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All analytic solutions for geodesic motion in axially symmetric space-times

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Abstract Recent observations of the orbits of star clusters around Sgr $A^\star$, imaging of black holes and gravitational waveforms of merging compact objects require a detailed understanding of the general relativistic geodesic motion. We came up with a method to provide all the possible geodesics in an axially symmetric space-time. The Kerr metric is explicitly worked out, recovering the Schwarzschild geodesics in the static limit. We also found the most general Killing tensor and its associated constant of motion for an axisymmetric space-time. The relevance of these results is crucial to understanding the different scenarios and the fundamental nature of the compact object at the galactic center.

1 Introduction

From the orbits of stars [1–3], and the imaging around supermassive black holes [4,5] through gravitational lensing [6], geodesic motions of particles and photons have a venerable history bringing results of General Relativity to observational grounds. Other important scenarios like frame dragging [7], radiation transport effects from accretion flows in the vicinity of relativistic stars [8], and gravitational waveform of merging compact objects [9], also require a detailed understanding of the geodesic motion in a general relativistic background.

Rotation is a crucial feature for celestial bodies from the an astrophysical viewpoint, and the Kerr solution of Einstein equations describes the gravitational field outside spinning compact objects and black holes [10]. Thus, geodesics across a Kerr gravitational background are very important and have a long history, since Carter's study of the existence of a new conserved quantity associated with each geodesic [11,12]. The astrophysical relevance of tracking particles & photons in Kerr space-times motivates a significant effort to obtain analytical and numerical trajectories in this gravitation background (see [13,14] and references therein).

We have recently implemented a tetrad formalism by an orthogonal splitting of the Riemann tensor, introducing a complete set of equations equivalent to the Einstein system and applying it to the spherical case [15–17]. This formalism provides coordinate-free results expressed in terms of structure scalars related to the kinematical and physical properties of the fluid.

We devise an exhaustive classification for all geodesic motion for any axisymmetric source establishing the equations which describe each alternative. Based on the tetrad scheme, we provide a method to integrate all possible orbits for any stationary axially symmetric solutions, illustrating each case with the Kerr metric.

We present the most general Killing tensor corresponding to any axisymmetric space-time and its associated constant of motion, which has not previously been obtained. This Killing tensor and its conserved quantity allow us to obtain the orbits by solving a system of algebraic equations. We recovered the famous Carter constant along the geodesic as a particular case.

2 The tetrad and kinematical variables

We shall consider stationary and axially symmetric sources with the line element written as

$$\begin{aligned} ds^2 &= -A^2 dt^2 + B^2 dr^2 + C^2 d\theta^2 \\ &\quad + R^2 d\phi^2 + 2\omega_3 dt\, d\phi\,, \end{aligned} \tag{1}$$

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with $A = A(r,\theta)$, $B = B(r,\theta)$, $R = R(r,\theta)$, and $\omega_3 = \omega_3(r,\theta)$.

In this case the tetrad is:

$$V^\alpha = \left(\frac{1}{A},0,0,0\right), K^\alpha = \left(0,\frac{1}{B},0,0\right),$$
$$L^\alpha = \left(0,0,\frac{1}{C},0\right), S^\alpha = \frac{1}{\sqrt{\Delta_2}}\left(\frac{\omega_3}{A},0,0,A\right)$$

where $\Delta_2 = A^2R^2 + \omega_3^2$.

2.1 The scalars and the tetrad covariant derivative

The covariant derivative of V_α in the 1+3 formalism can be written as $V_{\alpha;\beta} = -a_\alpha V_\beta + \Omega_{\alpha\beta}$, where the kinematical variables (a_α the acceleration and $\Omega_{\alpha\beta}$ the vorticity) can be written, in terms of the tetrad, as

$$a_\alpha = a_1 K_\alpha + a_2 L_\alpha\,, \tag{2}$$
$$\Omega_{\alpha\beta} = \Omega_2(K_\alpha S_\beta - K_\beta S_\alpha) + \Omega_3(L_\alpha S_\beta - L_\beta S_\alpha)\,. \tag{3}$$

Follows the covariant derivative of $\mathbf{K}$, i.e.

$$K_{\alpha;\beta} = -a_1 V_\alpha V_\beta + 2\Omega_2 V_{(\alpha}S_{\beta)} + (j_1 K_\beta + j_2 L_\beta)L_\alpha + j_6 S_\alpha S_\beta\,.$$

Now, the covariant derivative of $\mathbf{L}$, can be written as

$$L_{\alpha;\beta} = -a_2 V_\alpha V_\beta + 2\Omega_3 V_{(\alpha}S_{\beta)} - (j_1 K_\beta + j_2 L_\beta)K_\alpha + j_9 S_\alpha S_\beta\,.$$

Finally the covariant derivative of $\mathbf{S}$ is

$$S_{\alpha;\beta} = -2\Omega_2 V_{(\alpha}K_{\beta)} - 2\Omega_3 V_{(\alpha}L_{\beta)} - (j_6 K_\alpha + j_9 L_\alpha)S_\beta\,.$$

2.2 Scalars for a general axisymetric metric

Assuming $\omega_3 = A^2\psi$, the scalars for the axisymmetric metric (1) are:

$$a_1 = \frac{A_{,r}}{AB}\,,\quad a_2 = \frac{A_{,\theta}}{AC}\,,\quad j_1 = -\frac{B_{,\theta}}{BC}\,,\quad j_2 = \frac{C_{,r}}{BC}\,, \tag{4}$$
$$j_6 = \frac{(\ln(R^2 + A^2\psi^2))_{,r}}{2B}\,,\quad j_9 = \frac{(\ln(R^2 + A^2\psi^2))_{,\theta}}{2C}\,, \tag{5}$$
$$\Omega_2 = \frac{A\psi_{,r}}{2B\sqrt{R^2 + A^2\psi^2}} \quad \text{and} \quad \Omega_3 = \frac{A\psi_{,\theta}}{2C\sqrt{R^2 + A^2\psi^2}}\,. \tag{6}$$

3 All geodesic

To obtain all possible geodesics in any axially symmetric space-time, we define the tangent vector to the geodesic Z^α as

$$Z^\alpha \equiv \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} = z_0 V^\alpha + z_1 K^\alpha + z_2 L^\alpha + z_3 S^\alpha\,, \tag{7}$$

and, its norm, $\epsilon = Z_\alpha Z^\alpha = -z_0^2 + z_1^2 + z_2^2 + z_3^2$, represents photon ($\epsilon = 0$) and particle ($\epsilon = -1$) trajectories.

In what follows we shall make use of the geodesic equations $Z_{\alpha;\beta}Z^\beta = 0$, written in the tetrad formalism as

$$z_1 z_1^\dagger + z_2 z_1^\theta = j_1 z_1 z_2 + 2z_0 z_3 \Omega_2 - a_1 z_0^2 + j_2 z_2^2 + j_6 z_3^2 \tag{8}$$

and

$$z_1 z_2^\dagger + z_2 z_2^\theta = -j_2 z_1 z_2 + 2z_0 z_3 \Omega_3 - a_2 z_0^2 - j_1 z_1^2 + j_9 z_3^2\,. \tag{9}$$

Because the norm of the tangent vector Z^α is constant: $Z_\alpha Z^\alpha = \epsilon \Rightarrow \quad Z_{\alpha;\beta}Z^\alpha = 0$, and we get

$$z_1 z_1^\dagger + z_2 z_2^\dagger = j_6 z_3^2 + 2z_0 z_3 \Omega_2 - a_1 z_0^2 \quad \text{and} \tag{10}$$
$$z_1 z_1^\theta + z_2 z_2^\theta = j_9 z_3^2 + 2z_0 z_3 \Omega_3 - a_2 z_0^2\,; \tag{11}$$

where we have the derivatives $z^\dagger \equiv \frac{1}{B}z_{,r}$ and $z^\theta \equiv \frac{1}{C}z_{,\theta}$.

3.1 First order geodesic equations

Substracting (10) from (8) and (11) from (9) we have

$$z_2(z_2^\dagger - z_1^\theta + j_1 z_1 + j_2 z_2) = 0 \quad \text{and} \tag{12}$$
$$z_1(z_2^\dagger - z_1^\theta + j_1 z_1 + j_2 z_2) = 0\,. \tag{13}$$

This system describes all possible geodesic equations for any axially symmetric space-time, having Kerr and Schwarzschild metrics as particular cases.

3.2 Solutions for all geodesic cases

The four cases which solve the system (12)–(13):

1. $z_2 = 0$ and $z_1 = 0$, are circular orbits for $\theta = const$.
2. $z_2 = 0$ and $z_1^\theta = j_1 z_1$, are orbits for $\theta = const$.
3. $z_1 = 0$ and $z_2^\dagger = -j_2 z_2$, are spherical orbits.
4. $z_2^\dagger - z_1^\theta + j_1 z_1 + j_2 z_2 = 0$ is the most general case.

This is an exhaustive classification containing all possible cases for geodesic motions. Following subsections present the equations for each alternative and an application to the Kerr space-time.

3.3 Circular orbit on a plane

In the first case the system (12)–(13) is solved by $z_2 = 0$ and $z_1 = 0$, i.e the radial and the angular coordinate, θ, are constant. Thus, we obtain bounded orbits confined to a plane and the set of equations becomes,

$$j_6 z_3^2 + 2z_0 z_3 \Omega_2 - a_1 z_0^2 = 0\,, \tag{14}$$
$$j_9 z_3^2 + 2z_0 z_3 \Omega_3 - a_2 z_0^2 = 0\,, \tag{15}$$
$$z_1 = \frac{f_1'}{B} = 0\,, \quad \text{and} \quad z_2 = \frac{f_{2,\theta}}{C} = 0\,. \tag{16}$$

3.4 General orbital motion on a constant plane

The second solution for Eqs. (12)–(13), emerges from the conditions $z_2 = 0$ and $z_1^\theta = j_1 z_1$. Then, the geodesic equations reduce to

$$j_6 z_3^2 + 2z_0 z_3 \Omega_2 - a_1 z_0^2 - z_1 z_1^\dagger = 0\,, \tag{17}$$
$$j_9 z_3^2 + 2z_0 z_3 \Omega_3 - a_2 z_0^2 - j_1 z_1^2 = 0\,, \tag{18}$$
$$z_1 = \frac{f_1'}{B} \neq 0 \quad \text{and} \quad z_2 = \frac{f_{2,\theta}}{C} = 0\,. \tag{19}$$

3.5 General orbits on the two-sphere

The third set of solutions we shall consider are general orbits circumscribed on a 2-sphere ($\theta - \phi$), recently reported in reference [13]. This occurs having $z_1 = 0$ and $z_2^\dagger = -j_2 z_2$, and the corresponding geodesic equations are

$$j_6 z_3^2 + 2z_0 z_3 \Omega_2 - a_1 z_0^2 + j_2 z_2^2 = 0\,, \tag{20}$$
$$j_9 z_3^2 + 2z_0 z_3 \Omega_3 - a_2 z_0^2 - z_2 z_2^\theta = 0\,, \tag{21}$$
$$z_1 = \frac{f_1'}{B} = 0 \quad \text{and} \quad z_2 = \frac{f_{2,\theta}}{C} \neq 0\,. \tag{22}$$

3.6 The general case

The last case emerges from $z_2^\dagger - z_1^\theta + j_1 z_1 + j_2 z_2 = 0$. This is the most general case, and the system of geodesic equations becomes

$$j_6 z_3^2 + 2z_0 z_3 \Omega_2 - a_1 z_0^2 + j_2 z_2^2 - z_1 z_1^\dagger = 0\,, \tag{23}$$
$$\text{and} \quad j_9 z_3^2 + 2z_0 z_3 \Omega_3 - a_2 z_0^2 - j_1 z_1^2 - z_2 z_2^\theta = 0\,; \tag{24}$$
$$\text{with} \quad z_1 = \frac{f_1'}{B} \neq 0 \quad \text{and} \quad z_2 = \frac{f_{2,\theta}}{C} \neq 0\,. \tag{25}$$

The solution of the equation is

$$z_2^\dagger - z_1^\theta + j_1 z_1 + j_2 z_2 = 0 \Rightarrow z_1 = f^\dagger \text{ and } z_2 = f^\theta \tag{26}$$

with $f = f(r, \theta)$ an arbitrary function of its arguments. Consequently, $z_1^\theta = j_1 z_1$ and $z_2^\dagger = -j_2 z_2$, which in turn allows us to transform (8) and (9) into

$$z_1 z_1^\dagger = 2z_0 z_3 \Omega_2 - a_1 z_0^2 + j_2 z_2^2 + j_6 z_3^2 \tag{27}$$

and

$$z_2 z_2^\theta = 2z_0 z_3 \Omega_3 - a_2 z_0^2 - j_1 z_1^2 + j_9 z_3^2\,. \tag{28}$$

4 Symmetry and geodesic equations

In this section we shall discuss the consequences on imposing symmetries, i.e. Killing vectors and tensors, on the source generating the geodesic equations.

4.1 Killing vectors

From the Killing equation

$$\mathfrak{L}_X g_{\alpha\beta} = g_{\delta\alpha} X^\delta_{,\beta} + g_{\beta\delta} X^\delta_{,\alpha} + g_{\alpha\beta,\delta} X^\delta\,,$$

we can identify temporal and axial Killing vectors as

$$\tau^\alpha = \tau_0 V^\alpha \quad \Rightarrow \tau_0 = A \quad \text{and} \tag{29}$$
$$\xi^\alpha = \xi_0 V^\alpha + \xi_3 S^\alpha \Leftrightarrow$$
$$\xi_0 = -\frac{\omega_3}{A} \text{ and } \xi_3 = \frac{\sqrt{\Delta_2}}{A}\,. \tag{30}$$

These Killing vectors provide the energy $E = \tau^\alpha Z_\alpha$ and angular momentum $l = \xi^\alpha Z_\alpha$.

Thus, $z_{0;\alpha} = -z_0 a_1 K_\alpha - a_2 z_0 L_\alpha \quad \Rightarrow z_0 = -\frac{E}{A}$, and

$$z_{3;\alpha} = -(2z_0\Omega_2 + j_6 z_3) K_\alpha - (2z_0 \Omega_3 + j_9 z_3) L_\alpha\,, \tag{31}$$

with $z_3 = \frac{A^2 l + E\omega_3}{A\sqrt{\Delta_2}}$.

Now the set of equations for the parallel transport of the vector Z^α, can be written as

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{z_0}{A} + \frac{z_3\omega_3}{A\sqrt{\Delta_2}}\,, \quad \frac{\mathrm{d}r}{\mathrm{d}\lambda} = \frac{z_1}{B}\,, \tag{32}$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = \frac{z_2}{C} \quad \text{and} \quad \frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = \frac{z_3 A}{\sqrt{\Delta_2}}\,. \tag{33}$$

The above equations lead to the following characteristic expressions:

$$\frac{B\mathrm{d}r}{z_1} = \frac{C\mathrm{d}\theta}{z_2} = \frac{\Delta_2 \mathrm{d}\phi}{A^2 l + \omega_3 E}\,. \tag{34}$$

4.2 Killing tensor

Killing tensors are useful because they also provide conserved quantities for geodesic motion. The most famous is obtained for the Kerr space-time where the Killing tensor leads to the Carter constant [12].

For the stationary axially symmetric space-time, the Killing tensor $\xi_{\alpha\beta}$ satisfies $\xi_{\alpha\beta;\mu} + \xi_{\mu\alpha;\beta} + \xi_{\beta\mu;\alpha} = 0$, which can be written as

$$\begin{aligned}\xi_{\alpha\beta} &= \xi_{00} V_\alpha V_\beta + \xi_{11} K_\alpha K_\beta + \xi_{22} L_\alpha L_\beta \\ &+ \xi_{33} S_\alpha S_\beta + \xi_{03}(V_\alpha S_\beta + V_\beta S_\alpha)\end{aligned} \tag{35}$$

Integrating the Killing tensor equation we obtain

$$\xi_{00} = \xi(r) + \frac{\omega_3^2}{A^2}\,,\ \ \xi_{11} = C^2 + \xi(r)\,,\ \ \xi_{22} = \xi(r)\,, \tag{36}$$

$$\xi_{33} = \xi(r) + \frac{\Phi^2}{A^2} \quad \text{and} \quad \xi_{03} = \frac{\Phi\omega_3}{A^2}\,. \tag{37}$$

We have $\Phi = (r^2 - 2mr + a^2)\sin\theta$ and $\xi(r)$ an r-function.

Thus, the general conserved quantity Q for $\xi_{\alpha\beta}$ is

$$Q = \xi_{\alpha\beta} Z^\alpha Z^\beta \quad \Rightarrow \quad Q_{;\mu} Z^\mu = 0\,. \tag{38}$$

Since Z has a constant modulus we found

$$z_1^2 + z_2^2 = \epsilon + z_0^2 - z_3^2 \quad \text{and} \tag{39}$$

$$\xi_{11} z_1^2 + \xi_{22} z_2^2 = Q + 2\xi_{03} z_0 z_3 - \xi_{00} z_0^2 - \xi_{33} z_3^2\,, \tag{40}$$

obtaining that the scalars z_1 and z_2 are

$$z_1 = \frac{\sqrt{g_1(r,\theta)}}{C} \quad \text{and} \quad z_2 = \frac{\sqrt{g_2(r,\theta)}}{C}\,, \tag{41}$$

with

$$g_1(r,\theta) = Q - \xi_{22}\epsilon + 2\xi_{03} z_0 z_3 - (\xi_{00} + \xi_{22}) z_0^2 + (\xi_{22} - \xi_{33}) z_3^2, \tag{42}$$

and

$$g_2(r,\theta) = \xi_{11}\epsilon - Q - 2\xi_{03} z_0 z_3 + (\xi_{00} + \xi_{11}) z_0^2 + (\xi_{33} - \xi_{11}) z_3^2. \tag{43}$$

The new conserved quantity Q recovers the Carter constant Q_c for the Kerr metric [11,12]

The following sections implement all the previous cases to the particular example of the Kerr space-times.

5 The Kerr metric

To illustrate the different cases mentioned above, we consider $\Lambda = r^2 + a^2\cos^2\theta$ in the Kerr metric, i.e.

$$\mathrm{d}s^2 = -\left(1 - \frac{2mr}{\Lambda}\right)\mathrm{d}t^2 - \frac{4mar\sin^2\theta}{\Lambda}\mathrm{d}t\mathrm{d}\phi + \frac{\Lambda}{r^2 - 2mr + a^2}\mathrm{d}r^2 + \Lambda\mathrm{d}\theta^2 + \sin^2\theta\left(r^2 + a^2 + \frac{2ma^2 r\sin^2\theta}{\Lambda}\right)\mathrm{d}\phi^2. \tag{44}$$

5.1 Kerr killing tensor and geodesic motions

It is easy to verify that, assuming the metric (44), the general solution of the system (12)–(13), for $z_1 \neq 0$ and $z_2 \neq 0$ is

$$z_1^\theta = j_1 z_1 \quad \text{and} \quad z_2^\dagger = -j_2 z_2\,. \tag{45}$$

Thus, the separation constant method devised by Carter is equivalent to solve the geodesic equations (27) and (28), for the Kerr metric where Eqs. (42) and (43) become $g_1(r,\theta) = g_1(r)$ and $g_2(r,\theta) = g_2(\theta)$, i.e.

$$g_1(r) = Q_c + r^2\epsilon + \frac{E^2(r^4 + (2mr + r^2)a^2) + 4marEl + a^2 l^2}{r^2 - 2mr + a^2}$$

and $g_2(\theta) = -Q_c - \frac{l^2}{\sin^2\theta} + a^2(\epsilon + E^2)\cos^2\theta$, where Q_c is the Carter constant.

5.2 Kerr circular orbit on a plane

Considering the set of Eqs. (14)–(16) for the metric (44) with $\theta = \frac{\pi}{2}$, $Q_c = l^2$ we get:

$$(\epsilon + E^2)r^3 - 2m\epsilon r^2 + (a^2(\epsilon + E^2) - l^2)r + 2m(Ea + l)^2 = 0 \tag{46}$$

and

$$(4\epsilon + 5E^2)mr^3 - (6m^2(2\epsilon + E^2) - a^2(\epsilon + E^2) + l^2)r^2 + m(a^2(\epsilon + 4E^2) + 6aEl + 8\epsilon m^2 + 2l^2)r - 2m^2a^2(\epsilon - 2E^2) - 4aElm^2 = 0\,. \tag{47}$$

Equations (46) and (47) determine the radius of the circumference and related the physical constants. For instance, when we have the specific case $\epsilon = a = 0$ we obtain that $r = 3m$ and $l = \sqrt{27}Em$.

5.3 Kerr general orbital motion on a constant plane

Equations (17)–(19) with $\theta = \frac{\pi}{2}$, and $Q = l^2$ lead to

$$z_1^2 = \frac{(\epsilon + E^2)r^3 - 2m\epsilon r^2 + (a^2(\epsilon + E^2) - l^2)r + 2m(aE + l)^2}{r^3 - 2mr^2 + a^2 r}. \tag{48}$$

For a null geodesics and $a = -\frac{l}{E}$ we integrate (34) as

$$r = m + \sqrt{a^2 - m^2}\tan\left(\frac{\sqrt{a^2 - m^2}}{a}(\phi_0 - \phi)\right) \tag{49}$$

where ϕ_0 is a constant of integration.

Now for a time-like geodesic we have

$$\frac{x_1(r) + x_2(r)}{1 - x_1(r)x_2(r)} = \tan(\beta_0(\phi - \phi_0))\,, \tag{50}$$

with

$$x_1(r) = \frac{\beta_1 + \beta_2 r}{\beta_3\sqrt{(E^2 - 1)r^2 + 2mr - a^2}} \tag{51}$$

and $$x_2(r) = \frac{\beta_4 + \beta_5 r}{\beta_6\sqrt{(E^2 - 1)r^2 + 2mr - a^2}}\,; \tag{52}$$

where β_k $(k = 1, 2, 3, 4, 5, 6)$ are functions of the physical parameters.

5.4 Kerr general orbits on the two-sphere

In this case, from Eq. (22) we get

$$(\epsilon + E^2)r^3 \ -2m\epsilon r^2 + (a^2(\epsilon + E^2) - l^2)r + 2m(aE + l)^2 = 0 \quad (53)$$

and

$$f_{2,\theta} = \frac{\cos\theta\sqrt{\tilde{a}\sin^2\theta - l^2}}{\sin\theta}\,; \quad (54)$$

where we have redefined $\tilde{a} = a^2(\epsilon + E^2)$.

Next, substituting (54) into (34) we obtain

$$\frac{(F^2 l - \tilde{E})}{F^2\sqrt{\tilde{a} - l^2}}\arctan\left[\sqrt{\frac{\tilde{a}\sin^2\theta - l^2}{\tilde{a} - l^2}}\right] + \arctan\left[\frac{\sqrt{\tilde{a}\sin^2\theta - l^2}}{l}\right] + \phi_0 - \phi = 0\,, \quad (55)$$

with $\tilde{E} = 2marlE + a^2 l$ and $F^2 = r^2 - 2mr + a^2$.

5.5 The Kerr general case

The module of Z for the Kerr metric can be written as

$$(z_1^2 + z_2^2)C^2 = (\epsilon + E^2)a^2\cos^2\theta - \frac{l^2}{\sin^2\theta} + \epsilon r^2 + \frac{E^2(r^4 + (r^2 + 2mr)a^2) + 4marlE + a^2 l^2}{r^2 - 2mr + a^2}\,. \quad (56)$$

Substituting (26) into (56) we find

$$(r^2 - 2mr + a^2)(f')^2 + (f_{,\theta})^2 = (\epsilon + E^2)a^2\cos^2\theta - \frac{l^2}{\sin^2\theta} + \epsilon r^2 + \frac{E^2(r^4 + (r^2 + 2mr)a^2) + 4marlE + a^2 l^2}{r^2 - 2mr + a^2}\,, \quad (57)$$

which is an equation for the function f with a solution $f = f_1(r) + f_2(\theta)$ where

$$(f_1')^2 = \frac{-Q + r^2\epsilon}{r^2 - 2mr + a^2} + \frac{E^2(r^4 + (2mr + r^2)a^2) + 4marEl + a^2 l^2}{(r^2 - 2mr + a^2)^2}$$

$$\text{and } f_{2,\theta}^2 = Q - \frac{l^2}{\sin^2\theta} + a^2(\epsilon + E^2)\cos^2\theta\,. \quad (58)$$

Next, substituting (25) into (34) and considering $z_1^\theta = j_1 z_1$ and $z_2^\dagger = -j_2 z_2$, we obtain

$$\frac{\mathrm{d}\theta}{f_{2,\theta}} = \frac{\mathrm{d}r}{f_1'(r^2 - 2mr + a^2)} \quad \text{and}$$

$$\mathrm{d}\phi = \left(\frac{(F^2 - a^2\sin^2\theta)l - 2marE\sin^2\theta}{F^4 f_1'\sin^2\theta}\right)\mathrm{d}r\,, \quad (59)$$

where $F^2 = r^2 + a^2 - 2mr$.

Now, combining Eq. (59) we get

$$\frac{l\,\mathrm{d}\theta}{\sin^2\theta f_{2,\theta}} - \frac{(a^2 l + 2maEr)\mathrm{d}r}{F^4 f_1'} = \mathrm{d}\phi\,, \quad (60)$$

and by introducing

$$P(r) = a_0 r^4 + a_1 r^3 + a_2 r^2 + a_3 r + a_4\,, \quad (61)$$

we get

$$f_{2,\theta} = -\frac{\sqrt{P(\cos\theta)}}{\sin\theta} \quad \text{and} \quad f_1' = \frac{\sqrt{P(r)}}{r^2 - 2mr + a^2}\,. \quad (62)$$

Next, substituting $r = x + b$ into (61) we get

$$P(x) = (\sqrt{a_0}x + b_1)^2(x + b_2)(x + b_3)\,; \quad (63)$$

with

$$a_4 + a_3 b + a_2 b^2 + a_1 b^3 + a_0 b^4 = b_1^2 b_2 b_3\,, \quad (64)$$

$$a_3 + 2a_2 b + 3a_1 b^2 + 4a_0 b^3 = b_1^2(b_2 + b_3) + 2\sqrt{a_0}b_1 b_2 b_3, \quad (65)$$

$$a_2 + 3a_1 b + 6a_0 b^2 = (b_1 + \sqrt{a_0}b_2)(b_1 + \sqrt{a_0}b_3) \quad (66)$$

$$\text{and } a_1 + 4a_0 b = 2\sqrt{a_0}b_1 + a_0(b_2 + b_3)\,. \quad (67)$$

Consequently first integrals in r and θ of the Eq. (60) can be obtained as

$$\int \frac{(\kappa_1 x + \kappa_2)dx}{(x^2 + s_1 x + s_2)\sqrt{(\sqrt{a_0}x + b_1)^2(x + b_2)(x + b_3)}} = \alpha_1\arctan\gamma_1\Gamma(x) + \alpha_2\arctan\gamma_2\Gamma(x) + \alpha_3\arctan\gamma_3\Gamma(x)\,; \quad (68)$$

where $\Gamma(x) = \sqrt{\frac{b_2 + x}{b_3 + x}}$, where γ_i and α_j, with $i = 1, 2, 3$ and $j = 0, 1, 2, 3$.

Now, implementing the procedure described above for the polynomial (63) we get

$$P(r) = (\epsilon + E^2)r^4 - 2m\epsilon r^3 + (a^2(\epsilon + E^2) - l^2)r^2 + 2m(aE + l)^2 r\,, \quad (69)$$

and find $b = 0$, $b_3 = 0$, $a_0 = \epsilon + E^2$, $b_1^2 + \frac{2m\epsilon}{\sqrt{\epsilon + E^2}}b_1 + a^2(\epsilon + E^2) - l^2 = 0$ and $b_2 - \frac{2m(aE + l)^2}{b_1^2} = 0$.

Next, integrating the Eq. (60) we get

$$\arctan\left[\frac{\sqrt{\tilde{a}\sin^2\theta - l^2}}{\tilde{a} - l^2}\right] - \sqrt{\frac{\tilde{a} - l^2}{b_1(a_0 b_2 - b_1)}}\arctan\left[\sqrt{\frac{a_0 b_2 - b_1}{b_1}}\sqrt{\frac{r}{r + b_2}}\right] = C_1 \quad (70)$$

and

$$\arctan\left[\frac{\sqrt{\tilde{a}\sin^2\theta - l^2}}{l}\right] + \frac{l}{\sqrt{\tilde{a}-l^2}}\arctan\left[\sqrt{\frac{\tilde{a}\sin^2\theta - l^2}{l^2-\tilde{a}}}\right] - 2\alpha_1\arctan\left[\gamma_1\Xi\right] - \alpha_2\arctan\left[\gamma_2\Xi\right] - \alpha_3\arctan\left[\gamma_3\Xi\right] = C_2 \tag{71}$$

with $\Xi = \sqrt{\frac{r}{b_2+r}}$, where $G = G(C_1, C_2)$ becomes its general solution.

6 Conclusions

This work presents a method to classify and solve all geodesic motion analytically around any stationary axially symmetric source. The method summarises all these possible geodesic trajectories into two simple Eqs. (12) and (13), with ease in obtaining solutions. These distinct solutions allow us to classify the different trajectories for particles and photons. All the possible geodesics have been implemented for the Kerr metric, representing the gravitational field produced by a rotating compact object. In particular, those orbiting on a two-sphere surface could be especially relevant for describing observational data (see [3,18,19] and references therein). Now, it will be possible to build templates from exact General Relativistic analytical solutions, i.e. without any approximations for the orbits of stars [1–3], and the imaging of black holes [4,5,20]. The method shown here allows us to find solutions of the form $f_1(r,\theta) = C_1$ and $f_2(r,\theta) = C_2$ (equations (70) and (71)) which complement the standard elliptic integrals procedure handling in writing down the geodesics trajectories (see for example references [21–23]). It is clear that in the equations mentioned above (i.e.(70) and (71)), the elliptic integrals can be avoided only for very particular choices of α_i.

We found the most general for this Killing tensor corresponding to any axisymmetric space-time and its linked constant of motion. The existence this general constant of motion –along the geodesic– is clear from a simple system of algebraic equations (39) and (40). Again, the general expression for the constant along the geodesic could help to obtain solutions for the geodesic in a more general context where the Kerr metric may not adequately describe the gravitational field (see [24] and references therein). This new conserved quantity recovers the Carter constant for the Kerr metric ([11,12]).

Although we have considered Kerr space-time a helpful example, the equations for each case in our classification are general and valid for any axisymmetric metric. Analytic solutions for geodesic with more complex Kerr-like sources describing richer rotational compact objects could fit better the trajectories of the stars or represent more accurate black hole imaging or open the possibility of new information from gravitational wave astronomy.

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