



Are there BPS dyons in the generalized $SU(2)$ Yang–Mills–Higgs model?

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Abstract We use the well-known Bogomolny's equations, in general coordinate system, for BPS monopoles and dyons in the $SU(2)$ Yang–Mills–Higgs model to obtain an explicit form of BPS Lagrangian density under the BPS Lagrangian method. We then generalize this BPS Lagrangian density and use it to derive several possible generalized Bogomolny's equations, with(out) additional constraint equations, for BPS monopoles and dyons in the generalized $SU(2)$ Yang–Mills–Higgs model. We also compute the stress–energy–momentum tensor of the generalized model, and argue that the BPS monopole and dyon solutions are stable if all components of the stress-tensor density are zero in the BPS limit. This stability requirement implies the scalar fields-dependent couplings to be related to each other by an equation, which is different from the one obtained in Atmaja and Prasetyo (Adv High Energy Phys 2018:7376534, [arXiv: 1803.06122](https://arxiv.org/abs/1803.06122), 2018), and then picks particular generalized Bogomolny's equations, with no additional constraint equation, out of those possible equations. We show that the computations in [1] are actually incomplete. Under the Julia–Zee ansatz, the generalized Bogomolny's equations imply all scalar fields-dependent couplings must be constants, whose solutions are the BPS dyons of the $SU(2)$ Yang–Mills–Higgs model (Prasad and Sommerfield in Phys Rev Lett 35:760, 1975), or in another words there are no generalized BPS dyon solutions under the Julia–Zee ansatz. We propose two possible ways for obtaining generalized BPS dyons, where at least one of the scalar fields-dependent couplings is not constant, that are by using different ansatze, such as axially symmetric ansatz for higher topological charge dyons; and/or by considering the most general BPS Lagrangian density.

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1 Introduction

The natural extension of monopoles are dyons which are essentially monopoles with non-zero electrical charges. They were first proposed as an alternative to quarks by Julian Schwinger [3], whose quantum mechanical properties were studied by Zwanziger [4, 5]. Like monopoles, it is also natural for dyons to exist in the non-Abelian gauge theories. The first example of monopoles existence was shown in the $SU(2)$ Yang–Mills–Higgs model, also known as Georgi–Glashow model [6], by Polyakov and 't Hooft [7, 8]. It was later shown

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that the dyons could also exist in the same model by Julia and Zee [9].

The explicit solutions of 'tHooft–Polyakov monopoles and Julia–Zee dyons were presented by Prasad and Sommerfield by taking a special limit to the model [2]. These solutions turn out to be solutions of first-order differential equations, known as Bogomolny's equations, that were derived by Bogomolny [10].¹ The solutions saturate the non-trivial static energy bound which turns out to be proportional with the topological charge. Obtaining the Bogomolny's equations of a model is important in particular to study the topological stability of its solitons solutions. There have been some methods developed in these directions which are the first-order formalism [11, 12], FOEL (first-order-Euler–Lagrange) formalism by using the concept of strong necessary conditions [13, 14], the *On-Shell* method [15, 16], and the BPS Lagrangian method [17, 18].

The new studies on monopoles and dyons have been carried out recently that give rise to new features and dynamics. Some of those studies were based on modifications to the $SU(2)$ Yang–Mills–Higgs model [6]. One of the studies were done by inserting extra degrees of freedom along with additional global symmetries [19]. The other one is by adding scalar fields-dependent couplings to each of its kinetic terms, that we shall call as generalized $SU(2)$ Yang–Mills–Higgs model proposed in [20], in which the monopoles could be endowed with internal structures [21]. In condensed matter, this effective model could be important for the magnetic materials known as spin ice that has a capability to support exotic magnetic structures such as monopoles [22–24]. There is also a possibility of electric dipole existing in these monopoles [25]. This motivates us to study further about dyons in this effective model which may exist as exotic structures in spin ice. More recently, there is also a study by combining these two modifications that leads to monopoles endowed with some internal structures [26].

In this article we would like to study about BPS dyons in the generalized $SU(2)$ Yang–Mills model. In the generalized $SU(2)$ Yang–Mills–Higgs model the dynamics of overall system may differ from its corresponding canonical model, which are explicitly shown in the generalized Bogomolny's equations for BPS monopoles and for dyons. The first-order formalism has been used to derive the generalized Bogomolny's equations for BPS monopoles in which the solutions are called generalized BPS monopoles [20]. These Bogomolny's equations exist only if the scalar fields-dependent couplings are related to each other by an equation. On the other hand, the BPS Lagrangian method has been used to rederive the Bogomolny's equations for BPS

monopoles and it also managed to obtain the Bogomolny's equations for BPS dyons, which exist only if the scalar fields-dependent couplings are related to each other by a more general equation [1]. However, all those derivations rely on a particular hedgehog ansatz namely 't Hooft–Polyakov and Julia–Zee ansatz for monopoles and dyons respectively. It is then necessary to find the generalized Bogomolny's equations for BPS monopoles and dyons in general coordinate system that are independent of any ansatz in order to study other possible soliton solutions and configurations. We also would like to verify if the relations between scalar fields-dependent couplings for BPS monopoles and dyons, derived in [1] for the Julia–Zee ansatz, are still hold in the general coordinate system for any ansatz. Nevertheless the existence of these Bogomolny's equations in general coordinate system are inevitable for extending the corresponding model to its supersymmetric version.

For this matter, we will use the BPS Lagrangian method and generalize its procedures in order to work in general coordinate system. At first, we will employ it to the case of the $SU(2)$ Yang–Mills–Higgs model and derive its corresponding BPS Lagrangian density using the fact that we already had the well-known Bogomolny's equations, in the general coordinate system, for BPS monopoles and dyons at our disposal. We generalize the BPS Lagrangian density, by multiplying each term in the BPS Lagrangian density with arbitrary function of the scalar-fields, and then use this generalized BPS Lagrangian density to derive the generalized Bogomolny's equations for BPS monopoles and dyons in the generalised $SU(2)$ Yang–Mills–Higgs model. We write down all possible generalized Bogomolny's equations, that correspond to the generalized BPS Lagrangian density, for BPS monopoles and dyons in the general coordinate system and study their stabilities from the stress–energy–momentum density tensor. As an example we will apply the Julia–Zee ansatz into the generalized Bogomolny's equations, along with the constraint equations, and compare the results with the ones in [1].

2 The generalized $SU(2)$ Yang–Mills–Higgs model

In this article we will consider the generalized $SU(2)$ Yang–Mills–Higgs model with the following Lagrangian density [1, 20]:²

$$\mathcal{L} = -\frac{w(|\Phi|)}{2} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) + G(|\Phi|) \text{Tr}(D_\mu \Phi D^\mu \Phi) - V(|\Phi|), \quad (2.1)$$

where $w, G > 0$ and $V \geq 0$ are functions of scalar fields and are also $SU(2)$ invariant, with $|\Phi| = 2\text{Tr}(\Phi)^2$, $F_{\mu\nu} =$

¹ These solutions to Bogomolny's equations are generally called BPS solutions for monopoles and dyons, or briefly called BPS monopoles and BPS dyons.

² Here we follow the notations in [1].

$\partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu]$, $D_\mu \equiv \partial_\mu - ie[A_\mu, \cdot]$, and $\mu, \nu = 0, 1, 2, 3$ are spacetime indices with metric signature $(+ - - -)$. In terms of components, the gauge and scalar fields are

$$A_\mu = \frac{1}{2} \tau^a A_\mu^a, \quad \Phi = \frac{1}{2} \tau^a \Phi^a, \quad (2.2)$$

with $a = 1, 2, 3$ and τ^a are the Pauli matrices. The full Euler–Lagrange equations are

$$D_\mu (G D^\mu \Phi) = 2 \frac{\partial G}{\partial |\Phi|} \text{Tr}(D_\mu \Phi D^\mu \Phi) \Phi - \frac{\partial w}{\partial |\Phi|} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) \Phi - 2 \frac{\partial V}{\partial |\Phi|} \Phi, \quad (2.3a)$$

$$D_\nu (w F^{\mu\nu}) = -ieG[\Phi, D^\mu \Phi]. \quad (2.3b)$$

In the literature, the solutions for monopoles and dyons were mostly found by taking the following Julia–Zee ansatz

$$\Phi^a = f(r) \frac{x^a}{r}, \quad (2.4a)$$

$$A_0^a = \frac{j(r)}{e} \frac{x^a}{r}, \quad (2.4b)$$

$$A_i^a = \frac{1 - a(r)}{e} \epsilon^{aij} \frac{x^j}{r^2}, \quad (2.4c)$$

where $x^a \equiv (x, y, z)$, as well as $x^{i,j} \equiv (x, y, z)$, denotes the Cartesian coordinates. Here we shall call the function $f(r)$, $j(r)$, and $a(r)$ as effective fields of the scalar Higgs, the scalar potential, and the vector potential fields respectively. Notes that the Levi-Civita symbol ϵ^{aij} in (2.4) mixes the spatial indices and the group index. The ansatz (2.4) is actually defined for the Julia–Zee dyons while for the 't Hooft–Polyakov monopoles is defined by taking $j = 0$, or known as the 't Hooft–Polyakov ansatz. For the later purposes let us define $E_i = \frac{1}{2} \tau^a E_i^a \equiv F_{0i}$ and $B_i = \frac{1}{2} \tau^a B_i^a \equiv \frac{1}{2} \epsilon_{ijk} F_{jk}$ which represent electric fields and magnetic fields respectively.

3 BPS Lagrangian method in general coordinate system

The standard way to obtain Bogomolny's equations of a model is by considering its energy functional. We then rewrite it by completing the square in such a way that it contains "boundary" term. Using this Bogomolny's trick, we will be able to obtain the Bogomolny's equations by minimizing the energy functional of this form [10]. However the Bogomolny's trick may not be applicable to all models since there is no rigorous way to do it. Nevertheless even if we obtain the Bogomolny's equations, by using the Bogomolny's trick, we still need to verify if these Bogomolny's equations satisfy the full equations of motion trivially, e.g. in the case of $SU(2)$

Yang–Mills–Higgs model the Bogomolny's equations indeed satisfy the Gauss's law constraint equation trivially [28, 29].

A more rigorous way to obtain the Bogomolny's equations of a model is by using BPS Lagrangian method [17, 18]. Rather than considering the energy functional, the BPS Lagrangian method works directly to the action of a model. This method uses the fact that the Lagrangian density of most models contains up to quadratic in first-derivative of the fields. As an example Lagrangian density of a model with N -scalar fields, ϕ^i where $i = 1, \dots, N$, can be rewritten into the following form

$$\mathcal{L} = \sum_{i=1}^N g^i(\phi^j) \left(\partial_\mu \phi^i - f_\mu^i(\phi^j, \partial_\nu \phi^j) \right)^2 + \mathcal{L}_{BPS}, \quad (3.1)$$

where in general $g^i(\phi^j)$ is a function of fields ϕ^j 's, and f_μ^i is a function of fields ϕ^j 's and their first-derivative $\partial_\nu \phi^j$'s, with $j = 1, \dots, N$, but not of $\partial_\mu \phi^i$. Here we shall call \mathcal{L}_{BPS} as BPS Lagrangian density which in general is a function of fields ϕ^j 's and their first-derivative $\partial_\nu \phi^j$'s. The Bogomolny's equations are obtained from (3.1) in the limit where $\mathcal{L} - \mathcal{L}_{BPS} = 0$, or also known as the BPS limit condition, such that

$$\partial_\mu \phi^i = f_\mu^i(\phi^j, \partial_\nu \phi^j). \quad (3.2)$$

The BPS Lagrangian density plays an important role in the BPS Lagrangian method. Its Euler–Lagrange equations are called constraint equations,

$$\partial_\mu \left(\frac{\delta \mathcal{L}_{BPS}}{\delta (\partial_\mu \phi^i)} \right) = \frac{\delta \mathcal{L}_{BPS}}{\delta \phi^i}, \quad (3.3)$$

which must be considered, in addition to the Bogomolny's equations, in order to find the solitonic solutions. Depending on the choice of terms in the BPS Lagrangian density, its Euler–Lagrange equations could be all trivial. In this case the BPS Lagrangian density contains only "boundary" terms such that there are no additional constraint equations. Several possible "boundary" terms, that can be included in the BPS Lagrangian density, have been studied in [14]. For most of the known cases, their BPS Lagrangian densities were found to contain only "boundary" terms, under some particular ansatz [1, 17]. There has been a study on the BPS Lagrangian density containing "non-boundary" terms, under a particular ansatz, which results in solitonic solutions whose stress tensor are non-zero [18]. However, the existence "non-boundary" terms does not always imply additional constraint equations. In the BPS limit, these constraint equations could be trivially satisfied and thus can be neglected in finding the solitonic solutions. We will see that this is the case for BPS Lagrangian densities considered in this article.

Let us first consider the $SU(2)$ Yang–Mills–Higgs model by taking $G = w = 1$ into the Lagrangian density (2.1), which can be written in terms of E_i and B_i as [6]

$$\mathcal{L} = \text{Tr} (E_i)^2 - \text{Tr} (B_i)^2 + \text{Tr} (D_0 \Phi)^2 - \text{Tr} (D_i \Phi)^2 - V, \quad (3.4)$$

where $i = 1, 2, 3$ is the spatial indices. The next step in the BPS Lagrangian method is to write the BPS Lagrangian density. The BPS Lagrangian density initially consisted of terms that are linear in the first-derivative of fields with additional condition that they are “boundary” terms, which its Euler–Lagrange equations are trivial [17]. It was then extended to contain the terms that are quadratic in the first-derivative of fields [18]. Furthermore, it can be generalized to contain terms that are polynomial in the first-derivative of fields, or in general terms that are not necessary “boundary” terms as such its Euler–Lagrange equations are non-trivial [27]. These Euler–Lagrange equations will then be constraint equations that must be considered in finding the solutions. However so far the BPS Lagrangian density has been written under certain ansatzs, such as (2.4), in the spherical coordinate system [1]. Generalizing to general coordinate system would then implies the BPS Lagrangian density with massive terms and hence making the computation to be more complicated. For particular case of the $SU(2)$ Yang–Mills–Higgs model, we will make use of the well-known Bogomolny’s equations for monopoles and dyons [10, 28, 29] to derive the BPS Lagrangian that would lead to these Bogomolny’s equations.

Using the Bogomolny’s trick [10], one can obtain the well-known Bogomolny’s equations for monopoles and dyons by completing the square in the energy density [28, 29],

$$E_i = \sin \theta D_i \Phi, \quad B_i = \cos \theta D_i \Phi, \quad D_0 \Phi = 0, \quad V = 0, \quad (3.5)$$

with θ is a real constant. In addition there is one constraint equation that must be considered in order to find the solutions and that is the Gauss’s law constraint [28, 29],

$$D_i F_{0i} = ie [\Phi, D_0 \Phi], \quad (3.6)$$

which is essentially the Euler–Lagrange equations for the gauge scalar potential A_0 . Notes that the Gauss’s law constraint is trivially satisfied in the BPS limit and thus we only need to consider and solve the Bogomolny’s equations (3.5) in order to find the BPS monopole and dyon solutions. Using these Bogomolny’s equations, we can rewrite the Lagrangian density (3.4) to be

$$\begin{aligned} \mathcal{L} = & \text{Tr} (E_i - \sin \theta D_i \Phi)^2 - \text{Tr} (B_i - \cos \theta D_i \Phi)^2 \\ & + \text{Tr} (D_0 \Phi)^2 - V \\ & + 2 \sin \theta \text{Tr} (E_i D_i \Phi) - 2 \cos \theta \text{Tr} (B_i D_i \Phi) \\ & - 2 \sin^2 \theta \text{Tr} (D_i \Phi)^2. \end{aligned} \quad (3.7)$$

In the BPS Lagrangian method we set $\mathcal{L} - \mathcal{L}_{BPS} = 0$ in the BPS limit, which is the limit where the Bogomolny’s equations (3.5) are satisfied, and thus implies the BPS Lagrangian density

$$\begin{aligned} \mathcal{L}_{BPS} = & 2 \sin \theta \text{Tr} (E_i D_i \Phi) - 2 \cos \theta \text{Tr} (B_i D_i \Phi) \\ & - 2 \sin^2 \theta \text{Tr} (D_i \Phi)^2. \end{aligned} \quad (3.8)$$

So here we find that the BPS Lagrangian density consists of terms proportional to $B_i D_i \Phi$, $E_i D_i \Phi$, and $(D_i \Phi)^2$. Furthermore setting all terms in $\mathcal{L} - \mathcal{L}_{BPS}$ to be zero gives us the Bogomolny’s equations (3.5) in which here their solutions shall be called the standard BPS monopoles and dyons, respectively for $\sin(\theta) = 0$ and $\sin(\theta) \neq 0$.

Now let us write a slightly more general BPS Lagrangian density, than the previous one, as follows

$$\begin{aligned} \mathcal{L}_{BPS} = & -2\beta \text{Tr} (B_i D_i \Phi) + 2\alpha \text{Tr} (E_i D_i \Phi) \\ & - (\alpha^2 - \beta^2 + 1) \text{Tr} (D_i \Phi)^2, \end{aligned} \quad (3.9)$$

where now α and β are arbitrary constants. We would like to prove that the Bogomolny’s equations (3.5) and also the Gauss’s law constraint (3.6) can be rederived using this BPS Lagrangian density. Taking $\mathcal{L} - \mathcal{L}_{BPS} = 0$ and setting all terms to be zero gives us Bogomolny’s equations

$$E_i = \alpha D_i \Phi, \quad B_i = \beta D_i \Phi, \quad D_0 \Phi = 0, \quad V = 0. \quad (3.10)$$

One can show that Euler–Lagrange equation of the first term in the BPS Lagrangian density above is trivial using the Bianchi identity $D_i B_i = 0$ and a relation $[D_i, D_j] \Phi = -ie [F_{ij}, \Phi]$, and hence it is indeed a “boundary” term while the remaining terms turn out to be “non-boundary” terms which contribute to the Euler–Lagrange equations of the BPS Lagrangian density: for Φ ,

$$\alpha D_i F_{0i} - (\alpha^2 - \beta^2 + 1) D_i D_i \Phi = 0, \quad (3.11)$$

for A_i ,

$$\alpha (D_0 D_i \Phi - ie [F_{0i}, \Phi]) = ie (\alpha^2 - \beta^2 + 1) [\Phi, D_i \Phi], \quad (3.12)$$

and for A_0 ,

$$\alpha D_i D_i \Phi = 0. \quad (3.13)$$

The Eqs. (3.11)–(3.13) are additional constraint equations, in addition to the Bogomolny’s equations (3.10), that must be considered in finding solutions for monopoles and dyons. With these additional constraint equations, we seem to have more equations than the number of fields to be solved. In the BPS limit, in which the BPS equations (3.10) are satisfied, these additional constraint equations can be simplified, respectively, to

$$(1 - \beta^2) D_i D_i \Phi = 0, \quad (3.14a)$$

$$(1 - \alpha^2 - \beta^2) [D_i \Phi, \Phi] = 0, \quad (3.14b)$$

$$\alpha D_i D_i \Phi = 0, \quad (3.14c)$$

where we have used the fact that $[D_0, D_i] \Phi = -ie [F_{0i}, \Phi]$. We can simplify these constraint equations using the Bianchi identity $D_i B_i = 0$ which, after substituting the Bogomolny's equations (3.10), becomes $\beta D_i D_i \Phi = 0$. Requiring $\beta \neq 0$, the remaining constraint equation is³

$$(1 - \alpha^2 - \beta^2) [D_i \Phi, \Phi] = 0. \quad (3.15)$$

Solutions to this equation is $\alpha^2 + \beta^2 = 1$. In this BPS limit, the Gauss's law constraint (3.6) is trivial and thus in finding the BPS monopoles and dyons, there are no additional equations need to be considered beside the Bogomolny's equations (3.10). This is actually what we expected from the BPS Lagrangian method since the Bogomolny's equations (3.10) must satisfy trivially the Euler–Lagrange equations which the Gauss's law constraint is one of.

4 Generalized BPS monopoles and dyons

Following the previous sections now we may consider a more general BPS Lagrangian density to derive Bogomolny's equations for monopoles and dyons in the generalized $SU(2)$ Yang–Mills–Higgs model (2.1), which is given by

$$\mathcal{L}_{BPS} = 2\alpha \text{Tr}(E_i D_i \Phi) - 2\beta \text{Tr}(B_i D_i \Phi) - \gamma \text{Tr}(D_i \Phi)^2, \quad (4.1)$$

where now $\alpha \equiv \alpha(|\Phi|)$, $\beta \equiv \beta(|\Phi|)$, and $\gamma \equiv \gamma(|\Phi|)$ are arbitrary functions of $|\Phi|$. In this case

$$\begin{aligned} \mathcal{L} - \mathcal{L}_{BPS} = & w \text{Tr} \left(E_i - \frac{\alpha}{w} D_i \Phi \right)^2 - w \text{Tr} \left(B_i - \frac{\beta}{w} D_i \Phi \right)^2 \\ & + G \text{Tr}(D_0 \Phi)^2 \\ & - \left(-\gamma + \frac{\alpha^2}{w} - \frac{\beta^2}{w} + G \right) \text{Tr}(D_i \Phi)^2 - V. \end{aligned} \quad (4.2)$$

Now in the BPS limit $\mathcal{L} - \mathcal{L}_{BPS} = 0$ which implies all terms on the right hand side of (4.2) should be zero. Since $(G, w) \neq 0$, the first three terms should be identified as the Bogomolny's equations

$$E_i = \frac{\alpha}{w} D_i \Phi, \quad (4.3a)$$

$$B_i = \frac{\beta}{w} D_i \Phi, \quad (4.3b)$$

$$D_0 \Phi = 0, \quad (4.3c)$$

and the last term implies $V = 0$. The fourth term could be zero if we set $D_i \Phi = 0$, but this will make the Bogomolny's

equations (4.3a) and (4.3b) trivial and hence $D_i \Phi \neq 0$. So for this term we should take

$$\gamma = G + \frac{\alpha^2}{w} - \frac{\beta^2}{w}. \quad (4.4)$$

Additionally there are also constraint equations coming from the Euler–Lagrange equations of the BPS Lagrangian density (4.1), which are: for Φ ,

$$\begin{aligned} & 4 \frac{\partial \alpha}{\partial |\Phi|} [\text{Tr}(\Phi \partial_i \Phi) E_i - \text{Tr}(E_i D_i \Phi) \Phi] + \alpha D_i E_i \\ & - 4 \frac{\partial \beta}{\partial |\Phi|} [\text{Tr}(\Phi \partial_i \Phi) B_i - \text{Tr}(D_i \Phi B_i) \Phi] \\ & + 2 \frac{\partial \gamma}{\partial |\Phi|} [\text{Tr}(D_i \Phi)^2 \Phi - 2 \text{Tr}(\Phi \partial_i \Phi) D_i \Phi] \\ & - \gamma D_i D_i \Phi = 0, \end{aligned} \quad (4.5)$$

for A_i ,

$$\begin{aligned} & 4 \frac{\partial \alpha}{\partial |\Phi|} \text{Tr}(\Phi \partial_0 \Phi) D_i \Phi + \alpha (D_0 D_i \Phi - ie [E_i, \Phi]) \\ & + 4 \frac{\partial \beta}{\partial |\Phi|} \epsilon_{ijk} \text{Tr}(\Phi \partial_j \Phi) D_k \Phi - ie \gamma [\Phi, D_i \Phi] = 0, \end{aligned} \quad (4.6)$$

for A_0 ,

$$-4 \frac{\partial \alpha}{\partial |\Phi|} \text{Tr}(\Phi \partial_i \Phi) D_i \Phi - \alpha D_i D_i \Phi = 0. \quad (4.7)$$

As shown in the previous section, we write these constraint equations in the BPS limit namely by substituting the Bogomolny's equations (4.3a), (4.3b), $D_0 \Phi = 0$, and $V = 0$, together with the Eq. (4.4). The constraint equations are now simplified, respectively, to

$$\begin{aligned} & -4 \left(G' - \frac{\beta}{w} \beta' + \frac{\beta^2}{w^2} w' \right) \text{Tr}(\Phi D_i \Phi) D_i \Phi \\ & - \left(G - \frac{\beta^2}{w} \right) D_i D_i \Phi \\ & + 2 \left(G' - \frac{\alpha^2}{w^2} w' + \frac{\beta^2}{w^2} w' \right) \text{Tr}(D_i \Phi)^2 \Phi = 0, \end{aligned} \quad (4.8a)$$

$$\begin{aligned} & 4\beta' \epsilon_{ijk} \text{Tr}(\Phi D_j \Phi) D_k \Phi \\ & - ie \left(G - \frac{\alpha^2}{w} - \frac{\beta^2}{w} \right) [\Phi, D_i \Phi] = 0, \end{aligned} \quad (4.8b)$$

$$-4\alpha' \text{Tr}(\Phi D_i \Phi) D_i \Phi - \alpha D_i D_i \Phi = 0, \quad (4.8c)$$

where the apostrophe ' means taking derivative over $|\Phi|$.

4.1 The Bianchi identity

The equations of motion for the gauge fields are not only given by the Euler–Lagrange equations (2.3b), but also by the Bianchi identity,

$$\epsilon^{\sigma\rho\mu\nu} D_\rho F_{\mu\nu} = 0, \quad (4.9)$$

³ We can not take $[D_i \Phi, \Phi] = 0$ because it could imply $[\tau^a, \tau^b] = 0$, for arbitrary a and b , which is incorrect.

which can be divided into two equations

$$D_i B_i = 0, \quad (4.10a)$$

$$2D_0 B_i = \epsilon_{ijk} D_{[j} E_{k]}. \quad (4.10b)$$

In the BPS limit, by substituting the Bogomolny's equations (4.3), the Eq. (4.10a) becomes

$$\frac{\beta}{w} D_i D_i \Phi = -4 \left(\frac{\beta}{w} \right)' \text{Tr}(\Phi D_i \Phi) D_i \Phi, \quad (4.11)$$

while, for static cases, the Eq. (4.10b) becomes

$$\left(\frac{\alpha}{w} \right)' \epsilon_{ijk} \text{Tr}(\Phi D_{[j} \Phi) D_{k]} \Phi = 0. \quad (4.12)$$

Using these Bianchi identity equations, the constraint equations (4.8) can be simplified to

$$\begin{aligned} & -2 \left(G' - \frac{G}{\beta} \beta' + \frac{G}{w} w' \right) \text{Tr}(\Phi D_i \Phi) D_i \Phi \\ & + \left(G' - \frac{\alpha^2}{w^2} w' + \frac{\beta^2}{w^2} w' \right) \text{Tr}(D_i \Phi)^2 \Phi = 0, \end{aligned} \quad (4.13a)$$

$$\left(\frac{\alpha}{w} \right)' \left(G - \frac{\alpha^2}{w} - \frac{\beta^2}{w} \right) [\Phi, D_i \Phi] = 0, \quad (4.13b)$$

$$\alpha \left(\frac{\alpha'}{\alpha} - \frac{\beta'}{\beta} + \frac{w'}{w} \right) \text{Tr}(\Phi D_i \Phi) D_i \Phi = 0. \quad (4.13c)$$

Non-trivial solutions require $D_i \Phi \neq 0$, and so solutions to the Eq. (4.13c) are $\alpha = 0$ or $\alpha \neq 0$, or to be precise $\beta = c_\beta \alpha w$, with c_β is a constant. From now on, we require $\beta \neq 0$, along with $D_i \Phi \neq 0$, for BPS monopoles and dyons throughout this article, and hence $c_\beta \neq 0$.

4.2 BPS monopoles: $\alpha = 0$

Let us first consider the case of $\alpha = 0$, or $E_i = 0$, which correspond to BPS monopoles case. In this case, the constraint equations (4.13b) and (4.13c) are trivially satisfied and the remaining constraint equation (4.13a) can be simplified to

$$\begin{aligned} & -2 \left(G' - \frac{G}{\beta} \beta' + \frac{G}{w} w' \right) \text{Tr}(\Phi D_i \Phi) D_i \Phi \\ & + \left(G' + \frac{\beta^2}{w^2} w' \right) \text{Tr}(D_i \Phi)^2 \Phi = 0. \end{aligned} \quad (4.14)$$

This constraint equation can be trivial if we set $\beta = c_\beta G w$ and $w = \frac{1}{c_\beta^2 G} + c_g$, with $c_\beta \neq 0$ and c_g are constants. Without losing generality, we can fix these constants by comparing to the results of $SU(2)$ Yang–Mills–Higgs model, where $G = w = 1$, and thus we must set $c_g = 1 - \frac{1}{c_\beta^2}$. So the Bogomolny's equations for BPS monopoles are⁴

⁴ Here the scalar potential V is not absolutely zero, but it is $SU(2)$ invariant in which the model (2.1) can be spontaneously broken to $U(1)$ gauge symmetry, see [28, 29] for more detail.

$$E_i = 0, \quad B_i = c_\beta G D_i \Phi, \quad D_0 \Phi = 0, \quad V = 0 \quad (4.15)$$

with $w = \frac{1}{c_\beta^2 G} + 1 - \frac{1}{c_\beta^2}$ and $c_\beta \neq 0$ is a real constant. Setting $c_\beta^2 = 1$, we get back the results of [1, 20] where $w = 1/G$. In general we shall call the Bogomolny's equations (4.15), with(out) additional constraint equation (4.14), as the generalized Bogomolny's equations for BPS monopoles whose solutions, with w or G are non-constants, shall be called generalized BPS monopoles.

4.3 BPS dyons: $\alpha \neq 0$

As we mentioned previously here $\beta = c_\beta \alpha w$. Later on, the constraint equation (4.13b) implies $\alpha = c_\alpha w$, with $c_\alpha \neq 0$ is a constant, or $Gw = \alpha^2 + \beta^2 = \alpha^2 (1 + c_\beta^2 w^2)$.

4.3.1 The case of $\alpha = c_\alpha w$

In this case, the constraint equation (4.13a) can be simplified to

$$\begin{aligned} & -2 \left(G' - \frac{G}{w} w' \right) \text{Tr}(\Phi D_i \Phi) D_i \Phi \\ & + \left(G' + c_\alpha^2 \left(c_\beta^2 w^2 - 1 \right) w' \right) \text{Tr}(D_i \Phi)^2 \Phi = 0 \end{aligned} \quad (4.16)$$

which is trivially satisfied if G and w is constants, and thus lead us to the standard BPS dyon solutions. Therefore the generalized BPS dyons may exist as solutions to the constraint equation (4.16) in addition to the Bogomolny equations

$$\begin{aligned} E_i &= c_\alpha D_i \Phi, \quad B_i = c_\alpha c_\beta w D_i \Phi, \\ D_0 \Phi &= 0, \quad V = 0. \end{aligned} \quad (4.17)$$

Here, in general, the scalar fields-dependent couplings G and w are independent to each other, but their relation can be determined from the above constraint equation (4.16), which depend on the choice of ansatz.

4.3.2 The case of $G = \frac{\alpha^2}{w} (1 + c_\beta^2 w^2)$

In this case, the constraint equation (4.13a) can be simplified to

$$\begin{aligned} & \left(\alpha \left(c_\beta^2 w^2 - 1 \right) w' + w \left(c_\beta^2 w^2 + 1 \right) \alpha' \right) \\ & \left(\text{Tr}(\Phi D_i \Phi) D_i \Phi - \text{Tr}(D_i \Phi)^2 \Phi \right) = 0 \end{aligned} \quad (4.18)$$

which is trivially satisfied if $\alpha = \frac{c_\alpha w}{1 + c_\beta^2 w^2}$, with $c_\alpha \neq 0$ is a constant. So we have two possible Bogomolny's equations:

- $\alpha = \frac{c_\alpha w}{1 + c_\beta^2 w^2}$

In this case, the generalized BPS dyon solutions can be obtained by solving the Bogomolny's equations

$$E_i = \frac{c_\alpha}{1 + c_\beta^2 w^2} D_i \Phi, \quad B_i = \frac{c_\alpha c_\beta w}{1 + c_\beta^2 w^2} D_i \Phi, \\ D_0 \Phi = 0, \quad V = 0, \quad (4.19)$$

without additional constraint equation.

$$\bullet \alpha \neq \frac{c_\alpha w}{1 + c_\beta^2 w^2}$$

In this case, the generalized BPS dyon solutions can be obtained by solving the Bogomolny's equations

$$E_i = \frac{\alpha}{w} D_i \Phi, \quad B_i = c_\beta \alpha D_i \Phi, \\ D_0 \Phi = 0, \quad V = 0, \quad (4.20)$$

with additional constraint equation

$$\text{Tr}(\Phi D_i \Phi) D_i \Phi = \text{Tr}(D_i \Phi)^2 \Phi. \quad (4.21)$$

This constraint equation may only be satisfied trivially by some particular ansatz, and may not be useful to fix some of the scalar fields-dependent couplings. Since the main objective of this article is to derive Bogomolny's equations that are valid by any ansatz, and thus we may neglect this type of constraint equations.

Similar to the case of the $SU(2)$ Yang–Mills–Higgs model, one can easily show that the Gauss's law constraint equations of the generalized model (2.1),

$$4w' \text{Tr}(\Phi D_i \Phi) E_i + w D_i E_i = ieG[\Phi, D_0 \Phi], \quad (4.22)$$

is satisfied trivially in the BPS limit for the case of BPS monopoles, by taking $\alpha = 0$, and the case of BPS dyons, by taking $\beta = c_\beta \alpha w$. Here we find a more general relation between the scalar fields-dependent couplings, as shown in the Sect. 4.3.2 where $Gw = \alpha^2 (1 + c_\beta^2 w^2)$, than the one derived in the spherically symmetric system [1], which is a particular case with α is constant.

Now we will show that there are no generalized BPS dyons for the Julia–Zee ansatz (2.4) in all of the cases described above. Let us assume general case of $\alpha \neq 0$. Using the ansatz (2.4), the Bogomolny's equation (4.3a) yields

$$wj + e\alpha f = 0, \quad w \frac{dj}{dr} + e\alpha \frac{df}{dr} = 0, \quad (4.23)$$

where α and w are functions of f only, whose solutions are $j = -e \frac{\alpha}{w} f$, with $\frac{\alpha}{w} \neq 0$ is a constant. Without losing generality, we therefore just need to consider the case of $\alpha = c_\alpha w$ as in the Sect. 4.3.1. On the other hand the Bogomolny's equation (4.3b) implies

$$\frac{da}{dr} = e c_\alpha c_\beta a f w, \quad \frac{df}{dr} = \frac{a^2 - 1}{e c_\alpha c_\beta r^2 w}. \quad (4.24)$$

Substituting those Bogomolny's equations into the constraint equation (4.16) implies

$$(a^2 - 1)^2 \left(\frac{\partial w}{\partial f} (c_\alpha^2 w (c_\beta^2 w^2 - 1) + 2G) - w \frac{\partial G}{\partial f} \right) \\ + 2a^2 c_\alpha^2 c_\beta^2 e^2 f^2 w^3 \left(c_\alpha^2 (c_\beta^2 w^2 - 1) \frac{\partial w}{\partial f} + \frac{\partial G}{\partial f} \right) r^2 = 0, \quad (4.25)$$

where G is a function of f only. This equation can be solved by considering it as a polynomial equation of explicit radial coordinate r and then setting all its “coefficients” to zero,

$$\frac{\partial w}{\partial f} (c_\alpha^2 w (c_\beta^2 w^2 - 1) + 2G) - w \frac{\partial G}{\partial f} = 0, \\ c_\alpha^2 (c_\beta^2 w^2 - 1) \frac{\partial w}{\partial f} + \frac{\partial G}{\partial f} = 0. \quad (4.26)$$

Solutions to these equations are given by w and G are constants, and so we will get back the Bogomolny's equations for standard BPS dyons, whose solutions have been studied in [2, 28, 29]. Here we may conclude that the Julia–Zee ansatz (2.4) is not suitable for finding the generalized BPS dyon solutions of the Bogomolny's equations (4.3). However there might be generalized BPS dyon solutions under different ansatz, such as axially symmetric ansatz [30], which is beyond the discussion of this article and will be discussed elsewhere.

4.4 Stress–energy–momentum density tensor

Although we do not have explicit generalized BPS dyon solutions, we may still learn some of their features from the stress–energy–momentum density tensor due to the existence of the generalized Bogomolny's equations for BPS dyons. The stress–energy–momentum density tensor of the generalized model (2.1) is defined as

$$T_{\mu\nu} = 2G \text{Tr}(D_\mu \Phi D_\nu \Phi) - 2w \text{Tr}(F_{\lambda\mu} F_\nu^\lambda) \\ - \eta_{\mu\nu} \mathcal{L}. \quad (4.27)$$

It is usually argued that the static soliton solutions are stable if their total energy $E = \int d^3x \sqrt{-g} T_0^0$, in the BPS limit, is proportional to the topological charge [10, 28, 29]. Here we would like suggest that stability of the static solution solutions can be seen from their stress density tensor. Physically, the stress density tensor is related to (internal) pressures and shear stress of the solutions. It is natural to expected that all stable solutions have vanishing (internal) pressures and shear stress. Not surprisingly, one can check that all stress density tensor components of many well-known (stable) BPS solutions are zero in the BPS limit. There are some examples where the BPS solutions, with some of their stress density tensor components are non-zero, are found to be unstable either because the total static energy is not proportional to the topological charge or the solution does not exist or unphysical [18, 31]. To meet this requirement, we compute the stress

density tensor components of the generalized model (2.1) that, in the BPS limit, is given by

$$T_{ij} = 2 \left(G - \frac{\alpha^2}{w} - \frac{\beta^2}{w} \right) \text{Tr} (D_i \Phi D_j \Phi) - \delta_{ij} \left(G - \frac{\alpha^2}{w} - \frac{\beta^2}{w} \right) \text{Tr} (D_k \Phi)^2. \quad (4.28)$$

Therefore the generalized BPS monopole and dyon solutions, if they exist, are stable when $Gw = \alpha^2 + \beta^2$. This additional relation between the scalar fields-dependent couplings will result in stable generalized BPS monopole and dyon solutions.

4.4.1 Stable generalized BPS monopoles

For the case of generalized BPS monopoles, the constraint equation (4.14) is simplified to

$$\beta' \left(\text{Tr} (\Phi D_i \Phi) D_i \Phi - \text{Tr} (D_i \Phi)^2 \Phi \right) = 0. \quad (4.29)$$

Assuming $\text{Tr} (\Phi D_i \Phi) D_i \Phi \neq \text{Tr} (D_i \Phi)^2 \Phi$ implies β is constant and the relation $Gw = 1$, where β^2 has been normalized to unity. The Bogomolny's equations are then given by

$$E_i = 0, \quad B_i = \pm G D_i \Phi, \\ D_0 \Phi = 0, \quad V = 0. \quad (4.30)$$

4.4.2 Stable generalized BPS dyons

For the case of generalized BPS dyons, the remaining constraint equation (4.13a) simply becomes the constraint equation (4.18). Again, by assuming $\text{Tr} (\Phi D_i \Phi) D_i \Phi \neq \text{Tr} (D_i \Phi)^2 \Phi$, we obtain $\alpha = \frac{c_\alpha w}{1 + c_\beta^2 w^2}$ and the (normalized) relation⁵

$$G = \frac{w}{\sin^2(\theta) + \cos^2(\theta) w^2}. \quad (4.31)$$

The Bogomolny's equations are then given by⁶

$$E_i = \sin(\theta) \frac{G}{w} D_i \Phi, \quad B_i = \cos(\theta) G D_i \Phi, \\ D_0 \Phi = 0, \quad V = 0, \quad (4.32)$$

with $\cos(\theta) \neq 0$ and $\sin(\theta) \neq 0$, where θ is a real constant.

⁵ We normalize the constants c_α and c_β by comparing with the results of $SU(2)$ Yang–Mills–Higgs model, where $G = w = 1$, and so they are related by $1 + c_\beta^2 = c_\alpha^2$, or to be precise we chose $c_\alpha = \csc(\theta)$ and $c_\beta = \cot(\theta)$.

⁶ By including $\sin(\theta) = 0$, the Bogomolny's equations (4.32), together with the (normalized) relation of the scalar fields-dependent couplings (4.31), are also valid for the stable generalized BPS monopoles where $\cos(\theta) = \pm 1$.

The static energy density, in the BPS limit, is given by

$$T_{00} = \left(G + \frac{\alpha^2}{w} + \frac{\beta^2}{w} \right) \text{Tr} (D_i \Phi)^2. \quad (4.33)$$

The (stable) generalized BPS monopole and dyon solutions have energy density

$$T_{00} = 2G \text{Tr} (D_i \Phi)^2. \quad (4.34)$$

4.5 Topological charge

The integer topological charge is defined as [29]

$$N_\Phi = \frac{1}{8\pi} \int_{|x| \rightarrow \infty} dS^i \epsilon^{ijk} \epsilon^{abc} \phi^a \partial_j \phi^b \partial_k \phi^c, \quad (4.35)$$

where $\phi \equiv \frac{\Phi}{\sqrt{|\Phi|}}$. Non-trivial topological charge requires $|\Phi| \rightarrow v > 0$ near the spatial infinity and thus typical potential for the scalar fields is $V = \frac{\lambda}{4} (|\Phi| - v)^2$, with $\lambda = 0$ in the BPS limit. Furthermore the finite energy configuration requires near the spatial infinity $D_i \Phi$ fall faster than $|x|^{-3/2}$ and in addition we also require the functions G and w to be finite everywhere. Under those requirements, the topological charge (4.35) can be rewritten as

$$N_\Phi = -\frac{e}{2\pi} \int_{|x| \rightarrow \infty} dS^i \text{Tr} (\phi B_i). \quad (4.36)$$

In the BPS limit, by substituting B_i from the Eq. (4.32), it becomes

$$N_\Phi = -\frac{e \cos(\theta)}{2\pi \sqrt{v}} \int_{|x| \rightarrow \infty} dS^i G \text{Tr} (\Phi D_i \Phi). \quad (4.37)$$

Now the total static energy of BPS dyon is given by

$$E_{BPS} = 2 \int d^3x G \text{Tr} (D_i \Phi)^2, \\ = 2 \int_{|x| \rightarrow \infty} dS^i G \text{Tr} (\Phi D_i \Phi) - 2 \int d^3x \text{Tr} \\ \times (\Phi (G D_i D_i \Phi + 4G' \text{Tr} (\Phi D_i \Phi) D_i \Phi)). \quad (4.38)$$

The first term in the second line is obtained using the Gauss's theorem. The second term in the second line is equal to zero by the Bianchi identity equation (4.11), and thus the total static energy of BPS dyon is proportional to the integer topological charge,

$$E_{BPS} = 4\pi \sqrt{v} \left| \frac{N_\Phi}{e \cos(\theta)} \right|. \quad (4.39)$$

Here, we show that solutions to the BPS dyon equations (4.32) are indeed stable.

5 Conclusions

We made use of the well-known Bogomolny's equations (3.5) in the $SU(2)$ Yang–Mills–Higgs model in order to get all

possible terms of the corresponding BPS Lagrangian density (3.8) that would lead to these Bogomolny's equations. We generalized this BPS Lagrangian density by multiplying each of its terms with arbitrary function of the scalar fields, and then used the generalized BPS Lagrangian density (4.1) to derive the generalized Bogomolny's equations in the generalized $SU(2)$ Yang–Mills–Higgs model (2.1). In the case of BPS monopoles, the generalized Bogomolny's equations are given by the equations (4.15) with(out) additional constraint equation (4.14). In the case of BPS dyons, there are two possible generalized Bogomolny's equations which are given by the equations (4.17) with(out) additional constraint equation (4.16), and by the equations (4.20) with(out) additional constraint equation (4.18). We then argued that the BPS monopole and dyon solutions to those generalized Bogomolny's equations are stable if all components of the stress density tensor are zero. This additional stability requirement yields the generalized Bogomolny's equations (4.32) and the equation (4.31) relating the scalar fields-dependent couplings for BPS monopoles, where $\sin(\theta) = 0$, and for BPS dyons without any constraint equation. Moreover we showed that the total energy of BPS dyon is proportional to the topological charge, $E_{BPS} \propto |N_\Phi|$.

It is easy to show that by substituting the 't Hooft–Polyakov ansatz, which is (2.4) with $j = 0$, into the generalized Bogomolny's equations (4.30) we will get back the self-dual (BPS) equations, and also the same equation relating the scalar fields-dependent couplings, whose solutions are the generalized BPS monopoles studied in [20]. Therefore the generalized Bogomolny's equations (4.30) are indeed the general coordinate extension of those self-dual (BPS) equations. Unfortunately, for the case of BPS dyons, substituting the Julia–Zee ansatz (2.4) into the generalized Bogomolny's equations (4.32) implies $G \propto w$ which then, from the equation (4.31), further imply $\cos(\theta) = 0$, or $B_i = 0$. There are possible BPS dyon solutions, with $\cos(\theta) \neq 0$, if both w and G are constants, or known as the standard BPS dyons, and therefore there are no generalized Bogomolny's equations for BPS dyons under the Julia–Zee ansatz. However this contradicts with the results in [1] where there exist (spherically symmetric) generalized Bogomolny's equations for BPS dyons under the Julia–Zee ansatz. We also found the relation equation (4.31) is different from the one obtained in [1]. These contradictions appear because the computations did in [1] is in the effective description under the Julia–Zee ansatz and the potential scalar A_0^a is directly identified with the scalar fields Φ^a , or namely by taking $j \propto f$, in the effective Lagrangian density (A.1). In this way, the corresponding effective Gauss's law constraint equation of the equation (4.22) disappears from the Euler–Lagrange equations of the effective Lagrangian density (A.1). Furthermore, there will be no corresponding effective constraint equation of the equation (4.13c). Therefore the computations, for BPS dyons, in

[1] are actually incomplete. In Appendix A, we repeated the (effective description) computations in [1], for BPS dyons in the generalized model (2.1) under the Julia–Zee ansatz (2.4), without first taking the identification $j \propto f$. We arrived at the same conclusion that there are no generalized Bogomolny's equations for BPS dyons and thus no generalized BPS dyon solutions.

We could consider more general BPS Lagrangian density than (4.1) that could lead to the generalized BPS dyons solutions under the Julia–Zee ansatz. The BPS Lagrangian density (4.1) is not the most general BPS Lagrangian density. There are other possible terms, in terms of E_i , B_i , $D_i\Phi$, and $D_0\Phi$, that one can add to the BPS Lagrangian density (4.1). The first one is a term that are independent to all first-derivative of the fields, or basically an arbitrary function of the scalar fields $|\Phi|$. The second one, a term that is proportional to $\text{Tr}(D_0\Phi)$. The third ones are the remaining terms that are proportional to quadratic of first-derivative of the fields: $\text{Tr}(E_i)^2$, $\text{Tr}(B_i)^2$, $\text{Tr}(D_0\Phi)^2$, and $\text{Tr}(E_i B_i)$. Another possible way to find the generalized BPS dyons is by considering different ansatze such as axially symmetric ansatz studied in [30] for higher topological charge dyons. However those possibilities are beyond the study of this article and they will be investigated elsewhere.

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Appendix A: Generalized BPS dyons in the effective description: spherically symmetric

In the effective description, we will apply the Julia–Zee ansatz (2.4) directly into the Lagrangian density of generalized $SU(2)$ Yang–Mills–Higgs model (2.1) such that the

effective Lagrangian density is spherically symmetric and is given by

$$\mathcal{L}_{eff} = -G(f) \left(\frac{f'^2}{2} + \frac{a^2 f^2}{r^2} \right) + \frac{w(f)}{e^2} \left(\frac{j'^2}{2} + \frac{a^2 j^2}{r^2} \right) - \frac{w(f)}{e^2} \left(\frac{a'^2}{r^2} + \frac{(a^2 - 1)^2}{2r^4} \right) - V(f), \quad (\text{A.1})$$

where from now on the apostrophe ' means taking derivative over radial coordinate r , or over its argument if explicitly written. Here we will consider two forms of effective BPS Lagrangian density. The first one is suggested by the form of generalized BPS Lagrangian density (4.1) and the second one is following the form in [1] which is linear to all first-derivative of the effective fields.

A.1: Effective BPS Lagrangian density motivated by the BPS Lagrangian density (4.1)

Under the Julia–Zee ansatz (2.4), each terms in the generalized BPS Lagrangian density (4.1) are proportional to the first-derivative of the effective fields as follow

$$\text{Tr}(E_i D_i \Phi) \propto f' j' \quad (\text{A.2a})$$

$$\text{Tr}(B_i D_i \Phi) \propto \frac{a'}{r^2}, \quad \text{and} \quad \frac{f'}{r^2} \quad (\text{A.2b})$$

$$\text{Tr}(D_i \Phi)^2 \propto f'^2. \quad (\text{A.2c})$$

This then suggests that we should take the following effective BPS Lagrangian density⁷

$$\mathcal{L}_{BPS} = -Q_f(f, a, j) \frac{f'}{r^2} - Q_a(f, a, j) \frac{a'}{r^2} - X_1(f, a, j) f' j' - X_2(f, a, j) f'^2, \quad (\text{A.3})$$

where Q_f , Q_a , X_1 , X_2 are arbitrary functions of the effective fields with no explicit radial coordinate dependent. Solving $\mathcal{L}_{eff} - \mathcal{L}_{BPS} = 0$ implies Bogomolny's equations: for f ,

$$f' = \frac{Q_f + r^2 X_1 j'}{r^2 (G - 2X_2)}; \quad (\text{A.4a})$$

for a ,

$$a' = \frac{e^2}{2} \frac{Q_a}{w}; \quad (\text{A.4b})$$

and for j ,

$$j' = -e^2 \frac{Q_f X_1}{r^2 (w (G - 2X_2) + e^2 X_1^2)}. \quad (\text{A.4c})$$

⁷ Since the effective Lagrangian density (A.1) is spherically symmetric, the effective BPS Lagrangian density should also be spherically symmetric as well.

The residual equation of $\mathcal{L}_{eff} - \mathcal{L}_{BPS} = 0$ is

$$\frac{w \left(e^2 Q_f^2 - (a^2 - 1)^2 (e^2 X_1^2 + w(G - 2X_2)) \right)}{2r^4 e^2 (e^2 X_1^2 + w(G - 2X_2))} + \frac{4a^2 w (j^2 w - e^2 f^2 G) + e^4 Q_a^2}{4r^2 e^2 w} = V \quad (\text{A.5})$$

which can be solved by setting all the “coefficients” in its explicit radial coordinate r expansion to be zero such that $V = 0$,

$$Q_a = \pm \frac{2}{e^2} a \sqrt{w (e^2 f^2 G - j^2 w)}, \quad (\text{A.6})$$

and

$$X_2 = \frac{e^2}{2} \left(\frac{(a^2 - 1)^2 X_1^2 - Q_f^2}{(a^2 - 1)^2 w} + \frac{G}{e^2} \right). \quad (\text{A.7})$$

Now, there are still two remaining functions (Q_f and X_1) need to be determined by using Euler–Lagrange equations of the BPS Lagrangian density (A.3) for f , a , and j ;

$$\partial_r \left(\frac{\partial(r^2 \mathcal{L}_{BPS})}{\partial_r \phi} \right) - \frac{\partial(r^2 \mathcal{L}_{BPS})}{\partial \phi} = 0, \quad (\text{A.8})$$

where $\phi \equiv (f, a, j)$ is the effective field. Substituting all Bogomolny's equations (A.4) and explicit solutions for V , Q_a , and X_2 into those Euler–Lagrange equations, that shall now be called as constraint equations, then we can solve them similarly by setting all the “coefficients” in their explicit radial coordinate r expansion to be zero. In this way, each of the constraint equations can be written into terms with explicit power of radial coordinate r : r^{-2} - and r^0 -terms. The function Q_f can determined from the r^0 -term of the constraint equation for f by equation

$$\begin{aligned} & a^2 e^2 f \frac{G}{w} (f w'(f) + 2w) \pm \frac{aG}{Q_f^2} \left((a^2 - 1) \frac{\partial Q_f}{\partial a} - 4a Q_f \right) \\ & \times (a^2 - 1) \sqrt{w (e^2 f^2 G - j^2 w)} \\ & + a^2 (e^2 f^2 G'(f) - 2j^2 w'(f)) = 0, \end{aligned} \quad (\text{A.9})$$

which has solution

$$Q_f = \pm \frac{2(a^2 - 1) G w \sqrt{w (e^2 f^2 G - j^2 w)}}{w (e^2 f^2 G'(f) - 2j^2 w'(f)) + e^2 f G (f w'(f) + 2w)}. \quad (\text{A.10})$$

Using these solutions for Q_f , the function X_1 can be determined from the r^{-2} -term of the constraint equation for a ,

$$\begin{aligned} X_1 = \frac{1}{2j} & \left(\frac{4Gw (e^2 f^2 G - j^2 w)}{w'(f) (e^2 f^2 G - 2j^2 w) + e^2 f w (f G'(f) + 2G)} \right. \\ & \left. + \frac{w'(f) (2j^2 w - e^2 f^2 G)}{e^2 w} - f (f G'(f) + 2G) \right), \end{aligned} \quad (\text{A.11})$$

and also from the r^{-2} -term of the constraint equation for j ,

$$X_1 = -\frac{2jGw^2}{w'(f)(e^2f^2G - 2j^2w) + e^2fw(fG' + 2G)}. \quad (\text{A.12})$$

Both solutions are equal if $w = cw^2$, and $G = cg^2$ or $G = \frac{cg^2}{f^4}$, where cw^2 and cg^2 are non-zero real constants. Substituting them then all the remaining constraint equations are trivially satisfied. So we have two possible solutions, with $w = cw^2$,

1. $G = cg^2$

Bogomolny's equations are given by

$$f'(r) = \pm \frac{(a(r)^2 - 1)f(r)}{r^2} \frac{cw^2}{\sqrt{cw^2cg^2e^2f(r)^2 - cw^4j(r)^2}}, \quad (\text{A.13a})$$

$$a'(r) = \pm \frac{a(r)}{cw^2} \sqrt{cw^2cg^2e^2f(r)^2 - cw^4j(r)^2}, \quad (\text{A.13b})$$

$$j'(r) = \pm \frac{(a(r)^2 - 1)j(r)}{r^2} \frac{cw^2}{\sqrt{cw^2cg^2e^2f(r)^2 - cw^4j(r)^2}}. \quad (\text{A.13c})$$

Comparing the Eqs. (A.13a) with (A.13c), we may take $j = \sigma f$, where σ is non-zero real constant.

2. $G = cg^2/f^4$

Bogomolny's equations are given by

$$f'(r) = \mp \frac{(a(r)^2 - 1)f(r)}{r^2} \frac{cw^2}{\sqrt{\frac{cw^2cg^2e^2}{f(r)^2} - cw^4j(r)^2}}, \quad (\text{A.14a})$$

$$a'(r) = \pm \frac{a(r)}{cw^2} \sqrt{\frac{cw^2cg^2e^2}{f(r)^2} - cw^4j(r)^2}, \quad (\text{A.14b})$$

$$j'(r) = \pm \frac{(a(r)^2 - 1)j(r)}{r^2} \frac{cw^2}{\sqrt{\frac{cw^2cg^2e^2}{f(r)^2} - cw^4j(r)^2}}. \quad (\text{A.14c})$$

Comparing the Eq. (A.14a) with (A.14c), we may take $j = \sigma/f$, where σ is also a non-zero real constant.

One can easily show that both Bogomolny's equations, (A.13) and (A.14), trivially satisfy the full Euler–Lagrange equations (2.3) under the Julian–Zee ansatz (2.4).

A.2: Linear effective BPS Lagrangian density

It is perhaps more suggestive to have effective Lagrangian density that is linear to the first-derivative of the effective

fields, $\int d^3x \sqrt{-g} \mathcal{L}_{BPS} = \int dQ(f, a, j)$, as such its Euler–Lagrange equations are trivial and thus no additional constraint equations needed. In the earlier development of the BPS Lagrangian method, some of the well-known Bogomolny's equations, for BPS vortices, were rederived from this linear effective BPS Lagrangian density [17]. Later on, it was shown that the Bogomolny's equations for BPS monopoles and dyons in the $SU(2)$ Yang–Mills–Higgs model also can be rederived from the linear effective BPS Lagrangian as well [1]. In this case the effective BPS Lagrangian density takes the following form:

$$\mathcal{L}_{BPS} = -\frac{Q_f(f, a, j)}{r^2} f' - \frac{Q_a(f, a, j)}{r^2} a' - \frac{Q_j(f, a, j)}{r^2} j', \quad (\text{A.15})$$

where Q_f , Q_a , Q_j are arbitrary functions. In order for the Euler–Lagrange equations of the effective BPS Lagrangian density to be trivial, the arbitrary functions must be related to each other as such $Q_f \equiv \frac{\partial Q}{\partial f}$, $Q_a \equiv \frac{\partial Q}{\partial a}$, and $Q_j \equiv \frac{\partial Q}{\partial j}$, with $Q(f, a, j)$. However we first set them to be independent and later we will find the function Q . Solving $\mathcal{L}_{eff} - \mathcal{L}_{BPS} = 0$ implies Bogomolny's equations: for f ,

$$f' = \frac{Q_f}{r^2 G}; \quad (\text{A.16a})$$

for a ,

$$a' = \frac{e^2}{2} \frac{Q_a}{w}; \quad (\text{A.16b})$$

and for j ,

$$j' = -e^2 \frac{Q_j}{r^2 w}. \quad (\text{A.16c})$$

The residual equation of $\mathcal{L}_{eff} - \mathcal{L}_{BPS} = 0$ is

$$\frac{(e^2 Q_f^2 w - e^4 Q_j^2 G - (a^2 - 1)^2 w^2 G)}{2r^4 e^2 G w} + \frac{4a^2 w (j^2 w - e^2 f^2 G) + e^4 Q_a^2}{4r^2 e^2 w} = V \quad (\text{A.17})$$

which can be solved by setting all the “coefficients” in its explicit radial coordinate r expansion to be zero such that $V = 0$,

$$Q_a = \pm \frac{2}{e^2} a \sqrt{w(e^2 f^2 G - j^2 w)}, \quad (\text{A.18})$$

and

$$e^2 Q_f^2 w - e^4 Q_j^2 G = (a^2 - 1)^2 w^2 G. \quad (\text{A.19})$$

To fix the functions Q_f and Q_j , we solve the Euler–Lagrange equations of the BPS Lagrangian density (A.15) for f , a , and j . Substituting all Bogomolny's equations (A.16) and explicit solutions of Q_a into those Euler–Lagrange equations, that shall be called constraint equations, then we can

solved them similarly by setting all the “coefficients” in their explicit radial coordinate r expansion to be zero. In this way, each of the constraint equations can be divided into terms with explicit power of radial coordinate r : r^{-4} , r^{-2} , and r^0 -terms. From the r^0 -term of the constraint equation for f , we have

$$\begin{aligned} e^2 \frac{\partial Q_f}{\partial a} \sqrt{w(e^2 f^2 G - j^2 w)} \\ = \pm a w \left(e^2 f^2 G'(f) - 2j^2 w'(f) \right) \\ \pm a e^2 f G (f w'(f) + 2w) \end{aligned} \quad (\text{A.20})$$

that gives general solutions to Q_f ,

$$\begin{aligned} Q_f = \pm (a^2 - c_f(f, j)) \\ \times \frac{(w'(f)(e^2 f^2 G - 2j^2 w) + e^2 f w (f G'(f) + 2G))}{2e^2 \sqrt{w(e^2 f^2 G - j^2 w)}}, \end{aligned} \quad (\text{A.21})$$

where c_f is an arbitrary function of f and j . Meanwhile, from the r^0 -term of the constraint equation for j , we have

$$2a^2 j w^2 \pm e^2 a \sqrt{w(e^2 f^2 G - j^2 w)} \frac{\partial Q_j}{\partial a} = 0, \quad (\text{A.22})$$

that gives general solutions to Q_j ,

$$Q_j = \mp \left(a^2 - c_j(f, j) \right) \frac{j w^2}{e^2 \sqrt{w(e^2 f^2 G - j^2 w)}}, \quad (\text{A.23})$$

where c_f is an arbitrary function of f and j . Now we have explicit a -dependent in all of the functions Q_f , Q_a , and Q_j . Therefore we can further expand all the constraint equations in the explicit a expansion and then solve them by setting all “coefficients” of explicit r coordinate and a expansions to zero. From the a^4 -term of the Eq. (A.19), we have

$$\begin{aligned} \left(w'(f) (e^2 f^2 G - 2j^2 w) + e^2 f w (f G'(f) + 2G) \right)^2 \\ - 4e^2 j^2 G w^3 = 4e^2 G w^2 (e^2 f^2 G - 2j^2 w). \end{aligned} \quad (\text{A.24})$$

Assuming f and j are independent each other, we solve it by rewriting it in the explicit j expansion and then setting all its “coefficients” to zero. Its j^4 -term implies $w = c_w^2$, where $c_w \neq 0$ is a real constant. Fixing the function w , its j^0 -term implies $G = c_g^2$ or $G = c_g^2/f^4$, where $c_g \neq 0$ is a real constant. Substituting these solutions for w and G into the a^2 -term of the Eq. (A.19), we have two possible solutions for c_f : if $G = c_g^2$ then

$$c_f = \frac{c_w^2 j^2 c_j(f, j) + c_g^2 e^2 f^2 - c_w^2 j^2}{c_g^2 e^2 f^2}, \quad (\text{A.25})$$

and if $G = c_g^2/f^4$ then

$$c_f = \frac{c_w^2 f^2 j^2 c_j(f, j) + c_g^2 e^2 - c_w^2 f^2 j^2}{c_g^2 e^2}. \quad (\text{A.26})$$

It turns out from the a^0 -term of the Eq. (A.19), with $w = c_w^2$,

$$\begin{aligned} \frac{e^2 f^2 (f G'(f) + 2G(f))^2 c_f(f, j)^2 - 4c_w^2 j^2 G(f) c_j(f, j)^2}{G(f) (e^2 f^2 G(f) - c_w^2 j^2)} \\ - 4 = 0, \end{aligned} \quad (\text{A.27})$$

both solutions of G , and also c_f , above yield the same $c_j = 1$ which surprisingly also leads to the same $c_f = 1$ for both. Now, we can compute the function Q which is given by $Q = \pm \frac{1}{e^2} (a^2 - 1) \sqrt{w(e^2 f^2 G - j^2 w)}$. At the end we will get the same Bogomolny’s equations (A.13) and (A.14) respectively for $G = c_g^2$ and $G = c_g^2/f^4$. As discussed previously at the end of Sect. 1, only the Bogomolny’s equations (A.13), with $c w^2 = c g^2 = 1$, satisfy the full Euler–Lagrange equations.

A.3: Solutions

1. $G = c g^2$

Solutions to the Bogomolny’s equations (A.13) are

$$f(r) = \pm \left(\frac{\omega^2}{r} - \coth \left(\frac{r}{\omega^2} \right) \right), \quad (\text{A.28a})$$

$$a(r) = \frac{r}{\omega^2} \frac{1}{\sinh \left(\frac{r}{\omega^2} \right)}, \quad (\text{A.28b})$$

$$j(r) = \sigma f(r), \quad (\text{A.28c})$$

with $\omega^2 = \frac{c w^2}{\sqrt{c w^2 c g^2 e^2 - c w^4 \sigma^2}}$ and $\sigma < \left| \frac{c g}{c w} e \right|$ is a real constant. Normalized $c w^2 = c g^2 = 1$ we get back the standard BPS dyon solutions [2, 28, 29].

2. $G = c g^2/f^4$

Solutions to the Bogomolny’s equations (A.14) are

$$f(r) = \pm \left(\frac{\omega^2}{r} - \coth \left(\frac{r}{\omega^2} \right) \right)^{-1}, \quad (\text{A.29a})$$

$$a(r) = \frac{r}{\omega^2} \frac{1}{\sinh \left(\frac{r}{\omega^2} \right)}, \quad (\text{A.29b})$$

$$j(r) = \frac{\sigma}{f(r)}, \quad (\text{A.29c})$$

with $\omega^2 = \frac{c w^2}{\sqrt{c w^2 c g^2 e^2 - c w^4 \sigma^2}}$ and $\sigma < \left| \frac{c g}{c w} e \right|$ is a real constant. These solutions imply only static energy density component of the stress–energy–momentum density tensor (4.27) is non-zero and is given by $E_{static} = 4\pi \omega^2$.

Although the total static energy is finite, unfortunately the solutions for $f(r)$ are singular near the origin as such $f(r \rightarrow 0) \rightarrow \mp\infty$, and so they are unphysical.⁸ Furthermore, the Bogomolny's equations (A.14) are not equivalent with the Bogomolny's equations (4.3), under the Julia–Zee ansatz, since $j \propto 1/f$. This also means that the general coordinate extension of the Bogomolny's equations (A.14) may come from a different BPS Lagrangian density than the generalized BPS Lagrangian density (4.1). Notice that the effective BPS Lagrangian density (A.3) is slightly more general than the generalized BPS Lagrangian density (4.1) which under the Julia–Zee ansatz is given by

$$\mathcal{L}_{BPS} \approx -2 \frac{\beta(f)}{e} \left(a^2 - 1 \right) \frac{f'}{r^2} - 4 \frac{\beta(f)}{e} a f \frac{a'}{r^2} - 2 \frac{\alpha(f)}{e} f' j' - \gamma(f) f'^2. \quad (\text{A.30})$$

Considering the most general BPS Lagrangian density would result in most general Bogomolny's equations, than the Bogomolny's equations (4.3), that may have physical generalized BPS dyon solutions under the Julia–Zee ansatz.

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⁸ Under the Julia–Zee ansatz (2.4), we expect that at the origin, $r = 0$, scalar fields Φ^a to be single-valued as such $f(0) = 0$.