



Localizing branes with bifurcating bulks

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Abstract We study the problem of evolution of bulk 5-fluids having an embedded braneworld with a flat, de Sitter, or anti-de Sitter geometry. We introduce new variables to express the Einstein equations as a dynamical system that depends on the equation of state parameter γ and exponent λ . For linear fluids (i.e., $\lambda = 1$), our formulation leads to a partial decoupling of the equations and thus to an exact solution. We find that such a fluid develops a transcritical bifurcation around the value $\gamma = -1/2$, and study how this behaviour affects to stability of the solutions. For nonlinear fluids, the situation is more diverse. We find an overall attractor at $\lambda = 1/2$ and draw enough phase portraits to exhibit in detail the overall dynamics. We show that the value $\lambda = 3/2$ is structurally unstable and typical for other forms of λ . Consequently, we observe a noticeable dependence of the qualitative behaviour of the solutions on different ‘polytropic’ forms of the fluid bulk. In addition, we prove the existence of a Dulac function for nonlinear fluids, signifying the impossibility of closed orbits in certain subsets of the phase space. We also provide ample numerical evidence of gravity localizing solutions on the brane which satisfy all energy conditions.

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1 Introduction

It has been known for some time that singularity-free solutions are possible for the scale factor and thermodynamical quantities describing a 4-dimensional braneworld embedded in a 5-dimensional space with a fluid analogue depending on the extra spatial coordinate and with a specific equation of state [1–10]. The problem is to find the most general circumstances that allow such solutions with the properties of satisfying the energy conditions and localising gravity on the brane.

In a series of works, we have been able to find special families of asymptotic solutions that satisfy all the above-mentioned properties (cf. [11, 12] and refs. therein). Although this search constitutes a viable approach to the cosmological constant problem through the mechanism of self-tuning, the search for the simplest solutions with the desired properties does not reveal the structure of the whole space of interesting solutions.

In this paper, we assume that the bulk is non-compact and we provide a detailed study of half of the space. In analogy with Ref. [4], we look at the support of curved branes that localize gravity and satisfy the energy conditions. We use a bulk fluid-analog to make a model-independent analysis without specific field representation.

This setup leads us to consider the general problem of a 4-braneworld embedded in a five-dimensional bulk space filled with a linear or nonlinear fluid from a more qualitative, dynamical viewpoint. This provides us with an insight into the global geometry of the orbits and the dynamics in the

phase space of the problem. In addition, a more careful definition of the meaning of singularity-free solutions allows us to look into the problem from a more precise point of view.

The main difference of the present approach with earlier analyses is that although previously our solutions were obtained as functions of the fifth coordinate Y , in the present work we look for the global behaviour of solutions as functions of the initial conditions. To achieve this goal, we introduce new variables and a novel formulation of the basic brane-bulk equations.

These variables are analogues of the Hubble and the density parameters H , Ω of relativistic cosmology, and are given as functions of a suitable monotone reparametrization of Y . Also since the new variables contain the scale factor, its first derivative, as well as the density, they are able to provide a more precise picture of the possible singular solutions. The resulting formulation transforms the whole setup into a dynamical systems problem that can then be studied using qualitative methods.

The reduction of the dynamics to the aforementioned form allows interesting dynamical properties to be studied here for the first time in a brane-bulk phase space context. These include the topological nature and bifurcations of equilibria, the phase portraits of the dynamics, the question of existence of closed orbits, as well as the dependence of the Planck-mass integral on initial conditions.

The plan of this paper is as follows. In the next Section, we rewrite the problem in terms of new variables and arrive at dynamical equations describing bulk fluids with an equation of state, and describe general features of the dynamics in Section 3. In Sections 4, 5, we analyse the structure of dynamical solutions for linear and nonlinear equations of state. In Section 6, we discuss the problem of localization of gravity on the brane, and we conclude with summarizing our results in the last Section.

2 Dimensionless formulation

In this Section, we rewrite the basic dynamical equations in a new dimensionless formulation for both the linear and the nonlinear fluid cases.

The five-dimensional Einstein equations on the bulk are given by,

$$G_{AB} = R_{AB} - \frac{1}{2}g_{AB}R = \kappa_5^2 T_{AB}, \quad (2.1)$$

and we assume a bulk-filling fluid analogue with an energy-momentum tensor,

$$T_{AB} = (\rho + p)u_A u_B - pg_{AB}, \quad (2.2)$$

where the indices run from 1 to 5, the ‘pressure’ p and the ‘density’ ρ are functions only of the fifth coordinate Y , and

the fluid velocity vector field $u_A = \partial/\partial Y$ is parallel to the Y -dimension. We also choose units such that $\kappa_5 = 1$. We consider below the evolution of this model for a brane-bulk metric given by

$$g_5 = a^2(Y)g_4 + dY^2, \quad (2.3)$$

where $a(Y)$ is a warp factor with $a(Y) > 0$, while the brane metric g_4 is taken to be the four-dimensional flat, de Sitter or anti-de Sitter standard metric.

With this setup, the Einstein equations split into the conservation equation,

$$\rho' + 4(p + \rho)\frac{a'}{a} = 0, \quad (2.4)$$

the Raychaudhuri equation,

$$\frac{a''}{a} = -\frac{1}{6}(2p + \rho), \quad (2.5)$$

and the Friedmann equation,

$$6\frac{a'^2}{a^2} = \rho + \frac{6k}{a^2}, \quad (2.6)$$

which is a first integral of the other two when $a' \neq 0$. Here, the constant k is zero for a flat brane, $k = 1$ for a de Sitter brane, and $k = -1$ for an anti de Sitter brane. For the brane-bulk problem, there is also the junction condition which in general describes a jump discontinuity in the first derivative of $a(Y)$ and takes the generic form,

$$a'(0^+) - a'(0^-) = f(a(0), \rho(0)), \quad (2.7)$$

where the brane tension f is a continuous, non-vanishing function of the initial values of a , ρ (for examples of this in specific solutions, see Ref. [11, 12]).

Inspired by standard cosmology, we introduce the following ‘observational parameters’ related to the extra ‘bulk’ dimension Y : The *Hubble scalar* H , measuring the rate of the expansion (a prime ‘ means d/dY),

$$H = \frac{a'}{a}, \quad (2.8)$$

the *deceleration parameter* q , which measures the possible speeding up or slowing down of the expansion in the Y dimension,

$$q = -\frac{a''a}{a'^2}, \quad (2.9)$$

and the *density parameter* Ω describing the bulk matter density effects,

$$\Omega = \frac{\rho}{3H^2}. \quad (2.10)$$

While Ω , q are dimensionless, the Hubble scalar H has dimensions $[Y]^{-1}$.

Using these variables and dividing both sides by $3H^2$, the Friedmann equation (2.6) is,

$$2 - \Omega = \frac{2k}{H^2 a^2}, \quad (2.11)$$

and we conclude that evolution of the 5-dimensional models with

- $\Omega > 2$ corresponds to those having an AdS brane ($k = -1$)
- $\Omega = 2$ corresponds to those having a flat brane ($k = 0$)
- $\Omega < 2$ corresponds to those having a dS brane ($k = +1$).

(By redefining $\lambda_5^2 = \kappa_5^2/2$, we would have obtained the usual trichotomy $\Omega \gtrless 1$ relations here, but this would have also changed various coefficients in the other field equations, so we prefer to leave it as above.)

We set Y_0 for some arbitrarily chosen reference value of the bulk variable Y , and $a_0 = a(Y_0)$. The evolution will be described not in terms of Y but by a new dimensionless bulk variable τ in the place of Y , with,

$$a = a_0 e^\tau. \quad (2.12)$$

Then we have,

$$a' = a \frac{d\tau}{dY}, \quad (2.13)$$

and hence,

$$\frac{dY}{d\tau} = \frac{1}{H}. \quad (2.14)$$

We shall assume that the brane lies at $\tau = 0$, or $a = a_0$ (we may assume without loss of generality that $Y_0 = 0$). In this case, the junction condition (2.7) expressed in terms of the new variables H, Ω implies that,

$$H(0^+) - H(0^-) = f(x(0), 0), \quad x = H^2 \Omega, \quad (2.15)$$

where the brane tension f is a function of the variables x, τ . The variable x is regular from the definition (2.10), and τ is regular from (2.14) because H has only a finite discontinuity at 0 as it follows from the junction condition (2.7).

Having a brane located at the length scale value a_0 , with $0 < a < +\infty$, we consider two intervals, $\tau \in (-\infty, 0)$ - the 'left-side' evolution, and the 'right-side' interval $\tau \in (0, +\infty)$. The latter is equivalent to the left-side interval under the transformation $\tau \rightarrow -\tau$, and so without loss of generality we shall restrict our attention only to the τ -range $(-\infty, 0)$. All our results involving the dimensionless variable τ can be transferred to the right-side interval by taking $\tau \rightarrow -\tau$. This will be important later.

We now show that the dynamics of the system (2.4)–(2.6) can be equivalently described by a simpler dynamical system in terms of H and the dimensionless variables q, τ , and Ω .

From the above definitions for H and q , we are led to the following evolution equation for H , namely,

$$\frac{dH}{d\tau} = -(1 + q)H. \quad (2.16)$$

In the following we shall assume that fluids in the bulk satisfy linear or nonlinear equations of state. Then the q -equation for a linear equation of state (EoS),

$$p = \gamma \rho, \quad (2.17)$$

is found to be,

$$q = \left(\frac{1}{2} + \gamma\right) \Omega, \quad (2.18)$$

while for the nonlinear equation of state,

$$p = \gamma \rho^\lambda, \quad (2.19)$$

for some parameter λ , we have,

$$q = \left(\frac{1}{2} + \gamma \rho^{\lambda-1}\right) \Omega. \quad (2.20)$$

We note here the usual fluid parameter γ will be constrained later using the energy conditions. On the other hand, the parameter λ , the ratio c_P/c_V of the specific heats of the bulk fluid under constant pressure and volume, is commonly taken to satisfy $\lambda > 1$ in other contexts, most notable in standard stellar structure theory (cf. e.g., [13], chap. IV). Although we generally put no constraint on it, we shall find that in the present problem there is a clear preference for the 'polytropic' values $\lambda = 1 + 1/n$, for integer n , valid for the entire bulk. Such polytropic changes provide ample differences as compared to the 'collisionless' case $n = \infty$, for small values of n .

The evolution equation for the Hubble scalar for the linear EoS case (2.17), becomes,

$$\frac{dH}{d\tau} = -\left(1 + \left(\gamma + \frac{1}{2}\right) \Omega\right) H, \quad (2.21)$$

whereas the nonlinear equation of state case (2.19), gives the Hubble evolution equation in the form,

$$\frac{dH}{d\tau} = -H - \frac{\Omega H}{2} - 3^{\lambda-1} \gamma H^{2\lambda-1} \Omega^\lambda. \quad (2.22)$$

Let us lastly consider the continuity equation (2.4). This equation, assuming that $H \neq 0$, in the linear-fluid case becomes,

$$\frac{d\Omega}{d\tau} = 2(q - 2\gamma - 1)\Omega. \quad (2.23)$$

On the other hand, in the nonlinear-fluid case, assuming again that $H \neq 0$, and using Eq. (2.22), we find the following evolution equation for Ω , namely,

$$\frac{d\Omega}{d\tau} = -2\Omega + \Omega^2 + 2\gamma 3^{\lambda-1} H^{2\lambda-2} \Omega^{\lambda+1}$$

$$-4\gamma 3^{\lambda-1} H^{2\lambda-2} \Omega^\lambda. \quad (2.24)$$

Summarizing, in our new formulation of the bulk-brane problem, the basic dynamical systems are given by the Friedman constraint Eq. (2.11), together with evolution equations in the following forms.

Case A: Nonlinear EoS, $p = \gamma\rho^\lambda$. In this case, we have a 2-dimensional dynamical system, namely,

$$\frac{dH}{d\tau} = -H - \frac{\Omega H}{2} - 3^{\lambda-1} \gamma H^{2\lambda-1} \Omega^\lambda, \quad (2.25)$$

$$\frac{d\Omega}{d\tau} = -2\Omega + \Omega^2 + 2\gamma 3^{\lambda-1} H^{2\lambda-2} \Omega^{\lambda+1} - 4\gamma 3^{\lambda-1} H^{2\lambda-2} \Omega^\lambda. \quad (2.26)$$

Case B: Linear EoS, $p = \gamma\rho$. This is the special case with $\lambda = 1$. We have the H equation,

$$\frac{dH}{d\tau} = -\left(1 + \left(\gamma + \frac{1}{2}\right)\Omega\right)H, \quad (2.27)$$

and a single, decoupled evolution equation for Ω , namely,

$$\frac{d\Omega}{d\tau} = 2\left(\left(\gamma + \frac{1}{2}\right)\Omega - 2\gamma - 1\right)\Omega. \quad (2.28)$$

3 General properties

The 5-dimensional fluid solutions are then given in terms of the (H, Ω) variables which satisfy these evolution equations together with the Friedman constraint Eq. (2.11). For their physical interpretation, it is helpful to use a new classification in terms of H , Ω and k . We shall say that the bulk fluid is:

1. a *dS (resp. an AdS) fluid*, when $k = +1$ (resp. $k = -1$)
2. a *flat fluid*, when $k = 0$.
3. *Static*, when $H = 0$ (it is necessarily flat in this case).
4. *Expanding (resp. contracting)*, when $H > 0$ (< 0). (This means that the fluid is moving away (resp. towards) the brane for positive τ)

As we have noted after Eq. (2.11), the cases $\Omega < 2$, $= 2$, > 2 correspond to a dS, a flat, or a AdS fluid respectively, while in the case $\Omega = 0$, the bulk is empty, and the constraint equation (2.11) necessarily implies $k = +1$ for consistency, hence, $a(\tau) = \tau + C$ in this case. We shall refer to the $\Omega = 0$ case as an *empty dS bulk*.

We shall also invariably refer to any given phase point (H, Ω) as a ‘state’, for example the point $(0, 0)$ describes the state of a static, empty dS bulk, while the $(0, 2)$ state is a static, flat fluid. Dynamical (non-static) states require $H \neq 0$, and these are described as non-trivial orbits in the

(H, Ω) phase space. Further classification tags for each one of these models appear in the next Sections and depend on the ranges and values of the two fluid parameters γ, λ that appear in the evolution equations.

Some general remarks about the dynamical system (2.25)–(2.26), and its special case (2.27)–(2.28) are in order:

- Since the dynamical systems studied in this paper are two-dimensional (with a constraint), our search is for bifurcations, oscillations, or limit cycles, but no chaotic behaviour, strange attractors, or more complex phenomena can be present here.
- Equation (2.26) implies that the set $\Omega = 0$ is invariant under the flow of the dynamical system, i.e. $\Omega = 0$ is a solution of the system. Since no trajectories of the dynamical system can cross, we conclude that if initially the state of the system is on the line $\Omega = 0$ (that is if we start with an ‘empty dS bulk’), it will remain on this line for ever. Therefore, if initially Ω is positive, it remains positive for ever. We emphasize that this result holds for all $\lambda \geq 0$.
- If $\lambda \geq 1$, Eq. (2.25) implies that the set $H = 0$ is also invariant under the flow of the dynamical system. Therefore, for $\lambda \geq 1$, assuming that initially $H > 0$, then $H(\tau) > 0$ for all $\tau \geq 0$. We conclude that any trajectory starting at the first quadrant $H \geq 0, \Omega \geq 0$, cannot cross the axes and therefore, cannot escape out of this quadrant. For instance, expanding AdS fluids, and expanding empty dS bulks remain always expanding, and static AdS fluids always remain static.
- For the linear EoS case, it is important that the Ω -equation (2.28) decouples, that is, it does not contain the H . This decoupling is due to our choice of new variables (cf. Eq. (2.12)). So now the Ω -equation can be treated separately as a logistic-type equation, and this simplifies the analysis considerably. In this case, solving the Ω -equation and substituting in the H -equation, we have a full solution of the system. This feature is absent from the nonlinear EoS fluid equations, which comprise a truly coupled 2D system. This is studied more fully below.
- The necessity of satisfying the energy conditions (cf. [11]) leads in general to restrictions on the γ range. For a fluid with a linear EoS the intersection of the requirements that follow from the weak, strong, or null energy conditions lead to the typical range $\gamma \in [-1, 1]$. We shall assume this restriction as a minimum requirement for our acceptance of a solution property.

Below, with an slight abuse of language, we shall use the term *linear (nonlinear) fluid* when the respective EoS is linear (nonlinear).

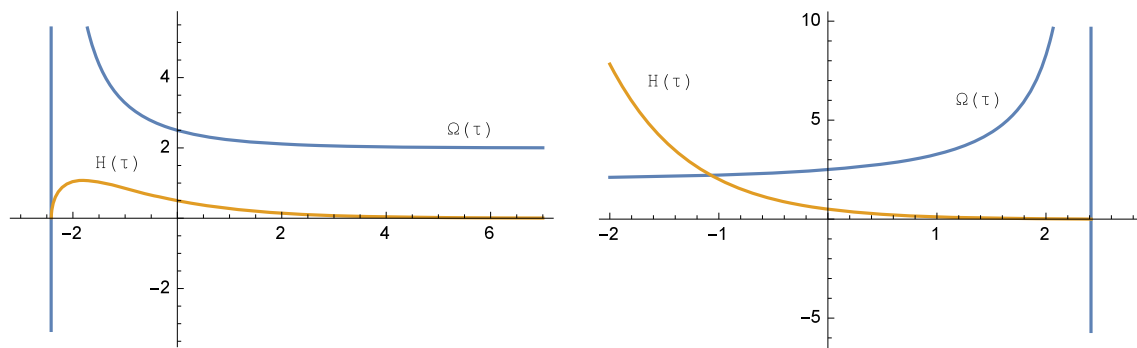


Fig. 1 Solutions (4.2) for $\gamma < -1/2$ and $\gamma > -1/2$ when $\Omega_0 > 2$

4 Linear fluids and their bifurcations

This Section provides a study of the behaviour of bulk fluids with the linear EoS given by Eq. (2.17) and described by the nonlinear system (2.27), (2.28), (2.11).

This system can be solved exactly and the asymptotic properties of the solutions displayed graphically. Equation (2.28) has the form

$$\frac{d\Omega}{d\tau} = A\Omega^2 + B\Omega, \quad (4.1)$$

with $A = 2(\gamma + 1/2)$, $B = -2A$. The Ω -solution from Eq. (4.1) with initial condition $\Omega(0) = \Omega_0$ is given by

$$\begin{aligned} \Omega(\tau) &= \frac{B/A}{\left(\frac{B}{A\Omega_0} + 1\right)e^{-B\tau} - 1} \\ &= \frac{2\Omega_0}{\Omega_0 + (2 - \Omega_0)e^{2(2\gamma+1)\tau}}, \end{aligned} \quad (4.2)$$

The resulting H -solutions are found by substituting in (2.27) (which is a linear differential equation), yielding,

$$H(\tau) = \frac{H_0 e^{-\tau} \sqrt{\Omega_0 e^{-2(2\gamma+1)\tau} - \Omega_0 + 2}}{\sqrt{2}}, \quad (4.3)$$

where $H_0 = H(0)$.

From these solutions it follows that dS fluids ($2 - \Omega_0 > 0$) are real solutions for all values of the fluid parameter γ and signs of τ . However, the AdS solutions ($2 - \Omega_0 < 0$) become complex when $\gamma > -1/2$, $\tau > 0$ or when $\gamma < -1/2$, $\tau < 0$, since in these cases the exponential in the right-hand-side of Eq. (4.3) decays. Hence, AdS fluids are real only when $\gamma < -1/2$, $\tau > 0$, or when $\gamma > -1/2$, $\tau < 0$, and we shall consider AdS solutions only in these ranges.

For dS fluids, we can see that $\Omega(\tau)$ approaches zero when $\gamma > -1/2$, $\tau > 0$ or $\gamma < -1/2$, $\tau < 0$, and approaches two when $\gamma < -1/2$, $\tau > 0$, or when $\gamma > -1/2$, $\tau < 0$. All AdS fluids develop Ω blow up singularities for $\gamma < -1/2$, $\tau > 0$, or $\gamma > -1/2$, $\tau < 0$.

On the other hand, to disclose the asymptotic nature of the H -solutions we can look at the monotonicity properties of

the function $H(\tau)$ and its possible dependence on different γ ranges. The results show a further sensitive dependence on the γ parameter around its $\gamma = -1$ value¹. For $\gamma > -1$, $H(\tau)$ always decreases and approaches zero. At the critical value $\gamma = -1$ the solution $H(\tau)$ decreases and approaches the constant $H_0 \sqrt{1 - \Omega_0/2}$, provided that $\Omega_0 < 2$. We note that the behaviour of the solutions (4.2) and (4.3) is insensitive on the initial value H_0 . The asymptotic behaviours of the H , Ω solution is shown in Figs. 1 and 2.

The nature of the solutions is further revealed by studying the stability of the equilibrium solutions. This is effected by formally setting $X = (H, \Omega)$, and think of the system (2.27)–(2.28) as one of the form,

$$\frac{dX}{d\tau} = F(X, \gamma), \quad F = (F_1, F_2), \quad (4.4)$$

where $\gamma \in (-\infty, \infty)$, and with the F_i , $i = 1, 2$, being the right-hand-sides of Eqs. (2.27) and (2.28) respectively. With this notation, we now show that the solutions of the system undergo a transcritical bifurcation when $\gamma = -1/2$ at the origin which is a non-hyperbolic equilibrium. This means that bulk fluids exchange their stability when the EoS parameter γ passes through $-1/2$.

Returning to Eq. (2.28), when $\gamma = -1/2$, the system is $\Omega' = 0$, $H' = -H$, with immediate solution,

$$\Omega(\tau) = \Omega_0 = \text{const.}, \quad H(\tau) = H_0 e^{-\tau}. \quad (4.5)$$

In this case, every point on the Ω axis is a non-hyperbolic equilibrium and every other point on the phase plane approaches the corresponding point of the Ω axis as shown in Fig 3. Therefore in this case we have expanding universes with ever-decreasing rate and collapsing ones with an ever-increasing rate both approaching a static dS bulk of a constant density. As two of us have shown elsewhere (cf. Ref. [11],

¹ For $\gamma < -1$, the term $-2(2\gamma + 1)$ is positive (and > 2), therefore, the term $e^{-\tau}$ dominates over $\sqrt{e^{-2(2\gamma+1)\tau}}$ for $\tau < 0$, while the opposite happens for $\tau > 0$. Thus, the solution $H(\tau)$ is decreasing for $\tau < 0$ and is increasing for $\tau > 0$. As discussed already, solutions in this range of γ are not acceptable as they do not satisfy the energy conditions, and so we shall not consider them further.

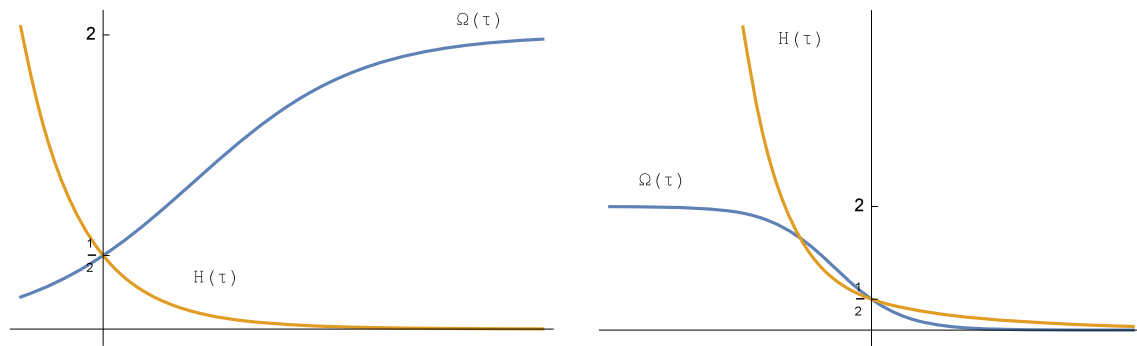


Fig. 2 Solutions for $\gamma = -2/3$ and $\gamma = 1/2$. In the first case the solution (4.2) increases following the usual logistic curve and quickly approaches the constant value 2

Sect. 4.1.1), for this value of γ the system satisfies all the energy conditions.

The case $\gamma \neq -1/2$ is shown in Figs. 4 and 5. When $\gamma \geq -1/2$, we have $F_2'' \geq 0$, and so $F_2(\Omega)$ is a convex or a concave function respectively. In this case, there are two equilibria, one at $\Omega = 0$, and a second one at $\Omega = 2$. When $\gamma > -1/2$, the equilibrium at the origin is stable while the one at $\Omega = 2$ is unstable, and they exchange their stability when $\gamma < -1/2$. For initial conditions with $\Omega_0 < 2$, that is for bulk models with a dS brane, and for the case $\gamma > -1/2$ (the left diagram in Fig. 4), the solution $\Omega(\tau)$ decreases approaching zero, the ‘Milne state’ for positive τ , whereas for $\gamma < -1/2$ (the right diagram in Fig. 4), the solution (4.2) increases and approaches the constant value 2 which corresponds to a flat brane ($k = 0$). The situation is different if initial conditions with $\Omega_0 > 2$, that is for bulk models with a AdS brane ($k = +1$) are considered. For $\gamma > -1/2$, the solution $\Omega(\tau)$ increases without bound, whereas for $\gamma < -1/2$ the solution $\Omega(\tau)$ decreases to the flat state at $\Omega = 2$. Therefore we have a transcritical bifurcation occurring at the parameter value $\gamma = -1/2$, so that the two equilibria switch their stability without disappearing after the bifurcation, see Fig. 5 for the full phase portrait of the system.

These results when translated to the bulk-brane language imply that the evolution of bulk fluids with a linear equation of state depends of the γ parameter and is organized around the two simplest equilibria, namely, the empty bulk and the flat fluid, which exchange their stability because of the transcritical bifurcation as the nature of the fluid changes (depending on γ). A typical bulk fluid evolves either towards or away from the equilibrium states ‘empty bulk’ and ‘flat fluid’ depending on whether it has $\gamma \geq -1/2$ as shown in Fig. 5.

A last special case of importance is that of a ‘bulk dust’. The corresponding behaviour of a bulk fluid with a linear EoS is also shared in this case with all bulk fluids having a nonlinear EoS (see next Section). For $\gamma = 0$, the system

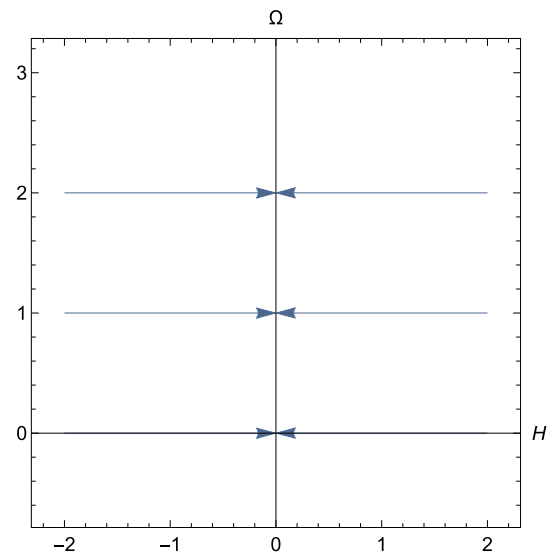


Fig. 3 Phase portrait of the system (2.27)–(2.28) for $\gamma = -1/2$. Every point on the Ω axis is an equilibrium

(2.25)–(2.26) reduces to

$$\begin{aligned} \frac{dH}{d\tau} &= -H - \frac{\Omega H}{2}, \\ \frac{d\Omega}{d\tau} &= -2\Omega + \Omega^2, \end{aligned} \quad (4.6)$$

for all λ . The phase portrait of the system (4.6) indicates that all solutions with initial values $\Omega_0 < 2$ (this corresponds to an dS dust fluid) and H_0 arbitrary, asymptotically approach the node $(0, 0)$ (that is a static, empty dS bulk), see Fig. 6. Hence, we find that dust-filled dS bulks rarefy to empty ones. This can be proved by noting that the formal solution of the first of (4.6) can be written as

$$H(\tau) = H(0) \exp\left(-\int_0^\tau (1 + \Omega(s)/2) ds\right), \quad (4.7)$$

which goes to zero as $\tau \rightarrow \infty$. By the same formula, we can see again here (like in Fig. 5, right phase portrait) that AdS

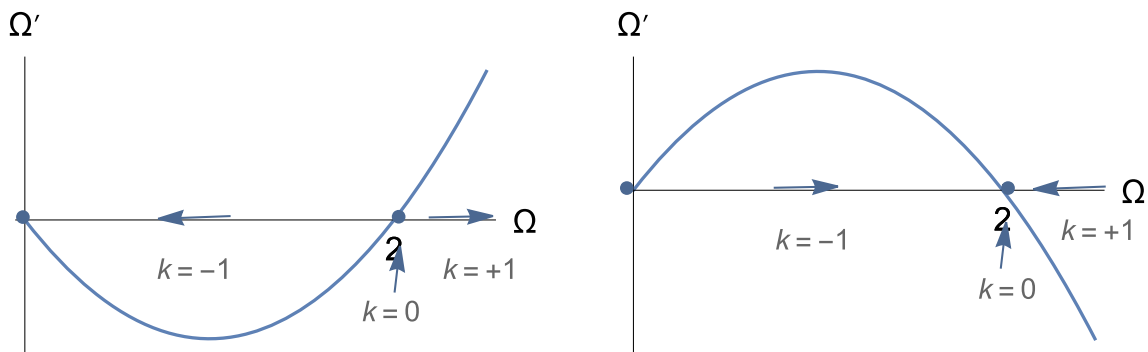
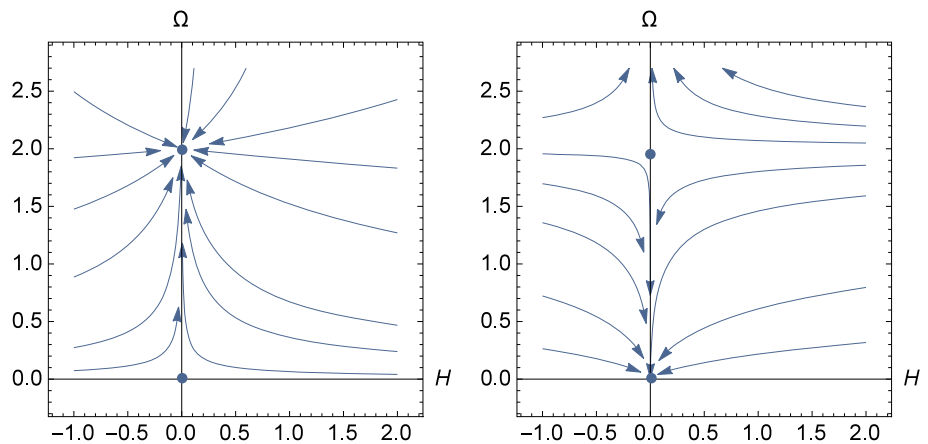


Fig. 4 The 1-dimensional phase space (Ω -line) in the linear fluid case. The Figure on the left (right) shows the case $\gamma > -1/2$ ($\gamma < -1/2$). The arrows correspond to evolution in positive τ

Fig. 5 Phase portrait of the system (2.27)–(2.28) for $\gamma < -1/2$ and $\gamma > -1/2$. The stable node at $(0, 2)$ (left), exchanges stability with the saddle $(0, 0)$ (right) as the γ values change



trajectories starting above the line $\Omega_0 = 2$ approach the Ω axis, while $\Omega(\tau)$ diverges.

5 Nonlinear fluids: Regularity and stability

Let us now move to discuss properties of the nonlinear, two-dimensional system (2.25)–(2.26). In distinction to the linear case treated above, this is a genuine, coupled, two-dimensional dynamical system and this results in two important effects that we discuss below. We first discuss the nature of the equilibria of the system and study the phase portrait. We then find a suitable Dulac function for the dynamics of the nonlinear fluid-brane system and show that there can be no closed (periodic) orbits for the system in certain parts of the phase space.

The nature of the (H, Ω) -solutions of the system (2.25)–(2.26) is strongly dependent on the ranges of the λ -parameter present in the fluid's nonlinear equation of state Eq. (2.19), in particular, on the three ranges, $\lambda < 0$, $\lambda \in (0, 1)$, and $\lambda \geq 1$.

When $\lambda < 0$, there are no finite equilibria for the system (2.25)–(2.26). In this case, the dynamics is transferred to points at infinity, a more complicated problem that we do not consider in this paper, since it requires a deeper analysis of the ‘companion system’ to (2.25)–(2.26), cf. [14, 15]. Because of

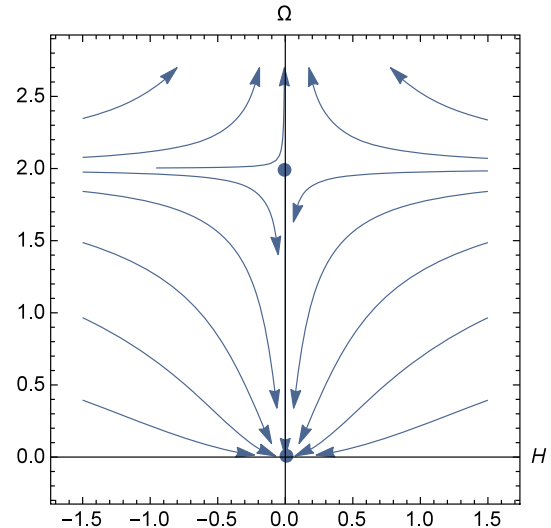


Fig. 6 Phase portrait of the system (2.25)–(2.26) for arbitrary λ and $\gamma = 0$

the presence of denominators in the vector field that defines the system (2.25)–(2.26) when $\lambda < 0$, an analysis of this case will help to further clarify the question of the existence of stable singularity-free solutions of the system. We only further note that the ‘dynamics at infinity’ in this case may

be realized through the Poincaré sphere compactification as a boundary dynamics in the framework of *ambient cosmology*, an extension of brane cosmology wherein the brane lies at the conformal infinity of the bulk [16].

Next, for the case $\lambda > 0$, we must distinguish between the two cases $\lambda \in (0, 1)$ and $\lambda \geq 1$. For $\lambda \geq 1$, there are always the two γ -independent equilibria at $(0, 0)$ and $(0, 2)$. In addition, there are γ -dependent equilibria being generally complex:

$$(H_*, \Omega_*) = \left(\exp \left(\frac{-i\pi + \lambda \ln 6 + \ln \frac{\gamma}{6}}{2 - 2\lambda} \right), 2 \right). \quad (5.1)$$

These equilibria are real provided λ takes the values

$$\lambda(n) = 1 \pm \frac{1}{2n}, \quad n = 1, 2, \dots \quad (5.2)$$

It is interesting that this case falls into the polytropic index form of the λ exponent (cf. [17], section 8). For λ taking the values $\lambda(n) = 1 - 1/2n$, the equilibria (5.1) all correspond to flat bulk fluids as they lay on the line $\Omega = 2$, and at the points where,

$$H_* = \left\{ -\frac{\gamma}{\sqrt{6}}, \frac{\gamma^2}{\sqrt{6}}, -\frac{\gamma^3}{\sqrt{6}}, \frac{\gamma^4}{\sqrt{6}}, \dots \right\}. \quad (5.3)$$

For the remaining values of λ in the case where $\lambda \in (0, 1)$, there are no other finite equilibria, and the presence of denominators in the vector field, as in the case of $\lambda < 0$, makes this case more complicated dynamically.

To construct a typical phase portrait for $\lambda = 1 - 1/2n$, we may use $\lambda(1) = 1/2$. The only equilibrium of the system is $(-\gamma/\sqrt{6}, 2)$ as implied by (5.3); for all γ the Jacobian matrix at this point is

$$\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix},$$

therefore $(-\gamma/\sqrt{6}, 2)$ is a stable improper node ('a sink star' in other terminology). It is interesting that this node represents a flat bulk fluid with a γ that falls inside the range of acceptable values as dictated by the energy conditions. It turns out that for all γ , this equilibrium is a global attractor of all solutions of (2.25)–(2.26), see Fig. 7, where the attractor is shown for the cases of a cosmological constant ($\gamma = -1$) and a massless scalar field bulk ($\gamma = 1$).

We now proceed with the analysis of the case $\lambda \geq 1$. As already mentioned, the system (2.25)–(2.26) has two equilibrium points, located at the origin and at the phase point $(0, 2)$, that is the whole bulk dynamics is organized around a static empty dS bulk and a static flat bulk fluid. We note that the linearized system around $(0, 0)$ becomes,

$$\begin{bmatrix} H' \\ \Omega' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} H \\ \Omega \end{bmatrix}, \quad (5.4)$$

therefore the point $(0, 0)$ is a stable node, and the local phase portrait of the full nonlinear system (2.25)–(2.26) is topologically equivalent to that of the linear system (5.4). By inspection of the Jacobian matrix at $(0, 2)$ we see that its eigenvalues are ± 2 , therefore this equilibrium is a saddle point, that is, trajectories approaching this point eventually move away. A typical phase portrait for $\lambda \geq 1$ is given in Fig. 8 for a cosmological constant (for $\gamma = -1$) and a massless scalar field bulk (i.e., $p = \rho$).

For $\lambda(n) = 1 + 1/2n$, the equilibria (5.1) lay on the line $\Omega = 2$, at the points where,

$$H_* = \left\{ -\frac{1}{\sqrt{6}\gamma}, \frac{1}{\sqrt{6}\gamma^2}, -\frac{1}{\sqrt{6}\gamma^3}, \frac{1}{\sqrt{6}\gamma^4}, \dots \right\}. \quad (5.5)$$

We take as typical example the case $\lambda = 3/2$. Apart from the attracting sink at the origin and the saddle at $(0, 2)$, the system has a third equilibrium located according to (5.5) at $(-1/(\sqrt{6}\gamma), 2)$, and belonging to the first quadrant for $\gamma < 0$, or to the second quadrant for $\gamma > 0$. The phase portrait of the system is shown in Fig. 9 for $\gamma = -1/\sqrt{6}$, $\gamma = 1/\sqrt{6}$.

There are three invariant lines, namely $H = 0$ and $\Omega = 0$ as discussed after (2.25)–(2.26), as well as the line $\Omega = 2$, corresponding to static, empty, and flat bulk fluids respectively. These lines are the boundaries of the following invariant sets. The strip between the lines $\Omega = 0$ and $\Omega = 2$ is an invariant set under the flow of the system, since every trajectory starting in this strip remains there forever. Similarly, the sets $\Omega > 2$, $H > 0$ and $\Omega > 2$, $H < 0$ (that is dynamic AdS bulks) are invariant sets. For both signs of γ , all solutions with initial values $\Omega_0 > 2$ and H_0 arbitrary, diverge to $\pm\infty$.

The only bounded solutions observed in Fig. 9 are those trajectories approaching the stable node at the origin. For example, expanding dS scalar field bulks (that is for $\gamma = 1$) with initial values $H_0 > 0$, $\Omega_0 < 2$ asymptotically approach $(0, 0)$, that is they become static and empty; however, the determination of the whole basin of attraction of a sink is not always possible. Finally, there are solutions with $\Omega(\tau)$ approaching the constant value $\Omega_* = 2$ while $H(\tau)$ is diverging to $\pm\infty$, depending on the sign of γ .

We conclude by giving in the following Table a summary of how the nature of the equilibria given in Eq. (5.1) depends on the type and range of γ for the first few values of n in Eq. (5.2):

n	$\lambda = 1 - 1/2n$	$\lambda = 1 + 1/2n$
1	sink for $\gamma \in [-1, 1]$	saddle for $\gamma \in [-1, 1]$
2	sink for $\gamma \in [-1, 0)$, saddle for $\gamma \in (0, 1]$	saddle for $\gamma \in [-1, 1]$
3	sink for $\gamma \in [-1, 0)$	saddle for $\gamma \in [-1, 0)$
4	saddle for $\gamma \in [-1, 1]$	saddle for $\gamma \in [-1, 1]$
5	sink for $\gamma \in [-1, 0)$	saddle for $\gamma \in [-1, 0)$

Fig. 7 Phase portrait of the system (2.25)–(2.26) for $\lambda = 1/2$ and $\gamma = -1$ and $\gamma = 1$

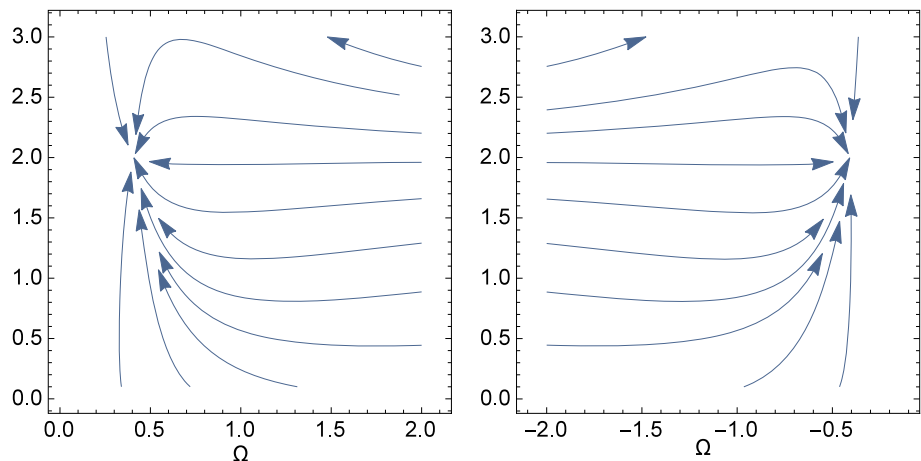


Fig. 8 Phase portrait of the system (2.25)–(2.26) for $\lambda = 2$ and $\gamma = -1$ and $\gamma = 1$. There are three saddles at $(-1, 2)$ and at $(1, 2)$ for $\gamma = -1$, but only one at $(0, 2)$ for $\gamma = 1$

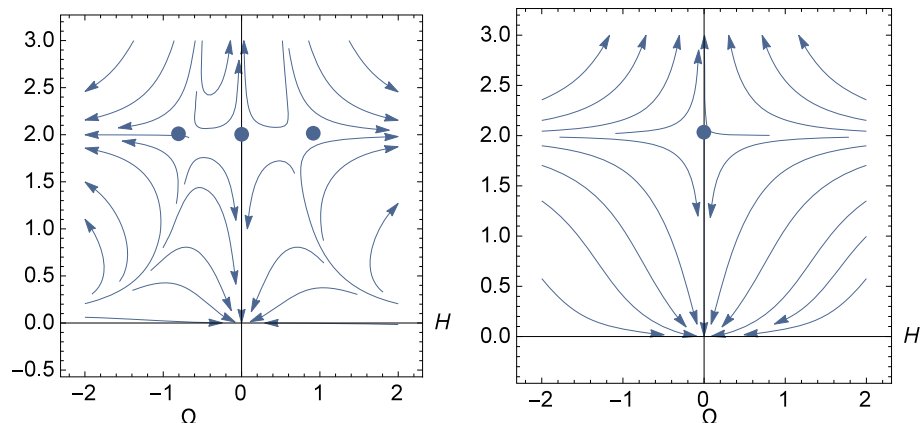
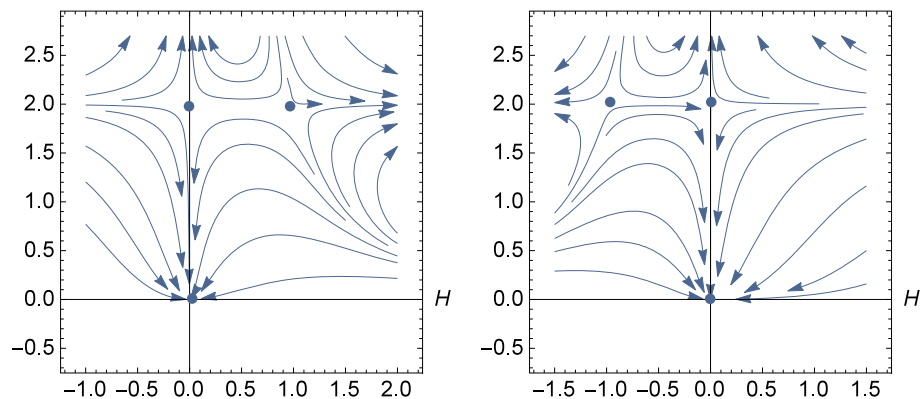


Fig. 9 Phase portrait of the system (2.25)–(2.26) for $\lambda = 3/2$ and $\gamma = -1/\sqrt{6}$ and $\gamma = 1/\sqrt{6}$. In each case there are three equilibrium points at $(0, 0)$, at $(0, 2)$ and at $(1/\sqrt{6}, 2)$ and $(-1/\sqrt{6}, 2)$ respectively



It is interesting to note that because of the presence of a *saddle connection* in Fig. 9 (the horizontal line $\Omega = 2$ connecting the two saddles), Peixoto theorem on structural stability is violated. It also clearly follows from Figs. 7, 8 and 9, that the three cases, two corresponding to the polytropic indices $\lambda(n) = 1 \pm 1/2n$, and the third case of λ unequal to those, are all qualitatively inequivalent.

We conclude this Section by showing the *impossibility of closed orbits* for the system (2.25)–(2.26) in the first quadrant of the $H - \Omega$ plane, that is for expanding, non-empty bulks. To see this, we introduce the function,

$$g = \frac{1}{H^a \Omega^b}, \quad (5.6)$$

and the problem is to use the system (2.25)–(2.26) to determine the constants a, b such that the divergence of the vector field given by the product of the function g times the vector field $(\dot{H}, \dot{\Omega})$, that is $g(\dot{H}, \dot{\Omega})$, is positive for certain ranges of λ, a, b, γ . The vector field $g(\dot{H}, \dot{\Omega})$ is given by

$$g(\dot{H}, \dot{\Omega}) = (f_1(H, \Omega), f_2(H, \Omega)), \quad (5.7)$$

where,

$$f_1(H, \Omega) = \frac{1}{H^{a-1}\Omega^b} - \frac{1}{2H^{a-1}\Omega^{b-1}} - 3^{\lambda-1}H^{2\lambda-1-a}\Omega^{\lambda-b}, \quad (5.8)$$

and

$$f_2(H, \Omega) = \frac{-2}{H^a\Omega^{b-1}} + \frac{1}{H^a\Omega^{b-2}} + 2\gamma 3^{\lambda-1}H^{2\lambda-2-a}\Omega^{\lambda+1-b} - 4\gamma 3^{\lambda-1}H^{2\lambda-2-a}\Omega^{\lambda-b}. \quad (5.9)$$

Then, setting $a = 1$, the divergence of this vector field is given by,

$$\text{div}[g(\dot{H}, \dot{\Omega})] = \frac{2(b-1)}{H\Omega^b} + \frac{2-b}{H\Omega^{b-1}} + 2\gamma 3^{\lambda-1}(2-b)H^{2\lambda-3}\Omega^{\lambda-b} - 4\gamma 3^{\lambda-1}(\lambda-b)H^{2\lambda-3}\Omega^{\lambda-b-1}. \quad (5.10)$$

The right-hand-side of this equation is positive provided,

$$\gamma > 0, \quad H > 0, \quad \Omega > 0, \quad 1 < b < 2, \quad \lambda < b, \quad (5.11)$$

so when these inequalities are all true, the divergence is strictly positive. The function g from (5.6) with this property is a *Dulac function*, but no algorithm in general exists for finding such functions. From the inequalities in (5.11) it then follows that on the simply connected domain,

$$\mathcal{D} = \{(H, \Omega) | H, \Omega > 0\}, \quad (5.12)$$

of the planar phase space, the vector field defined by (2.25)–(2.26), satisfies $(\dot{H}, \dot{\Omega}) \in \mathcal{C}^1(\mathcal{D})$, and $g \in \mathcal{C}^1(\mathcal{D})$, and the divergence $\nabla \cdot g(\dot{H}, \dot{\Omega})$ is strictly positive on \mathcal{D} . Then by the Bendixson-Dulac theorem, we conclude that there is no closed orbit lying entirely on \mathcal{D} .

6 Localisation

The basic condition for gravity localisation on the brane is that the 4-dimensional Planck mass proportional to the integral $\int_{-\infty}^0 a^2 dY$ is finite, that is the integral be convergent. We can use two different approaches to deal with this integral, firstly using the constraint to re-express the integral in terms of dimensionless variables, and secondly by direct evaluation.

To start with the first approach, we note that using the Friedmann constraint equation (2.11) which is reproduced here,

$$2 - \Omega = \frac{2k}{H^2 a^2},$$

we can express the ‘Planck mass integral’ $\int a^2 dY$ in terms of the dimensionless variables, namely,

$$2k \int \frac{d\tau}{H^3(2 - \Omega)}.$$

One may think that this integral expressing the Planck mass can be calculated explicitly, for $H(\tau)$ and $\Omega(\tau)$ given by the solutions (4.2) and (4.3), however, the constraint equation (2.11) provides a relation between the scale factor a and the variables H and Ω only for $k \neq 0$ models. Therefore we may instead choose to evaluate the integral directly,

$$\int a^2 dY = \int \frac{a^2}{H} d\tau. \quad (6.1)$$

Let us consider the case of the linear fluid first. By the definition (2.12), $a(\tau)$ is proportional to e^τ and $H(\tau)$ is given by the solution (4.3). We are interested to examine whether the Planck mass integral becomes finite on the interval $(-\infty, 0]$. In fact, we are able to show something more, namely, that it is finite on intervals of the form $(-\infty, \tau_1]$, with a suitable chosen positive τ_1 dependent of the initial conditions and γ .

It turns out that the integral expressing the Planck mass can be expressed explicitly in terms of the ordinary hypergeometric function ${}_2F_1(a, b; c; z)$. More precisely, the value of the indefinite integral (6.1) is

$$\frac{a_0^2 \sqrt{2} e^{3t} {}_2F_1(a, b; c; z)}{3H_0 \sqrt{2 - \Omega_0}}, \quad (6.2)$$

where

$$a = \frac{1}{2}, \quad b = -\frac{3}{4\gamma + 2}, \quad c = \frac{4\gamma - 1}{4\gamma + 2}, \quad z = \frac{\Omega_0 \exp(-2(2\gamma + 1)\tau)}{\Omega_0 - 2}. \quad (6.3)$$

For some particular values of γ , the integral (6.2) can be expressed as combination of elementary functions, although by complicated formulas. We treat the cases $\Omega_0 \leq 2$ separately below.

For $\Omega_0 < 2$ the improper integral $\int_{-\infty}^0 (a^2/H) d\tau$ exists, i.e. can be expressed in terms of the constants a_0, Ω_0, H_0 , at least for the representative values,

$$\gamma = -1, -2/3, -1/2, -1/3, 0, 1/2, 1. \quad (6.4)$$

For the critical value $\gamma = -1/2$, the hypergeometric function is not defined, but with the solution (4.5), i.e., $H \sim e^{-\tau}$, the integral (6.1) is elementary and $\int_{-\infty}^0 (a^2/H) d\tau = a_0^2/3H_0$. (For this value of γ , the system satisfies all the energy conditions, cf. Ref. [11], Sect. 5.)

Moreover, in all cases the integral is finite on intervals of the form $(-\infty, \tau_1]$, for some positive τ_1 . This fact can be understood, at least for $k \neq 0$, if we take into account the remarks after Eq. (4.3): the integrand $1/H^3(\Omega - 2)$ remains

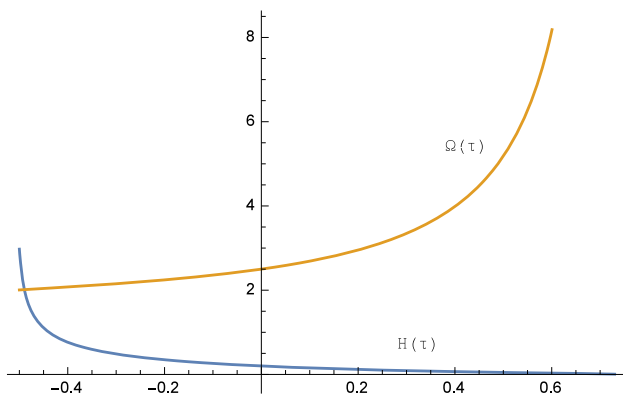


Fig. 10 Numerical solution for $\gamma = 1/2$, $\lambda = 2$ and $\Omega_0 > 2$. $\Omega(\tau)$ develops a singularity at about $\tau \sim 0.74$ and $H(\tau)$ has a singularity at $\tau \sim -0.5$

bounded and approaches zero, even if the functions $H(\tau)$ and $\Omega(\tau)$ take arbitrary large values.

The situation is different if $\Omega_0 > 2$. In that case, for $\gamma < -1/2$, $H(\tau)$ is real only when the expression inside the root in Eq. (4.3) is non-negative, that is when

$$\tau \geq \tau_*, \quad (6.5)$$

where,

$$\tau_* = \frac{1}{-2(2\gamma + 1)} \ln \left(\frac{\Omega_0 - 2}{\Omega_0} \right) < 0, \quad (6.6)$$

with equality in (6.5) giving the position of the brane. Then we find that the integral $\int_{\tau_*}^{\infty} (a^2/H) d\tau$ always diverges. For $\gamma \geq -1/2$, τ_* is positive, but in this case we require $\tau < \tau_*$ for the expression inside the square root in Eq. (4.3) to be positive. Thus we have to integrate in the range $(-\infty, \tau_*]$, and so we find that the integral $\int_{-\infty}^{\tau_*} (a^2/H) d\tau$ exists, at least for the representative values $\gamma = -1/3, -1/2, 0, 1/2, 1$.

To summarize our results for the linear fluid: if $\Omega_0 < 2$ the integral (6.2) allows for a finite Planck mass for all $\gamma \in [-1, 1]$. If $\Omega_0 > 2$, we have a finite Planck mass for all $\gamma \in [-1/2, 1]$. In this case, we choose the upper limit of the integral to be less than some $\tau_1 > 0$.

Next we consider the case of the nonlinear fluid, $\lambda \neq 1$. Equations (2.25) and (2.26) can be solved numerically for various values of the parameters γ and λ and initial values of the variables H and Ω . In all numerical evaluations the solutions develop finite time singularities, see for example Fig. 10.

Nevertheless, it seems that the integral (6.1) is finite in any interval between the singularities. This is due to the fact that the integrand $e^{2\tau}/H(\tau)$ remains bounded and approaches zero, even if the functions $H(\tau)$ and $\Omega(\tau)$ take arbitrary large values.

The numerical investigation described above indicates that even for the nonlinear fluid, the Planck mass expressed by

(6.1) may be finite, although we were unable to prove this result rigorously.

7 Discussion

In this paper we have introduced and studied the consequences of a new formulation for the dynamics of a 4-braneworld embedded in a bulk 5-space. This formulation transforms the problem into a two-dimensional dynamical system that depends on parameters such as the EoS parameter and the degree of nonlinearity of the fluid. This allows us to study the phase space of the model, and also consider in detail the importance of different states, points in phase space, such as the origin or the $(0, 2)$ -state for the overall dynamical features of the bulk fluid.

For the case of a bulk fluid with a linear equation of state, our new formulation leads to a partial decoupling of the dynamical equations of this model. This in turn implies that the linear fluid case can be solved exactly, and the asymptotic nature of the (H, Ω) solutions to be directly revealed as well as their dependence on the EoS parameter and the initial conditions to be explicitly shown. In addition, we find that the equilibria of the system depend on the fluid parameter γ and this has a major effect of the global dynamics of the system, not present in the simpler case of relativistic cosmologies. The main effect is the existence of a transcritical bifurcation around the $\gamma = -1/2$ value which change the nature of the local equilibria as well as their stability. We also concluded that the overall geometry of the orbits swirls around the two states we call empty bulk and flat fluid, as well as a number of other equilibria.

For the case of a nonlinear bulk fluid, the dynamics is organized differently for different λ -values, and shows a preference for polytropic bulk fluids. For instance, the existence on an overall attractor appears only for $\lambda = 1/2$, while the dynamics for a bulk having $\lambda > 1$ is characterized by portraits organized around nodes and saddle connections for the values of $\lambda = 1 + 1/n$. This means that the nonlinear case has a variety of instabilities as well as stable and saddle behaviours. The non-existence of closed orbits in the first quadrant is also a marked feature of the nonlinear bulk fluid.

However, as numerical evaluations show, despite the existence of singularities, the phenomenon of brane-localization in the sense of having a finite Planck mass is *self-induced* by the dynamics itself: restricting the dynamics on the orbifold leads generically to gravity localization on the brane. For this conclusion, although it follows clearly from various numerical evaluation that we have explicitly performed, we have not been able to provide a full analytic proof. We note, however, that for nonlinear fluids when the null energy condition is satisfied and $\gamma < 0$, there are indeed solutions without finite-distance singularities as two of us have shown

in [10] using different techniques such as representation of solutions through hypergeometric expansions and matching.

It would be interesting to extend some of these results further. For example, to the case of a bulk filled with a self-interacting scalar field instead of the fluid. Another extension is to study the ambient problem and allow for singularities at infinity using similar methods as those discussed here. It would also be interesting to study in detail the case $\lambda < 1$ where the vector field is non-polynomial. These more general problems will be given elsewhere.

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