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# A precisely feasible gauged model of chiral boson with its BRST cohomological perspectives

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**Abstract** We find that Siegel type chiral boson with a parameter-dependent Lorentz non-covariant masslike term for the gauge fields to be equivalent to the chiral Schwinger model with one parameter class of Faddeevian anomaly if the model is described in terms of Floreanini–Jackiw type chiral boson. By invoking the Wess–Zunino field gauge-invariant reformulation is made. It has been shown that the gauge-invariant model has the same physical content as its gauge non-invariant ancestor had. The BRST invariant effective action corresponding to this model has also been constructed. All the nilpotent symmetries associated with the BRST symmetry along with the bosonic, ghost, and discrete symmetries have been systematically studied. We establish that the nilpotent charges corresponding to these symmetries resemble the algebra of the de Rham cohomological operators in differential geometry. In the environment of conserved charges associated with the models, we study the Hodge decomposition theorem on the compact manifold.

## 1 Introduction

The chiral boson is the basic ingredient in the construction of heterotic string theory [1–4]. The theory of quantum Hall effect additionally got important input from the physics of chiral boson [5,6]. The chiral boson [7–9] and the gauged model of it [10] had been advanced independently. It became determined that the gauged model of the chiral boson and the chiral Schwinger had exciting connections regardless of the structural differences between them [11–14]. The Chiral Schwinger model is an interacting model which includes fermion and gauge field and the interaction among those is chiral in nature. This model [15] was extensively studied over the decades [15–20] and attracted extra attention while it became determined that it could be characterized through the expressions of chiral boson [11–14]. The model can be expressed in terms of chiral boson in view of the fact that in $(1+1)$ dimension exact bosonization is possible. However, the bosonization requires regularization which removes the hassles due to non-unitarity for which it had to suffer for quite a long period. Two independent regularizations are found within the literature for this model. The initial one is called the Jackiw–Rajaraman kind co-variant regularization [15] and the latter one is Mitra kind non-covariant regularization [19]. Mitra himself termed the anomaly in it as Faddeevian anomaly [21,22] in view of the fact that Gauss law commutator for this version rendered a non-trivial contribution [19]. Unlike Jackiw–Rajaraman's development, the masslike term for the gauge field was independent of any parameter in Mitra's development [19]. A few years later, a one-parameter involved improvement of Faddeevian regularization became advanced in [23]. In this article, an attempt has been made to establish the hidden link between the gauged chiral boson [10] and the bosonized version of chiral Schwinger with the one-parameter involved Faddeevian anomaly described in [23]. The description of the model [23] in terms of chiral boson was pursued in our article [14]. The investigation of the symmetry belonging to this version would be instructive since it is an anomalous gauge theory. The model itself has no gauge symmetry, however, gauge symmetry can be restored by enlarging the phasespace of the theory with the introduction of auxiliary fields. These fields render their incredible service in restoring the gauge symmetry keeping themselves allocated in the un-physical sector of the theory.

Since it is an anomalous theory, the study of Becchi–Rouet–Stora–Tyutin (BRST) is of interest since the BRST formalism in the context of covariant canonical quantization of gauge-invariant theories [24–31] play a pivotal role in the regime of formal field theory. It ensures unitarity at any arbitrary order of perturbation in the computations of physical

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processes. The gauge theories endowed with the anti-BRST as well as anti-co-BRST symmetries within the framework of BRST formalism can be shown to provide a set of tractable physical examples for the Hodge theory where the symmetries and the corresponding conserved charges provide the physical realization of the de Rham cohomological operators of differential geometry [32–34].

Batalin Fradkin vilkovisky (BFV) [28–31] formalism and its applications in different field theoretical models [35–48] has added a huge instructive and illuminating information in the field theoretical regime. So the attempt to construct the BRST invariant reformulations of this gauged chiral boson having a non-covariant para meter involved masslike term for gauge field taking the help of BFV formulation will add a new and instructive contribution to the regime of formal field theory.

The BRST cohomological aspects related to it will likewise be an significant extension. So along with the nilpotent BRST symmetry, we concentrate on the other nilpotent symmetries like ant-BRST, co-BRST, and anti-co-BRST symmetry in this framework systematically. There are a few investigations where endeavors have been made for various models to show that the generators of of the symmetries related to the BRST resembles the algebraic structure of de Rham cohomological operator of differential geometry [45,49–63]. A unique endeavor in this manner is made here to analyze whether the generators of these continuous symmetries satisfy algebra of de Rham cohomological operators of differential geometry. The foundation of the resemblance of the algebra of de Rham cohomological operators have been tried to administrator with the acquired insight from the past investigations. In the specific circumstance of conserved charges associated with the models, we study the Hodge decomposition theorem on the compact manifold and found that the gauged chiral boson that contrasts with masslike terms for the gauge field with a non-covariant parameter involved masslike terms have a place with the class of Hodge hypothesis

This article is organized as follows. Section 2 contains the formulation of gauged Floreanini–Jackiw type chiral boson that corresponds to the one-parameter class of Faddeevian anomaly. In Sect. 3, a review of the theoretical spectral of this model is made. In Sect. 4, the gauge-invariant version is constructed with the use of the Wess–Zumino field. In Sect. 5, an equivalence between and gauge non-invariant version is made. Section 6 contains a discussion of a similar type of model that contains one more chiral degree of freedom in the theoretical spectrum. In Sect. 7, like the previous cess with one less chiral degree of freedom here also an equivalence is made between the gauge-invariant and the gauge non-invariant version of this model. Section 8 is devoted to the BFV quantization of gauged FJ type chiral boson with a parameter-dependent non-covariant masslike term for gauge field. Sect. 9, contains discussions over the extended BRST symmetries. In Sect. 10, the BRST Cohomological aspect of the theory is studied. The final Sect. 11, contains a summary and discussions.

## 2 Formulation of gauged Floreanini–Jackiw type chiral boson that corresponds to one-parameter class of Faddeevian anomaly

A gauged model of Siegel type chiral boson that resembles the chiral Schwinger model with a one-parameter class of Jackiw–Rajaraman type regularization was mentioned in [14]. An extension that follows naturally is that the gauge model of Siegel type chiral boson with an appropriate selection of masslike terms for the gauge field is equivalent to the gauged Floreanini–Jackiw type chiral boson [64] with a one-parameter class of Faddeevian anomaly [14]. To formulate that let us proceed with the subsequent Lagrangian containing a suitable parameter-dependent Lorentz non-covariant masslike term for the gauge field.

$$
\begin{aligned}
L_{CH} = \int dx[&\frac{1}{2}(\dot{\phi}^2 - \phi'^2) + e(\dot{\phi} + \phi')(A_0 - A_1) \\
&+ \frac{\tau}{2}[(\dot{\phi} - \phi') + e(A_0 - A_1)]^2 + \frac{1}{2}(\dot{A} - A_0')^2 \\
&+ \frac{1}{2}e^2(A_0^2 + 2\alpha A_1 A_0 + (2\alpha - 1)A_1^2)]. \qquad (1)
\end{aligned}
$$

Here $\phi$ represents a scalar field. $A_0$and $A_1$ are the two components of the gauge field in $(1+1)$ dimension. Over-dot and over-prime indicate the time and space derivatives respectively. Note that there is a parameter-dependent Lorentz non-covariant masslike term for the gauge field within Lagrangian (1). Although the masslike term is lacking Lorentz covariance, it ultimately renders an interesting and physically sensible Lorentz invariant theory. What follows is the illustration to ascertain the physical sensibility of the model. To this endeavor, we need to work out the canonical momenta corresponding to the fields $A_0$, $A_1$, $\phi$, and $\tau$

$$\frac{\partial L}{\partial \dot{A}_0} = \pi_0 \approx 0, \qquad (2)$$

$$\frac{\partial L}{\partial \dot{A}_1} = \pi_1 = (\dot{A}_1 - A_0'), \qquad (3)$$

$$\frac{\partial L}{\partial \dot{\phi}} = \pi_\phi = (1+\tau)\dot{\phi} - \tau\phi' + e(1+\tau)(A_0 - A_1), \qquad (4)$$

$$\frac{\partial L}{\partial \dot{\tau}} = \pi_\tau \approx 0. \qquad (5)$$

The field $\tau$ is a lagrange multiplier field. Exploiting the Legendre transformation $H = \pi_0\dot{A}_0 + \pi_1\dot{A}_1 + \pi_\phi\dot{\phi} + \pi_\tau\dot{\tau} - L$ along with the use of the expression of momenta (2), (3), (4), (5) we compute the canonical Hamiltonian:

$$H_C = \int dx\mathcal{H}_C$$

$$= \int dx \left[ \frac{1}{2}\pi_1^2 + \pi_1 A_0' + \pi_\phi \phi' - e(\pi_\phi + \phi')(A_0 - A_1) + e^2[(\alpha-1)A_1^2 + (\alpha+1)A_0 A_1] + \frac{1}{2(1+\tau)}(\pi_\phi - \phi')^2 \right]. \quad (6)$$

The Eqs. (2) and (5) are the primary constraint of the theory since there is no time derivative in these equations. The preservation of these constraints leads to some additional constraints or it fixes the velocity. Repeating this preservation criterion on the usual constraint and the forthcoming secondary constraints we find that the phasespace of the system is endowed with the following five constraints.

$$\mathcal{C}_1 = \pi_0 \approx 0, \quad (7)$$
$$\mathcal{C}_2 = \pi_\tau \approx 0, \quad (8)$$
$$\mathcal{C}_3 = \pi_1' + e(\pi_\phi + \phi') + (\alpha+1)A_1 \approx 0, \quad (9)$$
$$\mathcal{C}_4 = \pi_\phi - \phi' \approx 0, \quad (10)$$
$$\mathcal{C}_5 = (\alpha+1)\pi_1 + 2\alpha(A_0' + A_1') \approx 0, \quad (11)$$

We use a gauge fixing condition

$$\mathcal{C}_6 = \tau + f(x) \approx 0. \quad (12)$$

Note that the constraint $\pi_\tau \approx 0$ is first class, and it generates the Siegel gauge symmetry, by the Anderson-Bergman algorithm, we can fix the gauge $\mathcal{C}_6 = \tau + f(x)$ where $f(x)$ is an arbitrary function [65]. Therefore, we can formulate the generating functional corresponding to the theory as follows:

$$Z = \int |det[\mathcal{C}_k, \mathcal{C}_l]|^{\frac{1}{2}} |dA_1 d\pi_1 d\phi d\pi_\phi d\Lambda d\pi_\Lambda dA_0 d\pi_0 \times e^{i\int d^2x(\pi_1\dot{A}_1 + \pi_\phi\dot{\phi} + \pi_\tau\dot{\tau} + \pi_0\dot{A}_0 - \mathcal{H}_C)} \times \delta(\mathcal{C}_1)\delta(\mathcal{C}_2)\delta(\mathcal{C}_3)\delta(\mathcal{C}_4)\delta(\mathcal{C}_5)\delta(\mathcal{C}_6). \quad (13)$$

The subscripts $k$ and $l$ run from 1 to 6. The simplification by the use of Gaussian integral leads us to

$$Z = \int d\phi dA_1 e^{i\int d^2x \mathcal{L}_{CH}}, \quad (14)$$

where

$$\mathcal{L}_{CH} = \dot{\phi}\phi' - \phi'^2 + 2e\phi'(A_0 - A_1) - e^2[(\alpha-1)A_1^2 - (\alpha+1)A_0A_1] + \frac{1}{2}(\dot{A}_1 - A_0')^2. \quad (15)$$

Thus, it manifests transparently that the Lagrangian (15) is the appropriate gauged Lagrangian of Siegel type chiral boson that corresponds to the gauged model of chiral boson with the one-parameter class of Floreanini–Jackiw type gauged chiral boson which can be generated from the chiral Schwinger model with one-parameter class Faddeevian anomaly [23] introducing a chiral constraint in the phase space of the theory [14].

## 3 Review of the theoretical spectrum

In this section, we describe the theoretical spectrum of this system in brief. The Lagrangian density with which we begin our analysis to find out the phasespace structure of the theory is

$$\mathcal{L}_{CH} = \dot{\phi}\phi' - \phi'^2 + 2e\phi'(A_0 - A_1) - e^2[(\alpha-1)A_1^2 - (\alpha+1)A_0A_1] + \frac{1}{2}(\dot{A}_1 - A_0')^2. \quad (16)$$

From the standard definition, the momenta $\pi_\phi$, $\pi_0$, and $\pi_1$ corresponding to the fields $\phi$, $A_0$, and $A_1$ are obtained:

$$\pi_\phi = \phi', \quad (17)$$
$$\pi_0 = 0, \quad (18)$$
$$\pi_1 = \dot{A}_1 - A_0'. \quad (19)$$

Using the above Eqs. (17), (18), and (19), it is straightforward to obtain the canonical Hamiltonian which reads

$$H_C = \int dx \left[ \frac{1}{2}\pi_1^2 + \pi_1 A_0' + \phi'^2 - 2e(A_0 - A_1)\phi' + \frac{1}{2}e^2[2(\alpha-1)A_1^2 + 2(\alpha+1)A_0A_1) \right]. \quad (20)$$

Equations (17) and (18) are the primary constraints of the theory. Therefore, the effective Hamiltonian is given by

$$H_{EFF} = H_C + u\pi_0 + v(\pi_\phi - \phi'), \quad (21)$$

where $u$ and $v$ are two arbitrary Lagrange multipliers. The constraints obtained in (17) and (18) have to be preserved in time to have a consistent physical theory. The preservation of the constraint (18), gives the Gauss law of the theory:

$$\mathcal{G} = \pi_1' + 2e\phi' + e^2(1+\alpha)A_1 = 0. \quad (22)$$

The consistency criterion of the constraint (17) although renders no new constraint it determines the velocity $v$ which is given by

$$v = \phi' - e(A_0 - A_1). \quad (23)$$

The Gauss law constraint, entails $\dot{\mathcal{G}} = 0$, to get preserved in time that results in a new constraint

$$(1+\alpha)\pi_1 + 2\alpha(A_0' + A_1') = 0. \quad (24)$$

The preservation of the constraint (24) does not give any new constraint. So we find that the phasespace of the theory is embedded with the constraints (17), (18), (22), and (24) and are all of these are weak conditions unto this stage. If the constraints are treated as strong conditions the following reduced Hamiltonian results in.

$$H_R = \int dx \left[ \frac{1}{2}\pi_1^2 + \frac{1}{4e_2}\pi_1'^2 + \frac{1}{2}(\alpha-1)\pi_1' A_1 + \frac{1}{2}e^2[(1+\alpha)^2 - 4\alpha)]A_1^2 \right]. \quad (25)$$

Since the constraints are treated here as strong condition to obtain the reduced Hamiltonian the usual Poisson's bracket becomes inadequate however, there is a remedy: the reduced Hamiltonian is known to be consistent with the Dirac brackets [66]. The Dirac brackets between the fields describing the reduced Hamiltonian $H_R$ are found out to be

$$[A_1(x), A_1(y)]^* = \frac{1}{2e^2}\delta'(x-y), \tag{26}$$

$$[A_1(x), \pi_1(y)]^* = \frac{(\alpha-1)}{2\alpha}\delta(x-y), \tag{27}$$

$$[\pi_1(x), \pi_1(y)]^* = -\frac{(1+\alpha)^2}{4\alpha}e^2\epsilon(x-y). \tag{28}$$

Using the reduced Hamiltonian (25), and the Dirac brackets (26), (27), and (28), a little algebra leads us to obtain the first-order equations of motion for $A_1$ and $\pi_1$:

$$\partial_-\pi_1 = \frac{(\alpha-1)^2}{\alpha}e^2 A_1, \tag{29}$$

$$\partial_+ A_1 = \frac{(\alpha-1)}{2\alpha}\pi_1 + \frac{1}{2\alpha}(\alpha+1)A_1', \tag{30}$$

and these first-order equations of motion reduce to the following second-order equation after little simplification. It is straightforward to see that the above two Eqs. (29) and (30) satisfy a Klein–Gordon type Equation

$$\left(\Box - \frac{(\alpha-1)}{\alpha}\right)\pi_1 = 0. \tag{31}$$

The Eq. (31), represents a massive boson with square of the mass $m^2 = -e^2\frac{(1-\alpha)^2}{\alpha}$. It is evident that the parameter $\alpha$ must be negative for the mass of the boson to be physical. Unlike the Abreu et al. [23] there is no massless degree of freedom in this situation since the phasespace of this theory contains one more constraint. After this brief review of the theoretical spectrum let us proceed to study the gauge symmetric properties of this model.

## 4 Gauge invariant version with the Wess–Zumino

The model in its usual phasespace is not gauge-invariant. The use of the Wess–Zumino field helps to get a gauge-invariant Lagrangian in the extended phasespace. The gauge-invariant Lagrangian corresponding to this gauged chiral boson that resembles the chiral Schwinger model with a one-parameter class Faddeevian anomaly is given by

$$L_{CHI} = \int dx[\mathcal{L}_{CH} + \mathcal{L}_{WZ}], \tag{32}$$

where $\mathcal{L}_{WZ}$ refers to the Wess–Zumino term [67]

$$\mathcal{L}_{WZ} = \alpha(\dot{\omega}\omega' + \omega'^2) + e(\alpha+1)(A_0\omega' - A_1\dot{\omega}) \\ -2e\alpha(A_0+A_1)\omega. \tag{33}$$

Therefore, the total Lagrangian density reads

$$\mathcal{L}_{CHI} = \dot{\phi}\phi' - \phi'^2 + 2e\phi'(A_0 - A_1) - e^2[(\alpha-1)A_1^2 \\ +(\alpha+1)A_0A_1] + \frac{1}{2}(\dot{A}_1 - A_0')^2 \\ +\alpha(\dot{\omega}\omega' + \omega'^2) + e(\alpha+1)(A_0\omega' - A_1\dot{\omega}) \\ -2e\alpha(A_0+A_1)\omega'. \tag{34}$$

Here $\omega$ represents the Wess–Zumino field [67]. The momenta corresponding to the field $A_0$, $A_1$, $\phi$, and $\omega$ respectively are

$$\frac{\partial L_{CHI}}{\partial \dot{A}_0} = \pi_0 \approx 0, \tag{35}$$

$$\frac{\partial L_{CHI}}{\partial \dot{A}_1} = \pi_1 = (\dot{A}_1 - A_0'), \tag{36}$$

$$\frac{\partial L_{CHI}}{\partial \dot{\phi}} = \phi', \tag{37}$$

$$\frac{\partial L_{CHI}}{\partial \dot{\omega}} = \pi_\omega = \alpha\omega' - e(1+\alpha)A_1 \approx 0. \tag{38}$$

The canonical Hamiltonian is obtained using the Eqs. (35), (36), (37), and (38) by the use of a Legendre transformation:

$$H_C = \int dx[\pi_1\dot{A}_1 + \pi_0\dot{A}_0 + \pi_\phi\dot{\phi} + \pi_\omega\dot{\omega}] - L_{CHI}]. \tag{39}$$

Therefore, the effective Hamiltonian reads

$$H_{CHI} = \int dx[\frac{1}{2}\pi_1^2 + \pi_1A_0' + \phi'^2 - 2e(A_0 - A_1)\phi' \\ -e^2[(\alpha-1)A_1^2 + (\alpha+1)A_0A_1] - \alpha\omega'^2 \\ -e(1+\alpha)A_0\omega' + 2e\alpha(A_0+A_1)\omega' \\ +w_1\pi_0 + w_2(\pi_\phi - \phi') \\ +w_3[\pi_\omega - \alpha\omega' + e(1+\alpha)A_1]. \tag{40}$$

The Gauss law constraint of the theory is computed using the preservation of the constraint (35), that comes out to be

$$G = \pi_1' + 2e\phi' + e(1-\alpha)\omega' + e^2(\alpha+1)A_1 \approx 0. \tag{41}$$

Here $w_1$, $w_2$, and $w_2$ are the Lagrange multipliers having dimension of velocity The velocities $w_2$ and $w_2$ are found out to be

$$w_2 = \phi' - e(A_0 - A_1), \tag{42}$$

$$w_3 = -\omega' + e(A_0 + A_1). \tag{43}$$

The velocity $w_1$ however remains undetermined. The preservation of the Gauss law constraint leads to a new constraint

$$\dot{G} = (\alpha+1)\pi_1 + 2\alpha(A_0 + A_1)' \approx 0. \tag{44}$$

So, it appears that the gauge-invariant system has the following five constraints:

$$\mathcal{K}_1 = \pi_0 \approx 0, \tag{45}$$

$$\mathcal{K}_2 = \pi_\phi - \phi' \approx 0, \tag{46}$$

$$\mathcal{K}_3 = \pi_\omega - \alpha\omega' + e(1+\alpha)A_1 \approx 0, \tag{47}$$

$$\mathcal{K}_4 = \pi_1' + 2e\phi' + e(1-\alpha)\omega' + e^2(\alpha+1)A_1 \approx 0, \quad (48)$$
$$\mathcal{K}_5 = (\alpha+1)\pi_1 + 2\alpha(A_0' + A_1') \approx 0. \quad (49)$$

Our next task is to make an equivalence between the gauge-invariant and gauge non-invariant version of the theory since gauge-invariant is made here in the extended phasespace with the introduction of a Wess–Zumino field.

## 5 To make an equivalence between the gauge invariant and gauge non-invariant version

To make an equivalence between the gauge invariant and the gauge non-invariant interpretation corresponding to this model let us proceed with the gauge symmetric Lagrangian. So we add up the Wess–Zumino term [67] with the usual Lagrangian.

$$L_{CHI} = \int dx[\mathcal{L}_{CH} + \mathcal{L}_{WZ}] \quad (50)$$
$$\begin{aligned}\mathcal{L}_{CHI} = {} & \dot{\phi}\phi' - \phi'^2 + 2e\phi'(A_0 - A_1) - e^2[(\alpha-1)A_1^2 \\ & +(\alpha+1)A_0A_1] + \frac{1}{2}(\dot{A_1} - A_0')^2 \\ & +\alpha(\dot{\omega}\omega' + \omega'^2) + e(\alpha+1)(A_0\omega' - A_1\dot{\omega}) \\ & -2e\alpha(A_0 + A_1)\omega' \end{aligned} \quad (51)$$

The canonical Hamiltonian in this situation reads

$$\begin{aligned} H_{CHI} = \int dx\Bigg[ & \frac{1}{2}\pi_1^2 + \pi_1 A_0' + \phi'^2 - 2e(A_0 - A_1)\phi' \\ & -e^2[(\alpha-1)A_1^2 + (\alpha+1)A_0A_1] \\ & -\alpha\omega'^2 - e(1+\alpha)A_0\omega' + 2e\alpha(A_0 + A_1)\omega'\Bigg] \end{aligned} \quad (52)$$

We will now follow the formalism developed in the article [68] to establish the required equivalence. To ensure it, we require two gauge fixing conditions. The appropriate gauge fixing conditions are

$$\mathcal{K}_6 = \omega' \approx 0, \quad (53)$$
$$\mathcal{K}_7 = \pi_\omega + e(\alpha+1)A_1 \approx 0. \quad (54)$$

There were five constraints in the phasespace of the theory. Those five constraints along with these two gauge fixing conditions form a second class set. It enables us to write down the generating functional:

$$\begin{aligned} Z = \int & [det[\mathcal{K}_m,\ \mathcal{K}_n]]^{\frac{1}{2}} dA_1 d\pi_1 d\phi d\pi_\phi dA_0 d\pi_0 d\omega d\pi_\omega \\ & \times e^{i\int d^2x(\pi_1\dot{A}_1 + \pi_\phi\dot{\phi} + \pi_\omega\dot{\omega} + \pi_0\dot{A}_0 - H_C)} \\ & \times \delta(\mathcal{K}_1)\delta(\mathcal{K}_2)\delta(\mathcal{K}_3)\delta(\mathcal{K}_4\delta(\mathcal{K}_5)\delta(\mathcal{K}_5)\delta(\mathcal{K}_7). \end{aligned} \quad (55)$$

Here $m$ and $n$ both runs from 1 to 7. Integrating out of the fields $\omega$ and $\pi_\omega$ we find that Eq. (55) reduces to

$$\begin{aligned} Z = N \int & dA_1 d\pi_1 d\phi d\pi_\phi dA_0 d\pi_0 \\ & \times e^{i\int d^2x(\pi_1\dot{A}_1 + \pi_\phi\dot{\phi} + \pi_0\dot{A}_0 - \mathcal{H}_{GSF})} \\ & \times \delta(\tilde{\mathcal{K}}_1)\delta(\tilde{\mathcal{K}}_2)\delta(\tilde{\mathcal{K}}_4)\delta(\tilde{\mathcal{K}}_5). \end{aligned} \quad (56)$$

where

$$\tilde{\mathcal{K}}_1 = F_1 = \pi_0 \approx 0, \quad (57)$$
$$\tilde{\mathcal{K}}_2 = F_2 = \pi_\phi - \phi' \approx 0, \quad (58)$$
$$\tilde{\mathcal{K}}_4 = \pi_1' + 2e\phi' + e^2(\alpha+1)A_1 \approx 0, \quad (59)$$
$$\tilde{\mathcal{K}}_5 = \tilde{\mathcal{K}}_f = (\alpha+1)\pi_1 + 2\alpha(A_0' + A_1') \approx 0. \quad (60)$$

These are the usual set of constraints of the gauge non-invariant version of the theory and the corresponding Hamiltonian is

$$\begin{aligned} \mathcal{H}_{GSF} = {} & \frac{1}{2}\pi_1^2 + \pi_1 A_0' + \phi'^2 - 2e(A_0 - A_1)\phi' \\ & -e^2[(\alpha-1)A_1^2 + (\alpha+1)A_0A_1. \end{aligned} \quad (61)$$

Again integrating out of the momenta $\pi_0$, $\pi_1$, and $\pi_\phi$ we land onto

$$Z = \tilde{N}\int d\phi dA_1 e^{i\int d^2x\mathcal{L}_{GSF}}, \quad (62)$$

where

$$\begin{aligned} \mathcal{L}_{GSF} = {} & \dot{\phi}\phi' - \phi'^2 + 2e\phi'(A_0 - A_1) - e^2[(\alpha-1)A_1^2 \\ & -(\alpha+1)A_0A_1] + \frac{1}{2}(\dot{A_1} - A_0')^2. \end{aligned} \quad (63)$$

Note that the system now contains the usual five constraints $\tilde{\mathcal{K}}_1, \tilde{\mathcal{K}}_2, \tilde{\mathcal{K}}_4$, and $\tilde{\mathcal{K}}_5$ and the Lagrangian density $L_{GSF}$ is identical to the usual Lagrangian $L_{CH}$ having the same Hamiltonian $H_{GSF} = H_R$. So the gauge invariant Lagrangian maps onto the gauge non-invariant Lagrangian described in the usual phasespace. It also ensures that the physical contents in both gauge-invariant and gauge non-invariant versions are identical.

## 6 A theory that contains one more chiral degrees of freedom

An alternative description of this theory is possible where the theory contains an extra chiral degree of freedom. If a chiral constraint is imposed in it by hand it transforms into the theory described in the earlier sections [14]. The theoretical spectra corresponding to the theory having one more chiral degree of freedom was discussed in [23].

$$L_{CB} = \int dx\bigg[\frac{1}{2}(\dot\phi^2 - \phi'^2) + e(\dot\phi + \phi')(A_0 - A_1) + \frac{1}{2}(\dot A - A_0')^2 + \frac{1}{2}e^2(A_0^2 + 2\alpha A_1 A_0 + (2\alpha - 1)A_1^2\bigg]. \tag{64}$$

The following is a brief review of the phasespace structure of this theory. The momenta of the fields describing the Lagrangian are given by

$$\pi_0 = 0, \tag{65}$$
$$\pi_1 = \dot A_1 - A_0'. \tag{66}$$
$$\pi_\phi = \dot\phi - e(A_0 - A_1), \tag{67}$$

It has been found that the theory contains three second class constraints.

$$\pi_0 \approx 0, \tag{68}$$
$$\pi_1' + e(\pi_\phi + \phi') + e^2(1+\alpha)A_1 \approx 0, \tag{69}$$
$$(1+\alpha)\pi_1 + 2e\alpha(A_0 + A_1)' \approx 0. \tag{70}$$

The theoretical spectrum is given by the flowing to second order equations of motion [23]

$$\left(\Box - e^2\frac{(\alpha-1)^2}{\alpha}\right)\pi_1 = 0, \tag{71}$$
$$\dot{\mathcal{F}} + \mathcal{F}' = 0, \tag{72}$$

where $\mathcal{F} = \phi - \frac{1}{e}\frac{\alpha}{1+\alpha}(\dot A_1 + A_1')$. It indicates that theoretical spectra of the theory contain one massive boson along with a massless chiral degree of freedom.

## 7 Gauge invariant version and making an equivalence with gauge variant version

Like the previous case introducing the Wess–Zumino field, a gauge-invariant version is possible to construct as follows. The gauge-invariant version in the extended phasespace is given by

$$L_{CBI} = \int dx\left[\frac{1}{2}(\dot\phi^2 - \phi'^2) + e(\dot\phi + \phi')(A_0 - A_1) + \frac{1}{2}(\dot A - A_0')^2 + \frac{1}{2}e^2(A_0^2 + 2\alpha A_1 A_0 + (2\alpha - 1)A_1^2\right] + \alpha(\dot\omega\omega' + \omega'^2) + 2e\alpha(A_0 + A_1)\omega' + (\alpha+1)(A_1\dot\omega - A_0\omega'). \tag{73}$$

The terms containing the field $\omega$ in Eq. (73) is known as Wess–Zumino term and $\omega$ is the corresponding Wess–Zumino field [67]. To show the equivalence between these two model we need to calculate the momenta corresponding to the field $A_0$, $A_1$, $\phi$, and $\omega$ like the previous case.

$$\frac{\partial L_{CBI}}{\partial \dot A_0} = \pi_0 \approx 0, \tag{74}$$
$$\frac{\partial L_{CBI}}{\partial \dot A_1} = \pi_1 = (\dot A_1 - A_0'), \tag{75}$$
$$\frac{\partial L_{CBI}}{\partial \dot\phi} = \pi_\phi + e(A_0 - A_1), \tag{76}$$
$$\frac{\partial L_{CBI}}{\partial \dot\omega} = \pi_\omega = \alpha\omega' - e(1+\alpha)A_1 \approx 0. \tag{77}$$

Here $\pi_\phi$, $\pi_0$, and $\pi_1$, and $\pi_\omega$ are the momentum corresponding to the field $A_0$, $A_1$, $\phi$, and $\omega$ The canonical Hamiltonian is is obtained through a Legendre transformation along with using the definition of momenta:

$$H_{CBI} = \int dx\bigg[\frac{1}{2}(\pi_1^2 + +\phi'^2 + \pi_\phi^2) + \pi_1 A_0' - e(A_0 - A_1)(\pi_\phi + \phi') - e^2[(\alpha-1)A_1^2 + (\alpha+1)A_0A_1] - \alpha\omega'^2 - e(1+\alpha)A_0\omega' + 2e\alpha(A_0 + A_1)\omega'\bigg]. \tag{78}$$

Note that $\pi_0 \approx 0$ and $\pi_\omega - \alpha\omega' + e(1+\alpha)A_1 \approx 0$ are the primary constraints of the theory. The system contains two more constraints. The constraints are explicitly given by

$$C_1 = \pi_0 \approx 0, \tag{79}$$
$$C_2 = \pi_\omega - \alpha\omega' + e(1+\alpha)A_1, \approx 0, \tag{80}$$
$$C_3 = \pi_1' + e(\pi_\phi + \phi') + e^2(\alpha+1)A_1 + (1-\alpha)\omega'. \tag{81}$$

It is convenient to write down the generating functional to make an equivalence keeping the required variables only, integrating out the rest of them. Here we need some gauge fixing conditions as suggested in the article [68]. The gauge fixing conditions that suit here are

$$C_4 = \omega' \approx 0 \tag{82}$$
$$C_5 = \pi_\omega - (1+\alpha)\pi_1 + 2\alpha(A_0 + A_1)' + e(1+\alpha)A_1 \approx 0 \tag{83}$$

We are now in a position to formulate the generating functional of the theory. It reads

$$Z = \int |det[C_k, C_l]^{\frac{1}{2}}|dA_1 d\pi_1 d\phi d\pi_\phi d\omega d\pi_\omega dA_0 d\pi_0 \, e^{i\int d^2x(\pi_1\dot A_1 + \pi_\phi\dot\phi + \pi_\Lambda\dot\Lambda + \pi_0\dot A_0 - \mathcal{H}_C)} \times\delta(C_1)\delta(C_2)\delta(C_3)\delta(C_4)\delta(C_5). \tag{84}$$

The subscripts $k$ and $l$ runs from 1 to 5. After simplification by the use of gaussian integral we land on to where the Liouville measure $[\mathcal{D}\mu] = d\pi_\phi d\phi d\pi_1 dA_1 d\pi_0 dA_0 d\pi_\omega d\omega$, and $m$ and$n$ run from 1 to 5. After integrating out of the field $\omega$ and $\pi_\omega$, we find that the generating functional turns into

$$Z = N\int [d\bar\mu]e^{i\int d^2x[\pi_1\dot A_1 + \pi_0\dot A_0 + \pi_\phi\dot\phi + -\tilde H_{CBS}]} \times\delta(\tilde C_1)\delta(\tilde C_2)\delta(\tilde C_3), \tag{85}$$

where $[\mathcal{D}\tilde{\mu}] = d\pi_\phi d\phi d\pi_1 dA_1 d\pi_0 dA_0$, and $N$ is a normalization constant having no significant physical importance, and $\tilde{H}_{CNS}$ is given by

$$\tilde{H}_{CBS} = \frac{1}{2}(\pi_1^2 + \phi'^2 + \pi_\phi^2) + \pi_1 A_0' + e(\pi_\phi + \phi')(A_0 - A_1) + e^2[(\alpha - 1)A_1^2 + (\alpha + 1)A_0 A_1], \tag{86}$$

and

$$\tilde{C}_1 = \pi_0, \tag{87}$$
$$\tilde{C}_2 = \pi_1' + e(\pi_\phi + \phi') + e^2(1 + \alpha)A_1, \tag{88}$$
$$\tilde{C}_3 = (1 + \alpha)\pi_1' + 2\alpha(A_0 + A_1), \tag{89}$$

which are the constraints of the theory in the gauge non-invariant situation as given in (68), (69), and (70). We land onto the required result after integrating out of the momenta $\pi_\phi$, $\pi_1$ and $\pi_0$:

$$Z = N \int d\phi dA_1 DA_0 e^{i\int d^2x \mathcal{L}_{CB}} \tag{90}$$

where

$$\mathcal{L}_{CB} = \frac{1}{2}(\dot{\phi}^2 - \phi'^2) + e(\dot{\phi} + \phi')(A_0 - A_1) - e^2[(\alpha - 1)A_1^2 - (\alpha + 1)A_0 A_1] + \frac{1}{2}(\dot{A}_1 - A_0')^2 \tag{91}$$

So it is now transparent that the Lagrangian (91) is the appropriate gauge-invariant Lagrangian density corresponding the Lagrangian of bosonized chiral Schwinger model with one-parameter class of Faddeevian anomaly [23].

## 8 BRST formulation of gauged FJ type chiral boson with non-covariant masslike term for gauge field

BRST formulation is instructive for any theory since it ensures the unitarity and renormalization of a physical theory. Batalin, Fradkin, and vilkovisky (BFV) formalism serves as an important tool to construct BRST invariant reformulation. We have got here a scope to exploit BFV formalism to write down the BRST invariant reformulation of the two nearly similar models, but having the differences in the number of constraints they possessed in the phasespaces. What follows next is an attempt to write down the BRST invariant Lagrangian for the model described in Eq. (63) using the flavors of BFV formulation as it was found in the article Fujiwara Igarashi and Kubo [27]. It was shown that using the improved version of BFV formalism the BF field needed to make a theory first-class turns into Wess–Zumino scalar [27]. So what we need is the first-class theory corresponding to this Lagrangian introducing the appropriate Wess–Zumino term. By the extension of phasespace by Wess–Zumino field $\omega$ we get the following gauge-invariant structure corresponding to this model:

$$\mathcal{L}_{WZ} = \alpha(\dot{\omega}\omega' + \omega'^2) + e(\alpha + 1)(A_0\omega' - A_1\dot{\omega}) - 2e\alpha(A_0 + A_1)\omega \tag{92}$$

Therefore, the total Lagrangian reads

$$\mathcal{L}_{CHI} = \dot{\phi}\phi' - \phi'^2 + 2e\phi'(A_0 - A_1) - e^2[(\alpha - 1)A_1^2 + (\alpha + 1)A_0 A_1] + \frac{1}{2}(\dot{A}_1 - A_0')^2 + \alpha(\dot{\omega}\omega' + \omega'^2) + e(\alpha + 1)(A_0\omega' - A_1\dot{\omega}) - 2e\alpha(A_0 + A_1)\omega' \tag{93}$$

The Lagrangian density has already been given in Eq. (34). The momenta corresponding to the field $A_0$, $A_1$, $\phi$ and $\sigma$ respectively are calculated in Eqs. (35), (36), (37), and (38). The canonical and the effective Hamiltonian are computed in Eqs. (39) and (40) respectively. The Gauss law constraint of the theory reads

$$\tilde{G} = \pi_1' + 2e\phi' + e(1 - \alpha)\omega' + e^2(\alpha + 1)A_1 \approx 0. \tag{94}$$

In Sect. 4, we have seen that the gauge invariant system has the five constraints. Let us now follow the result of the article [27] to write down the BRST invariant effective action without going to the formal construction using BFV formalism. With the information of the article [27] we can immediately write down a BRST invariant effective action since BF fields turns in to Wess–Zumino with appropriate choice of gauge fixing:

$$S_{eff} = \int d^2x \Big[\pi_\phi\dot{\phi} + \pi_1\dot{A}_1 + \pi_0\dot{A}_0 + \pi_\omega\dot{\omega} - \Big[\frac{1}{2}\pi_1^2 + \pi_1 A_0' + \phi'^2 - 2e(A_0 - A_1)\phi' - e^2[(\alpha - 1)A_1^2 + (\alpha + 1)A_0 A_1] - \alpha\omega'^2 - e(1 + \alpha)A_0\omega' + 2e\alpha(A_0 + A_1)\omega'\Big] + \dot{\mathcal{C}}\mathcal{P} + \dot{\bar{\mathcal{C}}}\bar{P} - [\mathcal{Q}_b, \Psi]\Big]. \tag{95}$$

Here $\tilde{Q}_b$ and $\Psi$ stand for the BRST charge and fermionic gauge fixing term respectively. The fields $\mathcal{C}$ and $\mathcal{P}$ is a pair of canonically conjugate ghost fields with ghost numbers 1 and $-1$ respectively, and the fields $\bar{\mathcal{C}}$, $\bar{\mathcal{P}}$ is a pair of canonically conjugate anti-ghost fields with ghost numbers $-1$ and 1 respectively. The constraints $\mathcal{K}_f = \mathcal{K}_4 - e\mathcal{K}_3 + e\mathcal{K}_2$ and $\mathcal{K}_1$ form a first-class set and these two form the generator of the gauge transformation. The gauge transformations corresponding to the fields $A_\mu$, $\phi$, and $\omega$ are the following:

$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\lambda, \quad \phi \to \phi + \lambda, \quad \omega \to \omega - \lambda, \tag{96}$$

which keep the Lagrangian invariant. The constraints $\mathcal{K}_f = \mathcal{K}_4 - e\mathcal{K}_3 + e\mathcal{K}_2$ and $\mathcal{K}_1$ constitute the BRST charge $\tilde{Q}_b$ for this theory:

$$\tilde{Q}_b = \frac{i}{e}\mathcal{C}[\pi_1' + e(\pi_\phi - \pi_\omega) + e(\phi' + \omega')] - \frac{i}{e}\pi_0\bar{\mathcal{P}}, \quad (97)$$

The fermionic gauge fixing term in this situation is

$$\Psi = e\left[\mathcal{P}A_0 + \bar{\mathcal{C}}\left(\frac{1}{2}\pi_0 + \partial_1 A^1\right)\right]. \quad (98)$$

The BRST transformations of the fields constituting the effective action $S_{eff}$ get generated from the BRST charger $Q_b$, which are given by

$$\begin{aligned} &s_b\phi = -\mathcal{C}, s_b\omega = \mathcal{C}, s_b\pi_\phi = -\mathcal{C}', s_b\pi_\omega = -\mathcal{C}',\\ &s_bA_0 = \frac{1}{e}\dot{\mathcal{C}}, s_bA_1 = \frac{1}{e}\mathcal{C}',\\ &s_b\mathcal{C} = 0, s_bp = \frac{1}{e}[\pi_1' + e(\pi_\phi - \pi_\omega) + e(\phi' + \omega')],\\ &s_b\bar{\mathcal{C}} = \frac{1}{e}\pi_0, s_b\bar{\mathcal{P}} = 0. \end{aligned} \quad (99)$$

With the use of Eq. (99) we find that

$$\begin{aligned} [Q_b, \Psi] = &A_0[\pi_1' + e(\pi_\phi - \pi_\omega) + e(\phi' + \omega')]\\ &+\pi_0\left(A_1' - \frac{1}{2}\pi_0\right) + \bar{\mathcal{P}}\mathcal{P} + \bar{\mathcal{C}}'\mathcal{C}' \approx -\pi_0\partial_1 A^1\\ &-\frac{1}{2}\pi_0^2 + \bar{\mathcal{P}}\mathcal{P} + \bar{\mathcal{C}}'\mathcal{C}'. \end{aligned} \quad (100)$$

So, the generating functional corresponding to the BRST invariant effective action takes the form

$$Z = \int D\phi e^{S_{eff}}, \quad (101)$$

where

$$\begin{aligned} S_{eff} = \int d^2x &\left[\pi_\phi\dot{\phi} + \pi_\omega\dot{\omega} + \pi_0(\dot{A}_0 - A_1') + \frac{1}{2}\pi_0^2\right.\\ &-\frac{1}{2}\pi_1^2 + \pi_1(\dot{A}_1 - A_0') - \phi'^2 + 2e\phi'(A_0 - A_1)\\ &-e^2[(\alpha - 1)A_1^2 + \quad (\alpha + 1)A_0A_1]\\ &\left.-\omega'^2 + 2e\omega'(A_0 + A_1) + \dot{\mathcal{C}}\mathcal{P} + \dot{\bar{\mathcal{C}}}\bar{\mathcal{P}} - \bar{\mathcal{P}}\mathcal{P} - \bar{\mathcal{C}}'\mathcal{C}'\right]. \end{aligned} \quad (102)$$

Performing integration over $\pi_1$, $p$, and $\bar{p}$ we land onto the following generating functional.

$$\begin{aligned} Z_{BR} = \int D\phi exp &\left[i\int d^2x\left[\pi_\phi\dot{\phi} + \pi_\omega\dot{\omega} + \pi_0(\dot{A}_0 - A_1')\right.\right.\\ &+\frac{1}{2}\pi_0^2 + \frac{1}{2}(\dot{A}_1 - A_0')^2 - 2e^2A_1^2\\ &-\phi'^2 + 2e\phi'(A_0 - A_1) - \alpha\omega'^2 - e(1 + \alpha)A_0\omega'\\ &\left.\left.+2e\alpha(A_0 + A_1)\omega' + \dot{\bar{\mathcal{C}}}\dot{\mathcal{C}} - \bar{\mathcal{C}}'\mathcal{C}'\right]\right]. \end{aligned} \quad (103)$$

Note that the field $\pi_0$ is playing the role of Nakanishi–Lautrup type auxiliary field. Let us call it as $B$ for convenience. The Lagrangian density now turns into

$$\begin{aligned} \mathcal{L}_{eff} = &\frac{1}{2}(\dot{A}_1 - A_0')^2 - 2e^2A_1^2 - \phi'^2 + \dot{\phi}\phi' + 2e\phi'(A_0 - A_1)\\ &+\alpha(\dot{\omega}\omega' + \omega'^2) + e(\alpha + 1)(A_0\sigma' - A_1\dot{\omega})\\ &-2e\alpha(A_0 + A_1)\omega + \frac{1}{2}\mathcal{B}^2 + \mathcal{B}\partial_\mu A^\mu + \partial_\mu\bar{\mathcal{C}}\partial^\mu\mathcal{C}. \end{aligned} \quad (104)$$

The action corresponding to this theory is invariant under the following transformations

$$\begin{aligned} &s_b\phi = -\mathcal{C}, s_b\omega = \mathcal{C}, s_bA_0 = \frac{1}{e}\dot{\mathcal{C}}, s_bA_1 = \frac{1}{e}\mathcal{C}',\\ &s_b\mathcal{C} = 0, s_b\bar{\mathcal{C}} = \frac{1}{e}\mathcal{B}, s_b\mathcal{B} = 0. \end{aligned} \quad (105)$$

Using the constraints of the theory we can recast the BRST charge in the following form

$$\begin{aligned} &\Omega = \pi_1' + e(\pi_\phi - \pi_\omega) + e(\phi' + \omega'),\\ &\mathcal{B} = \pi_0. \end{aligned} \quad (106)$$

It is straightforward to define define the anti-BRST charge for this system:

$$\mathcal{Q}_{ab} = \frac{i}{e}[\Omega\bar{\mathcal{C}} - \mathcal{B}\dot{\bar{\mathcal{C}}}], \quad (107)$$

which generate the following ant-BRST transformations of the fields describing the effective Lagrangian (104)

$$\begin{aligned} &s_{ab}\phi = -\bar{\mathcal{C}}, s_{ab}\omega = \bar{\mathcal{C}}, s_{ab}A_0 = \frac{1}{e}\dot{\bar{\mathcal{C}}}, s_{ab}A_1 = \frac{1}{e}\bar{\mathcal{C}}',\\ &s_{ab}\mathcal{C} = \frac{1}{e}\mathcal{B}, s_{ab}\bar{\mathcal{C}} = 0, s_{ab}\mathcal{B} = 0. \end{aligned} \quad (108)$$

Let us now calculate the equations of motion of the fields from the Lagrangian (104) by using the Euler–Lagrange equation in order to establish the important algebra between the nilpotent charges.

$$\dot{\phi}' - \phi'' + e(A_0' - A_1') = 0, \quad (109)$$

$$-\dot{\omega}' - \omega'' + e(A_0' + A_1') = 0, \quad (110)$$

$$\dot{\mathcal{B}} - \pi_1' - 2e(\phi' + \omega') = 0, \quad (111)$$

$$-\mathcal{B}' + \dot{\pi}_1 + 2e(\phi' - \omega') + 4e^2A_1 = 0, \quad (112)$$

$$\mathcal{B} + (\dot{A}_0 - A_1') = 0, \quad (113)$$

$$\ddot{\bar{\mathcal{C}}} - \bar{\mathcal{C}}'' = 0, \quad (114)$$

$$\ddot{\mathcal{C}} - \mathcal{C}'' = 0. \quad (115)$$

The Eqs. (109), (110), (111), (112), (113), (114), and (115) lead to the following useful relations

$$\dot{\mathcal{B}} = \Omega, \mathcal{B}'' = \dot{\Omega}. \quad (116)$$

It is straightforward to see that the BRST charge $Q_b$ and the anti-BRST $Q_{ab}$ satisfy the following relations

$$\dot{\mathcal{Q}}_b = \dot{\mathcal{Q}}_{ab} = 0, \{\mathcal{Q}_b, \mathcal{Q}_{ab}\} = 0, \mathcal{Q}_b^2 = 0, \mathcal{Q}_{ab}^2 = 0. \quad (117)$$

These algebras (117) guarantee the nilpotency of the BRST and anti-BRST charges. It completes the BRST and anti-BRST property of the model under consideration.

## 9 A discussion on the extended BRST symmetries of the model

A careful look reveals that apart from BRST and anti-BRST symmetry this model does have few other nilpotent symmetries. Let us now proceed to explore that. We should mention here that although these look nilpotent like the BRST symmetry, there are sharp differences with the BRST symmetry and that plays a pivotal role in the gauge symmetries of physically sensible theories. We are in a position to examine the symmetries one by one with special emphasis on the algebra of the charges corresponding to the symmetries.

### 9.1 Co-BRST and anti-Co-BRST symmetry

With this in view, we execute an investigation on the co-BRST and anti-co-BRST symmetry properties of this model. The algebra satisfied by the charges corresponding to these nilpotent symmetries also has been studied exhaustively. It is beneficial to mention at this stage that the total gauge fixing term remains invariant under the co-BRST symmetry transformations along with the invariance of the other terms involved in the theory. It is known that the origin of the gauge fixing term remains encoded in the co-exterior derivative $\delta = \pm * d*$ with $\delta^2 = 0$ of differential geometry as the operation of $\delta$ on a one-form produces the gauge-fixing term. The symbol $*$ indicates the Hodge duality operation on the $2D$ spacetime manifold. The $\pm$ sign refers to the dimensionality of the spacetime [34,69,70]. Thus, the nilpotent co-BRST symmetry transformations have their origin in the co-exterior derivative $\delta$ of differential geometry. We find that co-BRST transformations for the fields are

$$\begin{aligned} &s_d\phi = -\dot{\bar{\mathcal{C}}}, s_d\omega = \dot{\bar{\mathcal{C}}}, s_d A_0 = \frac{1}{e}\bar{\mathcal{C}}'', s_d A_1 = \frac{1}{e}\dot{\bar{\mathcal{C}}}', \\ &s_d c = \frac{1}{e}\Omega, s_d\bar{\mathcal{C}} = 0, s_d\mathcal{B} = 0. \end{aligned} \tag{118}$$

At this stage, it is straightforward to write down the conserved co-BRST charge of this theory which reads

$$\mathcal{Q}_d = \frac{i}{e}[\Omega\dot{\bar{\mathcal{C}}} - \pi_0\bar{\mathcal{C}}'']. \tag{119}$$

The action (104) is found to remain invariant under above co-BRST transformation with a little algebra. The anti-co-BRST charge is now written down as follows

$$\mathcal{Q}_{ad} = \frac{i}{e}[\Omega\dot{\mathcal{C}} - \pi_0\mathcal{C}'']. \tag{120}$$

A careful look reveals that the charges $\mathcal{Q}_d$ and $\mathcal{Q}_{ad}$ satisfy the following interesting relations.

$$\begin{aligned} s_d\mathcal{Q}_d &= -\{\mathcal{Q}_d, \mathcal{Q}_d\} = 0, \\ s_{ad}\mathcal{Q}_{ad} &= -\{\mathcal{Q}_{ad}, \mathcal{Q}_{ad}\} = 0, \\ s_d\mathcal{Q}_{ad} &= -\{\mathcal{Q}_{ad}, \mathcal{Q}_d\} = 0, \\ s_{ad}\mathcal{Q}_d &= -\{\mathcal{Q}_d, \mathcal{Q}_{ad}\} = 0. \end{aligned} \tag{121}$$

### 9.2 Bosonic symmetry

We have already discussed the BRST, anti-BRST, co-BRST, and anti-co-BRST symmetry. Besides, it is found that this theory has one more symmetry. This symmetry is indeed constituted with the aforesaid discussed BRST, anti-BRST, co-BRST, and anti-co-BRST symmetry. It is found that the following relations are satisfied.

$$\begin{aligned} &\{s_d, s_{ad}\} = 0, \\ &\{s_b, s_{ab}\} = 0, \\ &\{s_b, s_{ad}\} = 0, \\ &\{s_d, s_{ab}\} = 0 \\ &\{s_b, s_d\} = s_w, \\ &\{s_{ab}, s_{ad}\} = s_{\bar{w}}. \end{aligned} \tag{122}$$

Here $w$ corresponds to the bosonic symmetry. The field variables have the following transformations under the bosonic symmetry transformation:

$$\begin{aligned} s_w\phi &= -\frac{i}{e}(\dot{\mathcal{B}} + \Omega), \\ s_w\omega &= \frac{i}{e}(\dot{\mathcal{B}} + \Omega), \\ s_w A_0 &= \frac{i}{e}(\mathcal{B}'' + \dot{\Omega}), \\ s_w A_1 &= \frac{i}{e}(\dot{\mathcal{B}}' + \Omega'), \\ s_w\mathcal{C} &= 0, \\ s_w\bar{\mathcal{C}} &= 0, \\ s_w\mathcal{B} &= 0. \end{aligned} \tag{123}$$

However, the symmetry corresponding to $s_{\bar{w}}$ is not independent because we find that the operation of $s_w$ and $s_{\bar{w}}$ have the following linear algebraic relations between themselves:

$$s_w + s_{\bar{w}} = 0, ie, \{s_b, s_d\} = s_w = -\{s_{ab}, s_{ad}\} \tag{124}$$

This symmetry of course has a conserved charge. The conserved charge corresponding to this bosonic symmetry reads

$$\mathcal{Q}_w = \frac{i}{e^2}(\Omega^2 - \mathcal{B}\mathcal{B}''). \tag{125}$$

With the help of the equations of motion we land onto the following useful relations:

$$
\begin{aligned}
\dot{\mathcal{Q}}_w &= \frac{d\mathcal{Q}_w}{dt} = -\frac{i}{e^2}[\dot{\mathcal{B}}\mathcal{B}'' + \mathcal{B}\dot{\mathcal{B}}''] \\
&= \frac{i}{e^2}[\dot{\mathcal{B}}\mathcal{B}'' - \mathcal{B}\dot{\mathcal{B}}''] = 0,
\end{aligned} \tag{126}
$$

since $\Omega \approx 0$, and consequently $\dot{\Omega} \approx 0$. Therefore it appears the bosonic charge $\mathcal{Q}_w$ is a constant of motion of the theory. We have already seen that the theory is endowed with the BRST, anti-BRST, co-BRST, anti-co-BRST symmetry along with a bosonic symmetry. Apart from the presence of these symmetries, the theory has some extra symmetries. We now turn to observe that.

### 9.3 Ghost and discrete symmetry

We know that ghost and anti-ghost fields are designated by a specific number called ghost numbers. For the ghost field, the ghost number is 1 and the corresponding number for the anti-ghost field is $-1$. The matter, anti-matter, and gauge field of course have zero ghost numbers. The above fact provides a scale transformation that keeps the effective action of the theory invariant. We introduce the following scale transformation of the ghost field

$$
\begin{aligned}
&\phi \to \phi, \omega \to \omega, A_0 \to A_0, A_1 \to A_1, \mathcal{B} \to \mathcal{B}, \mathcal{C} \to e^{\lambda}\mathcal{C}, \\
&\bar{\mathcal{C}} \to e^{-\lambda}\bar{\mathcal{C}},
\end{aligned} \tag{127}
$$

where $\lambda$ is a global scale parameter and we find that the effective action of the theory remains invariant under these transformations. The above transformations in the infinitesimal limit take the following form

$$
\begin{aligned}
&s_g\phi = 0, s_g\omega = 0, s_g A_0 = 0, s_g A_1 = 0, s_g B = 0, s_g\mathcal{C} = \mathcal{C}, \\
&s_g\bar{\mathcal{C}} = -\bar{\mathcal{C}}.
\end{aligned} \tag{128}
$$

According to Noether's theorem, this symmetry must have a conserved charge and that conserved charge for this ghost symmetry is given by

$$
\mathcal{Q}_g = i[\dot{\bar{\mathcal{C}}}\mathcal{C} + \dot{\mathcal{C}}\bar{\mathcal{C}}]. \tag{129}
$$

The ghost sector is found to respect a discrete symmetry in addition to the above continuous symmetry transformation:

$$
\mathcal{C} \to \pm i\bar{\mathcal{C}}, \qquad \bar{\mathcal{C}} \to \pm i\mathcal{C}. \tag{130}
$$

This ends the discussion of the symmetry properties of this model. It is more interesting to investigate the geometrical cohomology corresponding to the symmetry of this model under consideration.

## 10 Cohomological aspect of the theory

In differential geometry, de Rham cohomological operators are known to obey the following important algebra

$$
d^2 = \delta^2 = 0, \Delta = (d+\delta)^2 = d\delta + \delta d = [d, \delta]_-, \tag{131}
$$

$$
[\Delta, \delta] = 0, [\Delta, d] = 0. \tag{132}
$$

Here $d$ and $\delta$ are respectively known as exterior and co exterior operator and $\Delta$ is known Laplace–Beltrami operator. Let us now look carefully at the algebra of the conserved charges corresponding to all these symmetries which the theory is possessing:

$$
\begin{aligned}
&\mathcal{Q}_b^2 = 0, \mathcal{Q}_{ab}^2 = 0, \mathcal{Q}_d^2 = 0, \mathcal{Q}_{ad}^2 = 0 \\
&\{\mathcal{Q}_b, \mathcal{Q}_{ab}\} = \{\mathcal{Q}_d, \mathcal{Q}_{ad}\} = \{\mathcal{Q}_b, \mathcal{Q}_{ad}\} = \{\mathcal{Q}_d, \mathcal{Q}_{ab}\} = 0, \\
&[\mathcal{Q}_g, \mathcal{Q}_b] = \mathcal{Q}_b, [\mathcal{Q}_g, \mathcal{Q}_{ab}] = -\mathcal{Q}_{ab}, [\mathcal{Q}_g, \mathcal{Q}_d] \\
&\quad = -\mathcal{Q}_d, [\mathcal{Q}_g, \mathcal{Q}_{ad}] = \mathcal{Q}_{ad}, \\
&\mathcal{Q}_w = -\{\mathcal{Q}_b, \mathcal{Q}_d\} = \{\mathcal{Q}_{ad}, \mathcal{Q}_{ab}\}, [\mathcal{Q}_w, \mathcal{Q}_\alpha] = 0.
\end{aligned} \tag{133}
$$

where $\mathcal{Q}_\alpha \equiv (\mathcal{Q}_b, \mathcal{Q}_{ab}, \mathcal{Q}_d, \mathcal{Q}_{ad}, \mathcal{Q}_g)$. The above relations (133) transpires that $\mathcal{Q}_w$ is the Casimir operator of the whole algebra.

We know that in differential geometry the role of an exterior derivative is to raise the degree of a form by one, i.e., $df_n = f_{n+1}$ whereas the role of the co-exterior derivative is the reverse, it lowers the degree of a form by one, i.e., $\delta f_n = f_{n-1}$. Here $f_n$ represents a form of degree $n$. Under the operation of $\Delta$ however, the degree of a form remains unaltered. Let us now define a state $|\eta\rangle$ with ghost number $\kappa$ in the Hilbert space of states of this BRST invariant theory as

$$
i\mathcal{Q}_g|\eta\rangle_\kappa = \kappa|\eta\rangle_\kappa. \tag{134}
$$

It is straightforward to verify the relations

$$
\begin{aligned}
i\mathcal{Q}_g\mathcal{Q}_b|\eta\rangle_\kappa &= (\kappa+1)\mathcal{Q}_b, |\eta\rangle_\kappa \\
i\mathcal{Q}_g\mathcal{Q}_{ad}|\eta\rangle_\kappa &= (\kappa+1)\mathcal{Q}_{ad}|\eta\rangle_\kappa, \\
i\mathcal{Q}_g\mathcal{Q}_d|\eta\rangle_\kappa &= (\kappa-1)\mathcal{Q}_d|\eta\rangle_\kappa, \\
i\mathcal{Q}_g\mathcal{Q}_{ab}|\eta\rangle_\kappa &= (\kappa-1)\mathcal{Q}_{ab}|\eta\rangle_\kappa, \\
i\mathcal{Q}_g\mathcal{Q}_w|\eta\rangle_\kappa &= \kappa\mathcal{Q}_w|\eta\rangle_\kappa.
\end{aligned} \tag{135}
$$

A careful look into the Eq. (135) transpires the following analogy:

$$
(\mathcal{Q}_b, \mathcal{Q}_{ad}) \to d, (\mathcal{Q}_d, \mathcal{Q}_{ab}) \to \delta, \mathcal{Q}_w \to \Delta. \tag{136}
$$

Note that $(\mathcal{Q}_b, \mathcal{Q}_{ad})$ raise the ghost number of the state by one whereas $(\mathcal{Q}_d, \mathcal{Q}_{ab})$ lower the ghost number by one, and $\mathcal{Q}_w$ keeps the ghost number unchanged. So $(\mathcal{Q}_b, \mathcal{Q}_{ad}), (\mathcal{Q}_d, \mathcal{Q}_{ab}), \mathcal{Q}_w$ resemble the algebra of $d, \delta, \Delta$. A closer look reveals that the analogy with the Hodge-de Rham decomposition theorem enables us to express any arbitrary state $|\eta\rangle_n$ in terms of the sets $(\mathcal{Q}_b, \mathcal{Q}_d, \mathcal{Q}_w)$ and $(\mathcal{Q}_{ad}, \mathcal{Q}_{ab}, \mathcal{Q}_w)$ as

$$|\eta\rangle_\kappa = |\sigma\rangle_\kappa + \mathcal{Q}_b|\chi\rangle_{\kappa-1} + \mathcal{Q}_d|\rho\rangle_{\kappa+1}, \tag{137}$$

$$|\eta\rangle_\kappa = |\sigma\rangle_\kappa + \mathcal{Q}_{ad}|\chi\rangle_{\kappa-1} + \mathcal{Q}_{ab}|\rho\rangle_{\kappa+1}, \tag{138}$$

where the most symmetric state is the harmonic state $|\sigma\rangle_\kappa$, that satisfies the following equations

$$\mathcal{Q}_w|\sigma\rangle_\kappa = 0, \mathcal{Q}_b|\sigma\rangle_\kappa = 0, \mathcal{Q}_d|\sigma\rangle_\kappa = 0, \mathcal{Q}_{ab}|\sigma\rangle_\kappa = 0, \\ \mathcal{Q}_{ad}|\sigma\rangle_\kappa = 0. \tag{139}$$

The charges $\mathcal{Q}_b, \mathcal{Q}_{ab}, \mathcal{Q}_d$, and $\mathcal{Q}_{ad}$ corresponding to the symmetries of a physically sensible theory must maintain the conditions

$$\mathcal{Q}_b|Phy\rangle = 0, \quad \mathcal{Q}_{ab}|Phy\rangle = 0, \\ \mathcal{Q}_d|Phy\rangle = 0, \quad \mathcal{Q}_{ad}|Phy\rangle = 0, \tag{140}$$

which lead to the following two conditions on the first-class constraints that generate the gauge as well as BRST symmetry of the theory:

$$\pi_0|Phy\rangle = 0, ie, \quad \mathcal{B}|Phy\rangle = 0 \tag{141}$$

$$[\pi_1' + e(\pi_\phi - \pi_\omega) + e(\phi' + \omega')]|Phy\rangle = 0, ie, \\ \Omega|Phy\rangle = 0. \tag{142}$$

## 11 Summary and discussion

We have taken into consideration a gauged Lagrangian with a Siegel type chiral boson with a parameter-involved non-covariant masslike term for the gauge field. The masslike term that was selected in [10] resulted in a gauged theory of Floreanini–Jackiw type chiral boson which may be derived from the Chiral Schwinger model with the Jackiw–Rajaraman type of the electromagnetic anomaly as became proven with the incredible concept of imposition of chiral constraint in the article of Harada [11]. To derive the gauged version of Floreanini–Jackiw type of chiral boson a parameter involved masslike term is brought right here in Eqn. (1) with the anticipation that we receive the same gauged model of chiral boson [14] obtained from the chiral Schwinger model with the one-parameter class of improved Faddeevian anomaly [23].

In the article [19], the author showed that the chiral Schwinger model remains physically sensible in all respect with a parameter-unfastened masslike term where the nature of anomaly belonged to the Faddeevian class and a one parameter-class of improved Faddeevian anomaly was generated in [23]. We have made here an equivalence between the gauge-invariant and gauge non-invariant version of this gauged Floreanini–Jackiw type chiral boson with a parameter-involved masslike term using the ingenious formalism developed in [68]. The role of gauge fixing is found very crucial here because an arbitrary but legitimist gauge fixing may lead to other effective theories which may fail to establish the equivalence. Using the formalism developed in [27] we write down the BRST invariant effective action without going through the formal BFV development of BRST quantization as it was followed in the article [49,53,63] to study the BRST symmetry and the corresponding BRST cohomology. In the article [53] however, BRST cohomology was not attempted. We study the extended version of the BRST algebra. We also notice that the nilpotent anti-BRST symmetry transformations always satisfy the absolute anti-commutativity. The nilpotency signalizes the fermionic nature of the anti-BRST symmetry transformations and the absolute anti-commutativity encodes the linear independence of these transformations. The gauge fixing term, owing to its origin in the co-exterior derivative remains invariant under co-BRST symmetry transformations. Thus, the co-exterior derivative can be realized in terms of the co-BRST symmetry of the theory. The anti-commutator of BRST and co-BRST transformations produce a bosonic symmetry which is an analog of the Laplacian operator. It is also found that the ghost terms of the theory remain invariant under the bosonic symmetry transformations. Finally, we have shown that, at the algebraic level, the aforesaid symmetry transformations resemble the identical algebra as it is found to be satisfied by the de Rham cohomological operators of differential geometry.

BRST quantization of the parameter-unfastened Faddeevian anomaly in a formal way using BFV was done earlier in [42] where we did not study the cohomological aspects. In this article, our main emphasis is to study the cohomological aspects that precisely require an extended version of the BRST algebra i.e., the algebra between the different transformation generators. To be precise, these are the BRST, anti-BRST, co-BRST, anti-co-BRST, and the conserved charge corresponding to the bosonic symmetry. We have been succeeded to establish that the model with parameter-dependent Faddeevian anomaly also belongs to the Hodge class.

One of the highlights of our present investigation is the observation that the Lagrangian density under consideration provides a model for the Hodge theory because the continuous symmetry operators of the specific Lagrangian density and the corresponding charges obey an algebra that is reminiscent of the algebra obeyed by the de Rham cohomological operators of differential geometry. In other words, the continuous symmetry operators provide the physical realizations of the cohomological operators of differential geometry. This happens because of the fact that the Lagrangian density respects five perfect symmetries. This is precisely the reason that four of the above-mentioned five symmetries of the theory obey an exact algebra that a reminiscent of the algebra obeyed by the de Rham cohomological operators of the differential geometry.

A cautious look exhibits that the version developed in [11] and [12] appear with quite different mathematical and structural forms. However these two [11] and [12] are essentially originated from the fermionic chiral Schwinger model which was described in [15]. A precise bosonization of this fermionic model within the $(1+1)$ dimensional has been found possible. A one-loop correction enters inside that via regularization during the course of the bosonization process. This regularization may be executed in unique methods and that ends up with unique counter terms. Parameter-unfastened Faddeevian regularization came in the literature due to Mitra [19]. The generalization with the one-parameter involved improved Faddeevian regularization was developed in [23]. It has been observed that each of the models may be expressed in terms of chiral boson [12,14], like Jackiw–Rajaraman's version of the chiral Schwinger model which was offered by Harada [11]. So it exhibits that those versions are the outcome of the use of various regularizations during the course of bosonization of the fermionic model proposed in [15]. Therefore, it has been found that unique regularization ends in the bosonized version that has an appearance quite unique in a structural sense. Most of these bosonized versions can be expressed in terms of chiral bosons and they have identical symmetry structure at the quantum level and they all belong to the magnificent Hodge class. In the article [46], it has been visible that for the bosonized model of the chiral Schwinger with different regularizations (the standard one and the parameter-unfastened Faddeevian) and also for the Schwinger model [63] the symmetry at the quantum level stays unchanged and it does no longer rely on the selection of regularization and certainly, the Hodge algebra becomes satisfied with the aid of the extended BRST symmetries. Here we observe that the bosonized model of the chiral Schwinger model with the one-parameter class of improved Faddeevian regularization additionally, respect the identical symmetry and belongs to the same magnificient Hodge class. So via case studies, it has been has been established that the symmetry of the chiral Schwinger model stays identical at the quantum level regardless of the character of anomaly. Of course, extra distinct research is needed to establish it in general.

**Data Availability Statement** This manuscript has no associated data or the data will not be deposited. [Authors' comment: No numerical computation is involved since it is an analytical work.]



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