum in the literature known as teleparallel gravity in which the tetrad e_{ϵ}^{i} is chosen as a basic physical variable [26]. Curvature is replaced by the notion of torsion through which geometric deformation creates a gravitational field. In other words, this modification leads to the torsion-based teleparallel connection instead of the curvature-based Levi-Civita connection. Since $g_{\epsilon\epsilon} = \eta_{ij} e^i \epsilon^j e^i \epsilon^j$, one may consider tetrad as the square root of the metric which seems analogous with the Dirac equation as the square root of Klein-Gorden equation. While working with tetrad, the inclusion of torsion is naturally used instead of curvature which yields the so-called the teleparallel equivalent of GR whose dynamical equivalence shows that they both are indistinguishable by classical experiments.

Although teleparallel gravity is equivalent to GR but conceptually quite different. In contrast to GR, teleparallel gravity is nicely motivated within a gauge theory context and can be beautifully framed as the gauge theory for the translation group [27]. In fact, like all other gauge theories, its Lagrangian density is quadratic in the torsion tensor, the field strength of the theory. This theory of gravity incorporates various features of GR like the possibility of studying nonvacuum solutions [28] and conserved currents [29] etc. Furthermore, this has also been a significant framework for studying nonsingular black holes [30], gravitational waves [31,32] and energy fluxes in cylindrical spacetimes [33]. The teleparallel gravity also makes it possible to come up with extensions of GR to address the challenges of dark matter [34,35] and cosmology [36]. One of the most astonishing features of the teleparallel theory that makes it simpler than GR is its first order Lagrangian as well as its Yang–Mills-like field equations. The other feature of this framework that makes it richer than that of GR is the fact that, unlike in GR, one can separate gravitational effects from inertial effects [37]. This arises due to the spin connection lying inside the Weitzenböck connection. These unique features are not shared by any other alternative theories.

Teleparallel gravity has also attained much interest the researchers regarding its applications to different astrophysical and cosmological scenarios as well as in the background of its extended versions. Krssak et al. [38] discussed the fully invariant formulation of teleparallel gravity and its generalizations using different assumptions on the frame and spin connection to present the covariant procedure. Singh et al. [39] studied Einstein's cluster mimicking compact star in the framework of teleparallel gravity and also discussed cluster formation in modified f(T) gravity by choosing an anisotropic fluid distribution. Hammad et al. [40] derived the Noether charge associated with diffeomorphism invariance and black hole entropy in the teleparallel gravity. They showed that the conformal issue that plagues the entropy-area law within GR does not arise in teleparallel gravity based on Wald's approach. Nashed and Capozziello [41] explored a charged non-vacuum solution for compact stars by taking a physically symmetric tetrad field in the context of teleparallel gravity. Asifa et al. [42,43] explored the existence of stars in extended teleparallel gravity and studied the regularity, anisotropy, energy conditions, stability and surface redshift of the model. Ditta et al. [44] also discussed a new exact model for anisotropic stars in the f(T) theory of gravity.

One of the most fascinating features of GR is the existence of hypothetical geometries with topological structures describing interstellar travel. A wormhole is a theoretical connection between remote regions of the Universe which reduces traveling time and distance. The concept of wormhole has a primal history starting with Flamm [45], who constructed the Schwarzschild solution of the field equations of GR as a non-traversable wormhole. Einstein and Rosen [46], proposed the existence of a bridge, known as the Einstein-Rosen bridge, by joining two copies of the Schwarzschild spacetime for which the wormhole throat implodes thus forming a singularity. Misner and Wheeler [47] termed these hypothetical characteristics of the field equations of GR as wormholes for the first time. Ellis [48] introduced the concept of the traversable wormhole with topological structure by coupling geometry and scalar field that creates a geodesically complete manifold with no horizon. Bronnikov [49] explored the scalar-electrovacuum configurations without scalar charge. Clement [50] gave a class of traversable wormholes in higher dimensions. Morris and Thorne [51] proposed the idea of a traversable wormhole by joining two distant cosmic regions (asymptotically flat) by a throat supported by an exotic matter violating the null energy condition (NEC) that keeps the wormhole throat open. The physical viability of wormhole configuration requires confining the usage of this matter, which is something controversial. There has been an extensive work for the construction of wormholes from black hole spacetimes and analysis of their various physical aspects [52–71].

Many researchers have discussed the stability of thin-shell wormholes (WHs) by using radial perturbation with different matter distributions. The stable configuration of thin-shell with linearized radial perturbation is analyzed by Poisson and Visser in [72]. Later, it is extended to explore the effects of the cosmological constant on the stability of thin-shell as discussed in [73]. Thin-shell Wh is developed by using cut and paste approach in the background of regular Hayward BH in [74] and a stable configuration of rotating thin-shell is explored in [75]. In the background of noncommutative geometry, the stability of charged thin-shell gravastar with linearized radial perturbation is studied in [76]. The effects of dark matter and dark energy on thin-shell WH under Lorentz symmetry breaking are discussed in [77]. These research articles are very interesting and motivate us to explore the stable configuration of thin-shell in the modified theory of gravity.

The search for a realistic source for wormhole construction leads to the extension of GR. In modified theories of gravity, normal matter supports the wormhole throat while the violation of energy conditions corresponds to the effective energy-momentum tensor. Lobo and Oliveira [78] examined the wormhole configurations without violation of energy conditions in the f(R) theory of gravity by taking particular choices for the shape function. Bronnikov et al. [79] discussed some remarkable aspects of wormhole geometry in f(R) gravity. Dehghani and Mehdizadeh [80] presented wormholes in Lovelock gravity and found that two different ranges of Lovelock coefficients satisfy the weak energy condition. Azizi [81] studied the wormhole solutions in f(R, T) gravity for which the matter supporting the wormhole throat might fulfill the energy conditions. Zubair et al. [82,83] obtained spherically symmetric wormhole solutions and found realistic consequences for anisotropic matter distribution under the umbrella of f(R, T) gravity. Garcia and Lobo [84] explored the exact wormhole solutions in the context of Brans-Dicke theory. Tayyaba et al., [85] investigated wormhole configurations by assuming symmetries in the f(G) theory of gravity. Farasat et al., [86] constructed noncommutative wormhole geometries in f(R, G) gravity and explored their equilibrium condition. Mustafa et al. [87] analyzed non-commutative wormholes through Lorentzian and Gaussian distributions in the framework of f(G, T) gravity. Javed et al. [88] presented an analysis of different physical characteristics of wormhole configurations by taking Rastall gravity. Recently, the study of WH configuration with two specific choices of shape function in f(Q) gravity is presented in [89].

Inspired by the remarkable features of teleparallel gravity, it is always of great fascination to investigate different astrophysical issues in this background. In this paper, we are interested to explore a new wormhole solution in teleparallel gravity. For this purpose, we consider two different generic shape functions. It is well known that any relativistic model will be physically interesting if it is stable under fluctuations, thus motivated by this fact we also study stability conditions for the respective wormhole solution. The format of paper is as follows. In the next section, we discuss some basic formulations of the teleparallel theory of gravity and wormhole geometry. In Sect. 3, we obtain new wormhole solutions for two different generic shape functions in the teleparallel theory of gravity. Here we consider two different cases describing relations between pressure components. In Sect. 4, we also analyze the thin shell around a wormhole geometry by using the cut and paste approach. In this context, we choose Schwarzschild's black hole as an exterior manifold. Section 5 deals with the stability of the thin-shell wormhole configurations through linearized radial perturbation for both cases. Finally, we conclude our results in the last section.

2 Teleparallel gravity and wormhole geometry

Usually, Minkowski metric is used for the tangential space indices with both upper and lowered, i.e., η_{ij} , for the current framework for the Riemannian metric is provided as:

$$g_{\xi\zeta} = e^l{}_{\xi} e^l{}_{\zeta} \eta_{ij}, \tag{1}$$

where, relation $e^i{}_{\xi}\partial^{\xi} = e^i(x_{\xi})$ denotes the tetrad portion. For the current analysis, we consider the orthogonal basis for tangential can be rearranged by using the nontrivial tetrad. A well-known connection, like Weitzenbock can be described via tetrad field as

$$\Gamma^{\alpha}{}_{\xi\zeta} = e_i{}^{\alpha}\partial_{\zeta}e^i{}_{\xi} = -e^i{}_{\xi}\partial_{\zeta}e_i{}^{\alpha}.$$
(2)

The covariant derivative with Weitzenbock for tetrad fields calculate as zero, i.e

$$\nabla \xi e^{i}{}_{\zeta} = \partial \xi e^{i}{}_{\zeta} - \Gamma^{\alpha}{}_{\xi\zeta} e^{i}{}_{\alpha} = 0.$$
(3)

From the above connection, the non-vanishing torsion and null curvature are expressed as

$$T^{\alpha}{}_{\xi\zeta} = \Gamma^{\alpha}{}_{\zeta\xi} - \Gamma^{\alpha}{}_{\xi\zeta} = e_i{}^{\alpha}(\partial_{\xi}e^i{}_{\zeta} - \partial_{\zeta}e^i{}_{\xi}). \tag{4}$$

The Levi–Civita and Weitzenbock connections can be related with the following relation:

$$\hat{\Gamma}^{\alpha}{}_{\xi\zeta} = \Gamma^{\alpha}{}_{\xi\zeta} - K^{\alpha}{}_{\xi\zeta},\tag{5}$$

In the above equation $\hat{\Gamma}^{\alpha}{}_{\xi\zeta}$ and $K^{\alpha}{}_{\xi\zeta}$ are given as:

$$K^{\alpha}{}_{\xi\zeta} = \frac{1}{2} (T_{\xi}{}^{\alpha}{}_{\zeta} + T_{\zeta}{}^{\alpha}{}_{\xi} - T^{\alpha}{}_{\xi\zeta}).$$
(6)

The torsion can be rewritten as:

$$T = S^{\alpha\xi\zeta} T_{\alpha\xi\zeta},\tag{7}$$

here super-potential $S^{\alpha\xi\zeta}$ is given as:

$$S^{\alpha\xi\zeta} = -S^{\alpha\zeta\xi} = \frac{1}{2} (K^{\xi\zeta\alpha} - g^{\alpha\zeta} T^{\gamma\xi}{}_{\gamma} + g^{\alpha\xi} T^{\gamma\zeta}{}_{\gamma}).$$
(8)

For teleparallel theory, the gravitational type Lagrangian become

$$L_G = \frac{e}{16\pi}T,\tag{9}$$

where, c = G = 1 and $e = det(e^{i}_{\xi})$. The action for teleparallel theory is given as:

$$S = -\int e(\frac{T}{16\pi} + L_m)d^4x,$$
 (10)

In the above action, L_m represents the matter term for Lagrangian. By variational approach Eq. (10) termed as:

$$e^{-1}e^{i}{}_{\xi}\partial_{\varrho}(ee_{i}{}^{\gamma}S_{\gamma}{}^{\zeta\varrho}) + T^{\gamma}{}_{\lambda\xi}S_{\gamma}{}^{\zeta\lambda} + \frac{1}{4}\delta^{\zeta}_{\xi}T = 4\pi\Theta^{\zeta}_{\xi}, \quad (11)$$

where Θ_{ξ}^{ς} mentions the fluid source, which can calculated as:

$$D_{\zeta}(e^{-1}e^{i}{}_{\xi}\partial_{p}(ee_{i}{}^{\gamma}S_{\gamma}{}^{\zeta p}) + T^{\gamma}{}_{\lambda\xi}S_{\gamma}{}^{\zeta\lambda} + \frac{1}{4}\delta^{\zeta}_{\xi}T) = 0, \quad (12)$$

where the covariant derivative in the considered modified theory is written as

$$D_{\zeta} V^{\xi} = \partial_{\zeta} V^{\xi} + (\Gamma^{\xi}{}_{\lambda\zeta} - K^{\xi}{}_{\lambda\zeta}) V^{\lambda}, \qquad (13)$$

From Eq. (11), we obtain

$$D_{\zeta}\Theta_{\xi}{}^{\zeta} = 0. \tag{14}$$

In the present manuscript, we consider the matter source in the following form:

$$\Theta_{\xi}{}^{\zeta} = (\rho + p_t)u_{\xi}u^{\zeta} - p_t\delta_{\xi}^{\zeta} + (p_r - p_t)v_{\xi}v^{\zeta}.$$
 (15)

Here, the line element of spherically symmetric static spacetime is given as:

$$ds^{2} = e^{\xi(r)}dt^{2} - e^{\chi(r)}dr^{2} - r^{2}(d\theta^{2} + \sin^{2}\theta d\varphi^{2}), \quad (16)$$

with

- $\xi(r) = 2\Phi(r)$ and the red-shift function is denoted with $\Phi(r)$.
- $e^{\chi(r)} = \left(\frac{r+b(r)}{r}\right)^{-1}$ and the shape function is represented with b(r).
- The position of wormhole throat is denoted with r_0 which connects two asymptotic regions and it follows $b(r_0) = r_0$.

- For the flaring-out condition, the following constraint must be verified as $\frac{b(r)-rb'(r)}{2b^2(r)} > 0$. At wormhole's throat, it is reduced to $b'(r_0) < 1$.
- For the signature of line element, the shape function of wormhole throat must follows 1 ^{b(r)}/_r > 0 if r > r₀.
- To obtain the asymptotically flat regions, the following functions φ(r) and b(r)/r approaches to zero as r → ∞. Otherwise, these constraints are relaxed for non-asymptotically flat structures.

By considering Eqs. (15-16) into Eq. (11), the respective field equations in considered modified gravity become:

$$\rho = \frac{1}{4\pi} \left(\frac{e^{-\chi(r)} \left(\xi'(r) + \chi'(r) \right)}{2r} + \frac{1}{2r^2} - \frac{T(r)}{4} \right), \quad (17)$$

$$p_r = \frac{1}{4\pi} \left(\frac{T(r)}{4} - \frac{1}{2r^2} \right),$$
(18)

$$p_{t} = \frac{1}{4\pi} \left(\frac{e^{-\chi(r)}}{2} \left(\frac{\xi''(r)}{2} + \left(\frac{\xi'(r)}{4} + \frac{1}{2r} \right) \right) \\ \left(\xi'(r) - \chi'(r) \right) \right).$$
(19)

where

$$T(r) = \frac{\left(2e^{-\chi(r)}\right)\left(\xi'(r) + \frac{1}{r}\right)}{r},$$
(20)

It is noted that the behavior of the redshift function must be non-vanishing as well as finite at the wormhole throat. In this regard, we have considered $\Phi(r) = constant$ and hence $\Phi'(r) = \xi(r) = 0$.

3 Wormhole solutions

Now, we explore the new wormhole models by employing two different relations between pressure components.

3.1 Case-I

In the current analysis, we consider new relations between pressure components [90], which is expressed as:

$$p_t = \alpha_1 p_r + \alpha_2 p_r^2 \tag{21}$$

where α_1 and α_2 are constants. For the current analysis, we take $\alpha_1 \in [0, 0.7]$ and $\alpha_2 = 5.0$. Using Eqs. (18–19) in (21) with $e^{\chi(r)} = \left(\frac{r+b(r)}{r}\right)^{-1}$ and $\Phi'(r) = \xi(r) = 0$, we get a differential equation as:

$$\frac{\alpha_2 b(r)^2 - 4\pi (2\alpha_1 + 1)r^3 b(r) + 4\pi r^4 b'(r)}{64\pi^2 r^6} = 0 \quad (22)$$



Fig. 1 Wormhole properties for Model-I (left) and Model-II (Right)

On solving Eq. (22), the shape function for the current proposed model yields

$$b(r) = \frac{8\pi(\alpha_1 - 1)r^{2\alpha_1 + 3}}{8\pi\mathcal{K}_1\alpha_1r^2 - 8\pi\mathcal{K}_1r^2 + \alpha_2r^{2\alpha_1}}$$
(23)

where \mathcal{K}_1 is a integrating constant. Using the calculated shape function under the proposed scenario in Eqs. (17-19), we have

$$\rho = \frac{8\pi \mathcal{K}_1 (\alpha_1 - 1)^2 (2\alpha_1 + 1)r^{2\alpha_1 + 2} + 3\alpha_2 (\alpha_1 - 1)r^{4\alpha_1}}{\left(8\pi \mathcal{K}_1 (\alpha_1 - 1)r^2 + \alpha_2 r^{2\alpha_1}\right)^2},$$
(24)

$$\frac{(\alpha_1 - 1)r^{2\alpha_1}}{(\alpha_1 - 1)r^{2\alpha_1}}$$

$$p_{t} = -\frac{(\alpha_{1} - 1)r^{2\alpha_{1}} \left(8\pi \mathcal{K}_{1}(\alpha_{1} - 1)\alpha_{1}r^{2} + \alpha_{1}r^{2\alpha_{1}}\right)}{(26)},$$

$$p_t = -\frac{(\alpha_1 - \gamma) (\alpha_1 - \alpha_1 - \gamma) (\alpha_1 - \alpha_1 - \gamma)}{(8\pi \mathcal{K}_1(\alpha_1 - 1)r^2 + \alpha_2 r^{2\alpha_1})^2}.$$
 (26)

 $p_r =$

In the current section we consider another new relations between pressure components [90], which is defined as:

$$p_t = \beta_1 p_r + \beta_2 r^{\beta_3} p_r^2 \tag{27}$$

where β_1 , β_2 and β_3 are constants. For the current analysis, we take $\beta_1 \in [0, 0.7], \beta_2 = 5.0$ and $\beta_3 = 0.2$. By plugging the Eqs. (18–19) in Eq. (27) with $e^{\chi(r)} = \left(\frac{r+b(r)}{r}\right)^{-1}$

 $p_r =$



and $\Phi'(r) = \xi(r) = 0$, we get another differential equation which as

r (km)

$$\frac{4\pi r^3 \left(2\beta_1 b(r) - rb'(r) + b(r)\right) - \beta_2 r^{\beta_3} b(r)^2}{64\pi^2 r^6} = 0 \qquad (28)$$

We obtain the specific form of shape function by solving Eq. (28) expressed as

$$b(r) = \frac{4\pi(\beta_3 + 2\beta_1 - 2)r^{2\beta_1 + 3}}{4\pi\mathcal{K}_2\beta_3 r^2 + 8\pi\mathcal{K}_2\beta_1 r^2 - 8\pi c_1 r^2 + \beta_2 r^{\beta_3 + 2\beta_1}},$$
(29)

where \mathcal{K}_2 is another constant of integration. According to Morris and Thorne [51], our calculated shape functions satisfy the essential properties of a wormhole. Both the calculated shape functions are seen positive with increasing nature, which can be confirmed from the left side of Fig. 1. The derivative of both the models with respect to the r, i.e., $\frac{d\beta}{dr}$ is confirmed as less than one, which confirmed the flaring out condition. The graphical development of $\frac{d\beta}{dr}$ can be verified from the right side of Fig. 1. The ratio of both the models with radial coordinate, i.e., $\frac{\beta}{r}$ is approaching zero as r approaches to infinity, which can be confirmed from the left side of Fig. 1. For the current study, the wormhole throat $r_0 = 0.2$ and $r_0 = 0.3$ for model-I and model-II respectively. By using the calculated shape function under the proposed scenario in Eqs. (17-19), we obtain

$$\rho = -\frac{(\beta_3 + 2\beta_1 - 2)r^{2\beta_1} \left((\beta_3 - 3)\beta_2 r^{L+2\beta_1} - 4\pi \mathcal{K}_2(2\beta_1 + 1)r^2(\beta_3 + 2\beta_1 - 2)\right)}{2 \left(4\pi \mathcal{K}_2 r^2(\beta_3 + 2\beta_1 - 2) + \beta_2 r^{L+2\beta_1}\right)^2},$$
(30)

20

$$= -\frac{(\beta_3 + 2\beta_1 - 2)r^{2\beta_1}}{2\left(4\pi\mathcal{K}_2\beta_3r^2 + 8\pi\mathcal{K}_2\beta_1r^2 - 8\pi\mathcal{K}_2r^2 + \beta_2r^{\beta_3 + 2\beta_1}\right)},\tag{31}$$

$$p_{t} = \frac{(\beta_{3} + 2\beta_{1} - 2)r^{2\beta_{1}} \left((\beta_{3} - 2)\beta_{2}r^{\beta_{3} + 2\beta_{1}} - 8\pi\mathcal{K}_{2}\beta_{1}r^{2}(\beta_{3} + 2\beta_{1} - 2)\right)}{4 \left(4\pi\mathcal{K}_{2}r^{2}(\beta_{3} + 2\beta_{1} - 2) + \beta_{2}r^{\beta_{3} + 2\beta_{1}}\right)^{2}}.$$
(32)



Fig. 2 ρ for Case-I (left) and Case-II (Right)



Fig. 3 $\rho + p_r$ for Case-I (left) and Case-II (Right)

The graphical behavior of all the energy constraints for both calculated shape functions is given in Figs. 2, 3, 4, 5 and 6. In order to explore the behavior of energy conditions, we use the particular values of involved parameters. With these particular values, the density profile remains positive in the current study for both models. Figure 3 shows the graphical behavior of condition $\rho + p_r$ for both considered models. The negative values of $\rho + p_r$ confirm the presence of exotic matter. In fact, exotic matter is the very basic requirement for the traversable of the wormhole.

4 Thin-shell around wormholes structure

Here, we are interested to construct thin-shell around wormhole geometries by taking two choices of generic shape functions in the background of teleparallel gravity. In this regards, we use interior manifold as a wormhole geometry and exterior region as a Schwarzschild black hoe spacetime. Mathematically, it can be expressed as

$$ds^{2} = -\Pi_{\pm}(r_{\pm})^{-1}dr_{\pm}^{2} - r_{\pm}^{2}d\theta_{\pm}^{2} -r_{\pm}^{2}\sin^{2}\theta_{\pm}d\phi_{\pm}^{2} + \Pi_{\pm}(r_{\pm})dt_{\pm}^{2},$$
(33)



the metric functions of inner (-) and outer (+) manifolds are given as

$$\Pi_{-}(r_{-}) = -\frac{b(r_{-})}{r_{-}} + 1, \qquad \Pi_{+}(r_{+}) = -\frac{2m}{r_{+}} + 1, \quad (34)$$

the mass m only appears as an integration constant in the Schwarzschild solution.

Visser introduced the cut and paste technique to construct thin-shell from joining of inner wormhole geometry and outer Schwarzschild black hole spacetime at hypersurface. Here, we use this approach to develop the geometry of thin-shell around wormhole spacetime. In this regard, we cut these spacetimes into the following regions as

$$\mathcal{M}^{\pm} = \left\{ r^{\pm} \le k, k > r_h \right\},\tag{35}$$

where k is known as the radius of thin-shell and r_h represents the position of the event horizon of the Schwarzschild black hole. The interior wormhole geometry and exterior black hole spacetime are connected at (2+1)-dimensional manifold referred to as hypersurface given as

$$\Sigma = \left\{ r^{\pm} = k, k > r_h \right\}. \tag{36}$$

This procedure gives a unique regular manifold and mathematically it can be expressed as $\mathcal{M} = \mathcal{M}^- \cup \mathcal{M}^+$. It is noted



Fig. 4 $\rho - p_r$ for Case-I (left) and Case-II (Right)



Fig. 5 $\rho + p_t$ for Case-I (left) and Case-II (Right)



Fig. 6 $\rho - p_t$ for Case-I (left) and Case-II (Right)

that the event horizon and singularity in the developed structure can be avoided by using the radius of the event horizon of the Schwarzschild black hole less than the shell radius. By considering the Darmois–Israel formalism, the coordinates of considered manifolds and hypersurface are in the following form $y_{\pm}^{\gamma} = (t_{\pm}, r_{\pm}, \theta_{\pm}, \phi_{\pm})$ and $\eta^i = (\tau, \theta, \phi)$, respectively. Here τ represents the proper time over the hypersurface. These coordinate systems are related to one another by using the following coordinate transformation





$$g_{ij} = \frac{\partial y^{\gamma}}{\partial \eta^i} \frac{\partial y^{\beta}}{\partial \eta^j} g_{\gamma\beta}.$$
(37)

The respective parametric equation for the hypersurface is defined as

$$\Sigma: R(r,\tau) = r - k(\tau) = 0.$$

The physical quantities of matter distribution are evaluated through the Einstein field equations at hypersurface referred

0.070



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$$S^{\alpha}_{\beta} = \frac{1}{8\pi} (\delta^{\alpha}_{\beta} \zeta^{\gamma}_{\gamma} - \zeta^{\alpha}_{\beta}), \tag{38}$$

where $\zeta_{\alpha\beta} = K_{\alpha\beta}^+ - K_{\alpha\beta}^-$ and $K_{\alpha\beta}^-$ represents the components of extrinsic curvature. For perfect fluid distribution, the stress energy tensor is expressed as $S^{\alpha}{}_{\beta} = diag(\rho, \mathcal{P}, \mathcal{P})$. The energy density and pressure of the matter distribution located at Σ is denoted by ρ and \mathcal{P} , respectively. The extrinsic curvature of interior and exterior geometries are defined as

$$K_{\alpha\beta}^{\pm} = -n_{\mu}^{\pm} \left[\frac{\partial^2 y_{\pm}^{\mu}}{\partial \eta^{\alpha} \eta^{\beta}} + \Gamma_{\lambda\nu}^{\mu} \left(\frac{\partial y_{\pm}^{\lambda}}{\partial \eta^{\alpha}} \right) \left(\frac{\partial y_{\pm}^{\nu}}{\partial \eta^{\beta}} \right) \right], \tag{39}$$

Further, we can define the unit normals as follow

$$n_{\pm}^{\mu} = \left(\frac{\dot{k}}{\Phi_{\pm}(k)}, \sqrt{\Phi_{\pm}(k) + \dot{k}^2}, 0, 0\right),\tag{40}$$

where the proper time derivative is represented with overdot.

By using Lanczos equations, we get

and

$$\sigma_0 + 2\mathcal{P}_0 = \frac{b'(k_0)\sqrt{1 - \frac{2m}{k_0}} + \sqrt{1 - \frac{b(k_0)}{k_0}} - \sqrt{1 - \frac{2m}{k_0}}}{4\pi k_0\sqrt{1 - \frac{b(k_0)}{k_0}}\sqrt{1 - \frac{2m}{k_0}}},$$
(45)

where surface energy density and pressure at equilibrium position are denoted by σ_0 and \mathcal{P}_0 , respectively.

It is noted that $\sigma_0 < 0$ which leads to the violation of week as well as dominant energy constraints. Such violations indicate that the obtained structure is filled with matter distribution having exotic nature. These matter distributions at thin-shell produce repulsion against collapse and also helps to keep it open. Hence, the developed structure is physically acceptable for the observer movement among this configuration.

5 Stability analysis through radial linear perturbation

Now, we are interested to explore the stability of constructed thin-shell around wormhole geometry through linearized

$$\sigma = -\frac{[K_{\theta}^{\theta}]}{4\pi} = -\frac{\sqrt{\dot{k}^2 - \frac{2m}{k} + 1} - \sqrt{-\frac{b(k)}{k} + \dot{k}^2 + 1}}{4\pi k},$$

$$P = \frac{[K_{\theta}^{\theta}] + [K_{\tau}^{\tau}]}{8\pi}$$
(41)

$$= \frac{1}{8\pi k} \left\{ \frac{k \left(b'(k) - 2 \left(\dot{k}^2 + \ddot{k}k + 1 \right) \right) + b(k)}{k \sqrt{-\frac{b(k)}{k} + \dot{k}^2 + 1}} + \frac{2 \left(\dot{k}^2 k + \ddot{k}k^2 + k - m \right)}{k \sqrt{\dot{k}^2 - \frac{2m}{k} + 1}} \right\},\tag{42}$$

while

$$\sigma + 2\mathcal{P} = \frac{b'(k)\sqrt{\dot{k}^2 - \frac{2m}{k} + 1} - \left(\dot{k}^2 + 2\ddot{k}k + 1\right)\left(\sqrt{\dot{k}^2 - \frac{2m}{k} + 1} - \sqrt{-\frac{b(k)}{k} + \dot{k}^2 + 1}\right)}{4\pi k\sqrt{-\frac{b(k)}{k} + \dot{k}^2 + 1}\sqrt{\dot{k}^2 - \frac{2m}{k} + 1}}.$$
(43)

Now, it is assumed that thin-shell of the developed geometry does not move along its radial direction at the equilibrium shell radius k_0 . Therefore, it is interesting to mention that the proper time derivative of shell radius vanish, i.e., $\dot{k_0} = 0 = \ddot{k_0}$. Hence, we have

$$\sigma_{0} = -\frac{1}{4\pi k_{0}} \left\{ \sqrt{-\frac{2m}{k_{0}} + 1} - \sqrt{-\frac{b(k_{0})}{k_{0}} + 1} \right\},$$

$$\mathcal{P}_{0} = \frac{1}{8\pi k_{0}^{2}} \left\{ \frac{k_{0} \left(b'(k_{0}) - 2 \right) + b(k_{0})}{\sqrt{1 - \frac{b(k_{0})}{k_{0}}}} + \frac{2(k_{0} - m)}{\sqrt{1 - \frac{2m}{k_{0}}}} \right\},$$
(44)

radial perturbation at $k = k_0$. For this purpose, we develop the equation of motion of the shell from Eq. (41) as

$$\dot{k}^2 + \mathcal{V}(k) = 0,$$
 (46)

here the effective potential function of thin-shell is denoted with $\mathcal{V}(k)$. It is defined as

$$\mathcal{V}(k) = \frac{mb(k)}{16\pi^2 k^4 \sigma^2} - \frac{b(k)^2}{64\pi^2 k^4 \sigma^2} - \frac{b(k)}{2k} - \frac{m^2}{16\pi^2 k^4 \sigma^2} - \frac{4\pi^2 k^2 \sigma^2}{k} + 1.$$
(47)

The components of stress-energy tensor follows the energy conservation constraints as

$$\mathcal{P}\frac{d}{d\tau}(4\pi k^2) + \frac{d}{d\tau}(4\pi k^2\sigma) = 0,$$



Fig. 7 Plots of the solution of Eq. (50) with different values of ω^2

which turns out to be

$$\sigma' = -\frac{2(\sigma + \mathcal{P}(\sigma))}{k}.$$

For stability analysis, we use Taylor's series to expand the effective potential upto second-order terms about equilibrium shell radius as

$$\mathcal{V}(k) = \mathcal{V}(k_0) + (k - k_0)\mathcal{V}'(k_0) + \frac{1}{2}(k - k_0)^2\mathcal{V}''(k_0) + O[(k - k_0)^3],$$

where $\mathcal{V}(k_0) = 0 = \mathcal{V}'(k_0)$. By taking $J = k - k_0$, we get

$$J^2 + \omega^2 J^2 \simeq 0, \tag{48}$$

where

$$\omega^2 = \frac{1}{2} \frac{d^2 \mathcal{V}}{dk^2}|_{k=k_0}.$$
(49)

Differentiating Eq. (48) with respect to proper time, it follows that

$$\ddot{J} + \omega^2 J \simeq 0. \tag{50}$$

This equation plays an important role to discuss stability of the developed structure which depends on ω^2 . It is found that thin-shell expresses oscillation about J = 0 for $\omega^2 > 0$ as shown in the left plot of Fig. 7. Hence, the developed structure indicates oscillation about the equilibrium shell radius ($J = 0 \Rightarrow k = k_0$) and remains stable. This leads to a stable configuration of a thin-shell. We can write this condition as

$$\frac{d^2 \mathcal{V}}{dk^2}|_{k=k_0} > 0.$$
(51)

If $\omega^2 < 0$, then the shell's radius represents the exponential behavior which corresponds to the unstable behavior as shown in the right plot of Fig. 7. The respective condition for the unstable configuration can be expressed as

$$\frac{d^2 \mathcal{V}}{dk^2}|_{k=k_0} < 0.$$
(52)



The developed structure become stable if $\mathcal{V}(k_0) = 0 = \mathcal{V}'(k_0)$. Hence, the above yields

$$\mathcal{V}(k) = \frac{1}{2}(k - k_0)^2 \mathcal{V}''(k_0).$$
(53)

Thin-shell mass and its radial derivatives in terms of energy density and pressure are given as

$$M(k_0) = 4\pi k_0^2 \sigma_0, \qquad M'(k_0) = -8\pi k_0 p_0, M''(k_0) = -8\pi \mathcal{P}_0 + 16\pi \xi_0^2 (\sigma_0 + \mathcal{P}_0),$$

here $\xi_0^2 = dp/d\sigma|_{k=k_0}$ is an EoS parameter. So, we get

$$\mathcal{V}''(k_0) = -\frac{1}{2k_0^4 M^4} \left\{ -k_0^4 M (2m - b(k_0)) \times \left(M''(2m - b(k_0)) - 4M'b'(k_0) \right) + k_0^4 M^2 \left((b(k_0) - 2m)b''(k_0) + b'(k_0)^2 \right) + k_0 M^4 \left(k_0 \left(k_0 b''(k_0) - 2b'(k_0) + (M')^2 \right) + 2b(k_0) + 4m \right) + 3k_0^4 \left(M' \right)^2 (b(k_0) - 2m)^2 + k_0 M^5 \left(k_0 M'' - 4M' \right) + 3M^6 \right\}.$$
(54)

Here, we consider $\mathcal{V}''(k_0) > 0$ to analyze the stability of the developed structure. Hence, Eq. (54) becomes

$$-\left(-\sigma_{0}(2m-b(k_{0}))\left(32\pi k_{0}\mathcal{P}_{0}b'(k_{0})\right.\right.\\\left.+\left(2m-b(k_{0})\right)\left(16\pi\xi_{0}^{2}(\mathcal{P}_{0}+\sigma_{0})-8\pi\mathcal{P}_{0}\right)\right)\\\left.+48\pi\mathcal{P}_{0}^{2}(b(k_{0})-2m)^{2}\right.\\\left.+4\pi k_{0}^{2}\sigma_{0}^{2}\left((b(k_{0})-2m)b''(k_{0})+b'(k_{0})^{2}\right)\right.\\\left.+64\pi^{3}k_{0}^{3}\sigma_{0}^{4}\left(k_{0}^{2}b''(k_{0})-2k_{0}b'(k_{0})+2b(k_{0})\right.\\\left.+64\pi^{2}k_{0}^{3}\mathcal{P}_{0}^{2}+4m\right)\right.\\\left.+2048\pi^{5}k_{0}^{6}\sigma_{0}^{5}\left(\left(2\xi_{0}^{2}+3\right)\mathcal{P}_{0}+2\xi_{0}^{2}\sigma_{0}\right)\right.\\\left.+3072\pi^{5}k_{0}^{6}\sigma_{0}^{6}\right)\left(128\pi^{3}k_{0}^{6}\sigma_{0}^{4}\right)^{-1}>0$$

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Fig. 8 Stable regions for the choice of first shape function (case(i)) for different values of α_1 . Left plots show the graphical behavior of λ_0 and right plots expressed the stable regions. The straight line in right plots

show the position of expected event horizon of the developed structure. Here, the shaded regions show the stable regions of developed structure

Further, we can write stability constraints in the following form

$$\mathcal{V}''(k_0) > 0 \implies \lambda(h_0)\xi_0^2 - \mathcal{E}_0 > 0.$$
 (55)

Here, the coefficient of EoS parameter is denoted by $\lambda(k_0) = \lambda_0$ and remaining terms are named as $\mathcal{E}(k_0) = \mathcal{E}_0$.

The geometrical configuration of thin-shell is explored by using stable regions which can be expressed as

(i) For
$$\lambda_0 < 0 \Rightarrow \xi_0^2 < \mathcal{E}_0/\lambda_0$$
:
(ii) For $\lambda_0 > 0 \Rightarrow \xi_0^2 > \mathcal{E}_0/\lambda_0$,

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where

$$\begin{aligned} \mathcal{E}_{0} &= b(k) \left(8 \left(m \mathcal{P}_{0}(6\mathcal{P}_{0} + \sigma_{0}) - 4\pi^{2}k_{0}^{3}\sigma_{0}^{4} \right) \\ &- k_{0}\sigma_{0} \left(k_{0}\sigma_{0}b''(k_{0}) + 8\mathcal{P}_{0}b'(k_{0}) \right) \right) \\ &+ k_{0}\sigma_{0} \left(2k_{0}\sigma_{0}b''(k_{0}) \left(m - 8\pi^{2}k_{0}^{3}\sigma_{0}^{2} \right) \\ &+ b'(k_{0}) \left(-k_{0}\sigma_{0}b'(k_{0}) + 32\pi^{2}k_{0}^{3}\sigma^{3} + 16m\mathcal{P}_{0} \right) \right) \\ &- 2\mathcal{P}_{0}b(k_{0})^{2}(6\mathcal{P}_{0} + \sigma_{0}) \\ &- 8 \left(32\pi^{4}k_{0}^{6}\sigma_{0}^{4} \left(4\mathcal{P}_{0}^{2} + 6\mathcal{P}_{0}\sigma_{0} + 3\sigma_{0}^{2} \right) \right) \end{aligned}$$



Fig. 9 Stable regions for the choice of first shape function (case(i)) for different values of \mathcal{K}_1 . Here, the shaded regions show the stable regions of developed structure

$$+ 8\pi^{2}k_{0}^{3}m\sigma_{0}^{4} + m^{2}\mathcal{P}_{0}(6\mathcal{P}_{0} + \sigma_{0})),$$

$$\lambda_{0} = 4\sigma_{0}(\mathcal{P}_{0} + \sigma_{0})$$

$$\times \left(4mb(k_{0}) - b(k_{0})^{2} + 256\pi^{4}k_{0}^{6}\sigma_{0}^{4} - 4m^{2}\right).$$

The characteristics of thin-shell around wormhole geometry are explored through stable regions for both generic shape functions as discussed in cases (i) and (ii).

5.1 For
$$b(k) = \frac{8\pi(\alpha_1 - 1)k^{2\alpha_1 + 3}}{8\pi\mathcal{K}_1\alpha_1k^2 - 8\pi\mathcal{K}_1k^2 + \alpha_2k^{2\alpha_1}}$$

Now, we consider the first case of shape function to discuss stability regions of the developed structure. For this purpose, we use the following functions λ_0 and \mathcal{E}_0/λ_0 to obtain the stability regions. If $\lambda_0 < 0$, the stability of thinshell is expressed with a light gray shaded region in the plot of \mathcal{E}_0/λ_0 as shown in Figs. 8 and 9. In each of the figures, the left plot expressed the graphical behavior of λ_0 and the right plot shows stable regions through plots of \mathcal{E}_0/λ_0 . If $\lambda_0 > 0$,



Fig. 10 Stable regions for the choice of first shape function (case(ii)) for different values of α_1 . Here, the shaded regions show the stable regions of developed structure

then the light blue region shows a stable configuration of the shell. Figure 8 explains the effects of coupling constant α_1 on the position of event horizon as well as stable regions. It is noted that stability regions are decreased as α_1 increases. The integration constant \mathcal{K}_1 greatly effect the geometrical configuration of the developed structure. The stable regions are decreased for the higher values of the integration constant.

5.2 For
$$b(k) = \frac{4\pi(\beta_3 + 2\beta_1 - 2)k^{2\beta_1 + 3}}{4\pi\mathcal{K}_2\beta_3 k^2 + 8\pi\mathcal{K}_2\beta_1 k^2 - 8\pic_1 k^2 + \beta_2 k^{\beta_3 + 2\beta_1}}$$

For the second case of shape function, we explore characteristics of geometrical structure for different values of the coupling constant β_1 and integration constant \mathcal{K}_2 . Stable regions are decreased for the higher values of β_1 and \mathcal{K}_2 as shown in Figs. 10 and 11.

6 Concluding remarks

This analysis has discussed the spherically symmetric wormhole geometries involving two different and most generic approaches in the background of the teleparallel gravity. In the evolution of some unique features of the wormhole structures under teleparallel gravity, we calculate the density ρ ,



Fig. 11 Stable regions for the choice of first shape function (case(ii)) for different values of \mathcal{K}_2 . Here, the shaded regions show the stable regions of developed structure

the tangential pressure p_t , and the radial p_r for the wormhole solution obtained here. It is noted that the calculated new shape functions are the most generic which are viable and meet the Morris and Thorne wormhole criterion. Further, we have calculated the exact shape functions for two different models and have discussed their few major aspects, such as energy conditions and their physical interpretation in the framework of the teleparallel gravity. The obtained shape functions have fulfilled all the necessary conditions for the existence of wormhole structures under two different sources. All the required conditions, including flaring out property, have been provided in Fig. 1. Some necessary expressions for defining the energy conditions, ρ , $\rho + p_r$, $\rho - p_r$, $\rho + p_t$, $\rho - p_t$, and $\rho + p_r + 2p_t$ are shown graphically in Figs. 2, 3, 4, 5 and 6 respectively. The violation of NEC is revealed, which causes the existence of exotic matter, which can be confirmed in Fig. 3 for both the models.

Moreover, we have discussed the stable configuration of the thin-shell around the wormhole geometry in the background of the teleparallel gravity. For this purpose, we have considered Schwarzschild black hole spacetime as an exterior geometry. We have obtained the wormhole solution with two choices of the shape function, taken as an interior manifold. We have explored the stability of the thin-shell around the wormhole geometry by using linearized radial perturbation about the equilibrium position of the shell radius. Stable regions have decreased for the higher values of β_1 and \mathcal{K}_2 as shown in Figs. 8 and 11 for both the models. It should be noted that stable regions are obtained for every choice of the shape function with suitable values of the physical parameter. The stable regions and position of the expected event horizon depend on the physical parameters.

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