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Letter

Conformally-rescaled Schwarzschild metrics do not predict flat galaxy rotation curves

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Abstract For conformally invariant gravity theories defined on Riemannian spacetime and having the Schwarzschild–de-Sitter (SdS) metric as a solution in the Einstein gauge, we consider whether one may conformally rescale this solution to obtain flat rotation curves, such as those observed in galaxies, without the need for dark matter. Contrary to recent claims in the literature, we show that if one works in terms of quantities that can be physically measured, then in any conformal frame the trajectories followed by 'ordinary' matter particles are merely the timelike geodesics of the SdS metric, as one might expect. This resolves the apparent frame dependence of physical predictions and unambiguously yields rotation curves with no flat region. We also show that attempts to model rising rotation curves by fitting the coefficient of the quadratic term in the SdS metric individually for each galaxy are precluded, since this coefficient is most naturally interpreted as proportional to a global cosmological constant. We further extend our analysis beyond static, spherically-symmetric systems to show that the invariance of particle dynamics to the choice of conformal frame holds for arbitary metrics, again as expected. Moreover, we show that this conclusion remains valid for conformally invariant gravity theories defined on more general Weyl–Cartan spacetimes, which include Weyl, Riemann–Cartan and Riemannian spacetimes as special cases.

The modelling of galaxy rotation curves in general relativity (GR) typically requires the inclusion of a dark matter halo in order to reproduce observations [1–3]. The family of rotation curves fitted to observations is, in fact, quite varied [4], but particular focus has historically been placed on modelling the approximately flat rotation curves observed in the outskirts of large spiral galaxies and, to a lesser extent, the rising rotation curves observed in smaller dwarf galaxies [5–9]. The absence of any direct experimental evidence for dark matter [10], however, has led to the consideration of various modified gravity theories, which may not require a dark matter component to explain the astrophysical data.

In its simplest form, the modelling of rotation curves in the outskirts of galaxies may be performed merely by considering the motion of stars in the region exterior to a spherically-symmetric representation of the galactic matter distribution. In any metric-based gravity theory, this simplified approach therefore considers the motion of massive test particles in a spacetime with line element that can be written in the form[1]

$$ds^2 = A(r)\,dt^2 - \frac{dr^2}{B(r)} - r^2(d\theta^2 + \sin^2\theta\,d\phi^2), \tag{1}$$

for given functions $A(r)$ and $B(r)$.

In GR with a cosmological constant Λ, the relevant line-element is Schwarzschild–de-Sitter (SdS), for which

$$A(r) = B(r) = 1 - \frac{2GM}{r} - kr^2, \tag{2}$$

where M is the galactic mass interior to the test particle orbit and $k = \frac{1}{3}\Lambda$, which is therefore a global constant unrelated to the galaxy under consideration. Particle rest masses in GR can be defined kinematically, so that massive (test) particles merely follow timelike geodesics of the SdS metric. In this case, for a circular orbit of coordinate radius r (in the equatorial plane $\theta = \pi/2$), the velocity v of the test particle (as measured by a stationary observer at that radius) satisfies

$$v^2 = \frac{r}{2B}\frac{dB}{dr} = \frac{GMr^{-1} - kr^2}{1 - 2GMr^{-1} - kr^2}. \tag{3}$$

[1] We adopt the following sign conventions: $(+,-,-,-)$ metric signature, $R^{\rho}{}_{\sigma\mu\nu} = 2(\partial_{[\mu}\left\{{\rho \atop \nu]\sigma}\right\} + \left\{{\rho \atop \lambda[\mu}\right\}\left\{{\lambda \atop \nu]\sigma}\right\})$, where the metric (Christoffel) connection $\left\{{\rho \atop \lambda\mu}\right\} = \frac{1}{2}g^{\rho\sigma}(\partial_\lambda g_{\mu\sigma} + \partial_\mu g_{\lambda\sigma} - \partial_\sigma g_{\lambda\mu})$, and $R^{\rho}{}_{\mu} = R^{\rho\sigma}{}_{\mu\sigma}$. We also employ natural units $c = \hbar = 1$ throughout, unless otherwise stated.

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In the weak-field limit appropriate for considering a galaxy rotation curve one has $B \approx 1$, so the two terms in the numerator determine its shape [11,12]. The first term recovers the standard Keplerian rotation curve $v^2 = GM/r$ and the second term contributes $v^2 = -kr^2 = -\frac{1}{3}\Lambda r^2$, so that for a typical galaxy with $M \sim 10^{11}$ $M_\odot$ and assuming $\Lambda \sim 10^{-52}$ m^{-2}, which is consistent with cosmological observations, one obtains a rotation curve that falls for all values of r until bound circular orbits are eliminated beyond the watershed radius $r = (3GM/\Lambda c^2)^{1/3} \sim 0.5$ Mpc [13]. Thus, the rotation curve has no flat region.

If one instead considers a conformally invariant gravity theory defined on Riemannian spacetime, then particle rest masses cannot be fundamental, but must arise dynamically [14]. This may be achieved by introducing a Dirac field with Weyl weight $w = -3/2$ to represent 'ordinary' matter, which is (Yukawa) coupled to a compensator scalar field φ with $w = -1$ by making the replacement $m\bar{\psi}\psi \to \mathscr{m}\varphi\bar{\psi}\psi$ in the Dirac action, where $\mathscr{m}$ is a dimensionless parameter but $\mathscr{m}\varphi$ has the dimensions of mass in natural units. Indeed, the full matter action in such theories is usually taken to have the form [14–17]

$$S_M = \int d^4x\,\sqrt{-g}\,[\tfrac{1}{2}i\bar{\psi}\gamma^\rho \overleftrightarrow{D}_\rho\psi - \mathscr{m}\varphi\bar{\psi}\psi + \tfrac{1}{2}(\partial_\rho\varphi)\,(\partial^\rho\varphi) - \lambda\varphi^4 + \tfrac{1}{12}\varphi^2 R], \tag{4}$$

where λ is another dimensionless parameter and the numerical factors ensure that S_M varies only by a surface term under a conformal transformation $g_{\mu\nu} \to \Omega^2(x)g_{\mu\nu}$, for which $\varphi \to \Omega^{-1}(x)\varphi$ and $\psi \to \Omega^{-3/2}(x)\psi$, where $\Omega(x)$ is any smooth positive function. In the kinetic term for the Dirac field in (4), we define $\bar{\psi}\gamma^\rho \overleftrightarrow{D}_\rho\psi \equiv \bar{\psi}\gamma^\rho D_\rho\psi - (D_\rho\bar{\psi})\gamma^\rho\psi$, where the spinor covariant derivative has the form $D_\mu\psi = (\partial_\mu + \Gamma_\mu)\psi$, the fermion spin connection $\Gamma_\mu = \frac{1}{8}([\gamma^\lambda, \partial_\mu\gamma_\lambda] - \left\{{\lambda \atop \nu\mu}\right\}[\gamma^\nu, \gamma_\lambda])$ and the position-dependent quantities $\gamma_\mu = e^a{}_\mu\gamma_a$ are related to the standard Dirac matrices γ_a using the tetrad components $e^a{}_\mu$. The total action is then $S_T = S_G + S_M$, where the free gravitational contribution S_G depends only on the metric. In Riemannian spacetime, the unique conformally invariant quadratic action is $S_G = \alpha \int d^4x\,\sqrt{-g}\,C_{\rho\sigma\mu\nu}C^{\rho\sigma\mu\nu}$, where α is a further dimensionless parameter and $C_{\rho\sigma\mu\nu}$ is the Weyl tensor. The resulting action S_T then describes so-called conformal gravity (also known as Weyl or Weyl-squared gravity) [16,18] coupled to Dirac matter and a compensator scalar field; the special case for which $\alpha = 0$ and $\psi = 0$ is often described as Einstein conformal gravity [14,19]. Alternative local [20,21] or non-local [22–24] higher-derivative conformally invariant gravitational actions S_G in Riemannian spacetime have also been proposed.

In any case, the introduction of the compensator scalar field φ is usually considered important for providing a means for spontaneously breaking the scale symmetry. Most commonly one uses local scale invariance to set the scalar field to a constant value $\varphi = \varphi_0$, which is often termed the Einstein gauge. This is usually interpreted as choosing some definite scale in the theory, thereby breaking scale-invariance. As we show in [25], however, this interpretation is questionable, since in such scale-invariant gravity theories the equations of motion in the Einstein gauge are identical in form to those obtained when working in scale-invariant variables, which involves no breaking of the scale symmetry. This suggests that one should introduce further scalar fields, in addition to the compensator field φ, to enable a true physical breaking of the scale symmetry. The primary role of the compensator field φ arises instead from its necessary inclusion into the calculation of physical quantities, which renders them invariant under local scale transformations.

In this note, we consider *any* conformally invariant gravity theory in Riemannian spacetime with the matter action (4) that has the SdS metric (2) as a solution in the Einstein gauge (this includes conformal gravity). As shown in [12], in this gauge the scalar field energy-momentum tensor derived from the matter action (4) vanishes only if $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + 6\lambda\varphi_0^2 g_{\mu\nu} = 0$, so that the only vacuum metric allowed (assuming that $\psi = 0$, apart from matter test particles) has the SdS form (2) with $k = -2\lambda\varphi_0^2$. Thus, in conformally-invariant gravity theories, unlike GR, the constant k in (2) may be system dependent, if one assumes that φ_0 may be so. Hence, there exists the possibility of attempting to model some (typically rising) rotation curves by using the expression (3) to fit for (negative values of) k separately for each galaxy, as in [26]. Such an assumption seems questionable when viewed in the Einstein gauge, however, where φ_0 is more naturally interpreted as a system-independent quantity that leads to a 'global' cosmological constant $\Lambda = -6\lambda\varphi_0^2$. In this case, one may therefore no longer fit for k separately for each galaxy, or at all if one considers Λ to be fixed by cosmological observations. It is also worth noting that, to obtain a positive cosmological constant Λ, one must have $\lambda < 0$, which requires a negative scalar field vacuum energy $\lambda\varphi_0^4$, at least with the usual sign conventions in the matter action (4).

Turning to the dynamics of matter test particles, in the Einstein gauge the rest mass $m = \mathscr{m}\varphi_0$ of Dirac particles is independent of spacetime position and so they follow timelike geodesics of the SdS metric, hence yielding rotation curves with no flat region. It has been suggested in [27,28], however, that in such theories one may perform a conformal transformation of this solution to a frame in which the orbital velocity of a massive particle in a circular orbit is asymptotically constant, thereby yielding a flat rotation curve in the outskirts of galaxies. Nonetheless, since such theories are (by construction) conformally invariant, such a transformation should not change the observable predictions, unless the conformal symmetry is broken in some way, either dynamically or by imposing boundary conditions. Merely rescaling the SdS solution to an alternative conformal frame (or scale gauge) in which the compensator scalar field φ no longer takes a constant value should preserve the predictions for physically measurable quantities, such as a rotation curve. We now demonstrate that this is indeed the case.

As we discuss in [12,25], one may construct an appropriate action for a spin-$\frac{1}{2}$ point particle and then transition to the full classical approximation in which the particle spin is neglected. In the presence of the above Yukawa coupling of the Dirac field to the scalar compensator field φ, this action is equivalent [30] to the standard action for a massive particle conformally coupled to the scalar field φ, namely

$$S_{\rm p} = -m \int d\xi\, \varphi \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\xi} \frac{dx^\nu}{d\xi}}, \tag{5}$$

where ξ is a parameterisation for which the length (squared) $u^2 \equiv u^\mu u_\mu$ of the tangent vector $u^\mu = dx^\mu/d\xi$ remains equal to unity along the worldline.

Assuming a static, spherically-symmetric system with $\varphi = \varphi(r)$ and a line-element of the form (1), one finds that for a massive particle worldline in the equatorial plane $\theta = \pi/2$, the t- and ϕ-equations of motion are

$$A\Omega^{-1} \frac{dt}{d\xi} = k, \qquad r^2 \Omega^{-1} \frac{d\phi}{d\xi} = h, \tag{6}$$

where k and h are constants, and we may replace the r-equation of motion with the much simpler first integral $u^\mu u_\mu = 1$, which reads

$$A\left(\frac{dt}{d\xi}\right)^2 - B^{-1}\left(\frac{dr}{d\xi}\right)^2 - r^2\left(\frac{d\phi}{d\xi}\right)^2 = 1. \tag{7}$$

Here $\varphi(r) = \Omega^{-1}(r)\varphi_0$ and the constants k and h are defined such that one recovers the familiar timelike geodesic equations in GR for an affine parameter ξ if $\varphi(r) = \varphi_0$ and so $\Omega = 1$.

As discussed in [12,30,31], however, the parameter ξ cannot be interpreted as the particle proper time, since it has Weyl weight $w(\xi) = 1$ and so it is not invariant under conformal transformations. Rather, the proper time interval is instead given by $d\tau \propto \varphi\, d\xi$, which is correctly invariant under conformal transformations. Indeed, one sees from (5) that the particle dynamics obeys a geodesic principle, but one where φ must be included in the definition of the path length to be extremised. Without loss of generality, one may choose the constant of proportionality such that $d\tau = (\varphi/\varphi_0)\, d\xi = \Omega^{-1}\, d\xi$, so $d\tau$ and $d\xi$ coincide if $\varphi(r) = \varphi_0$. When expressed in terms of the proper time τ of the particle, and denoting $d/d\tau$ by an overdot, the equations of motion (6–7) become

$$A\Omega^{-2}\dot{t} = k, \tag{8a}$$
$$r^2\Omega^{-2}\dot{\phi} = h, \tag{8b}$$
$$A\dot{t}^2 - B^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = \Omega^2. \tag{8c}$$

If there exists a conformal frame in which a solution for a static, spherically-symmetric system is given by the metric (1) and $\varphi(r) = \varphi_0$ (i.e. the Einstein frame), then $\Omega = 1$ and so the equations of motion (8) reduce to the familiar forms for timelike geodesics in the equatorial plane $\theta = \pi/2$ of the line-element (1) [32]. In the special case of the SdS metric, where (2) holds, one therefore recovers the rotation curve (3), which has no flat region.

Suppose one now performs a conformal transformation $\tilde{g}_{\mu\nu}(x) = \Omega^2(r) g_{\mu\nu}(x)$ of the metric (1) and also brings the angular part back into the standard form in (1) by making the (radial) coordinate transformation $r' = r\Omega(r)$ to obtain $\tilde{g}'_{\mu\nu}(x') = X^\rho{}_\mu X^\sigma{}_\nu \tilde{g}_{\rho\sigma}(x(x'))$, where $X^\rho{}_\mu = \partial x'^\rho/\partial x^\mu$. As discussed in [12], in so doing, one finds that the resulting line-element again has the form (1), but expressed in terms of the new radial coordinate r' and the metric functions

$$\tilde{A}'(r') = \Omega^2(r(r'))A(r(r')), \tag{9a}$$
$$\tilde{B}'(r') = f^2(r(r'))B(r(r')), \tag{9b}$$

where we have defined the function $f(r) \equiv 1 + r\frac{d\ln\Omega(r)}{dr}$. In this new conformal frame, the massive particle equations of motion are again given by (8), but with the replacements $r \to r'$, $A(r) \to \tilde{A}'(r')$ and $B(r) \to \tilde{B}'(r')$. On substituting the expressions $r' = r\Omega(r)$ and (9) into these equations of motion, however, one finds after a short calculation that one obtains precisely the *original* equations of motion (8) with $\Omega = 1$, thereby recovering the particle dynamics in the Einstein frame. Thus, for example, if $r = r(\phi)$ is the orbit equation for a particle in the equatorial plane $\theta = \pi/2$ in the Einstein frame, then the orbit equation in the new conformal frame is given simply by $r' = r'(r(\phi))$.

This finding therefore eliminates, as it must, any ambiguity whereby physical predictions appear to depend on the conformal frame in which the calculation is performed. Specialising to the case where (2) holds, this further demonstrates as unwarranted the recent claims in the literature [27,28] that one may obtain flat galaxy rotation curves by conformally-rescaling the SdS metric. It is worth pointing out that these claims arise from the use instead of the equations of motion (6–7), which are expressed in terms of the parameter ξ, but where the latter is interpreted as the particle proper time and implicitly assumed to be invariant under conformal transformations, despite having a Weyl weight $w(\xi) = 1$. In that case, on following an analogous procedure to that we have described above, one arrives at the erroneous conclusion that one does not recover the particle dynamics in the Einstein frame and, more generally, that particle trajectories depend on the conformal frame in which they are calculated, which contradicts conformal invariance.

Although we have demonstrated that, when using the equations of motion (8) expressed in terms of the appropriate conformally-invariant proper time τ, one cannot obtain flat galaxy rotation curves by *any* conformal rescaling of the SdS metric, it is worth discussing briefly the particular rescaling considered in [27,28]. It is suggested in [28] that there are physical reasons to require a metric of the form (1) to satisfy the special condition $A(r) = B(r)$, of which the SdS metric (2) is an example.[2] As shown in [12], the relations (9) imply that in order to preserve this special condition, such that $\tilde{A}'(r') = \tilde{B}'(r')$, one requires the conformal rescaling to have the unique form $\Omega(r) = (1 - ar)^{-1}$, where a is an arbitrary constant, in which case $r' = r/(1 - ar)$ (or, equivalently, $r = r'/(1 + ar')$ and $\Omega'(r') \equiv \Omega(r(r')) = 1 + ar'$). This matches the conformal rescaling and coordinate transformation adopted in [27,28] for $a > 0$. As shown in [11,12,33,34], however, these transformations convert the SdS metric into the Mannheim–Kazanas metric [35,36], so that the claim in [27,28] that one obtains flat galaxy rotation curves in this conformal frame is merely a restatement of the long-standing claims that conformal gravity predicts such rotation curves [37–42], although both claims are unjustified, as we have shown above. It is also worth noting that $r' \to \infty$ as $r \to 1/a$, so that the r' coordinate patch covers only a finite subset of the original r coordinate patch. Indeed, it is straightforward to show that only a finite interval of proper time τ is required for particle to travel radially from some radius $r' = r'_0 > 2GM$ to $r' = \infty$. This contradicts the claim in [28] that the r' coordinate patch is geodesically complete, which is based on the fact that to reach $r' = \infty$ requires an infinite interval of the parameter ξ, which is again mistakenly interpreted as the particle proper time.

So far, our analysis has been limited to static, spherically-symmetric systems, but our finding above that the particle dynamics is independent of the choice of conformal frame is, in fact, entirely general, as one might expect. As shown in [12], the action (5) leads to massive particle equations of motion in any conformal frame that are given by

$$u^\sigma u^\mu{}_{;\sigma} = (g^{\mu\sigma} - u^\mu u^\sigma)\varphi^{-1}\,\partial_\sigma\varphi, \tag{10}$$

where the semi-colon denotes the standard Riemannian spacetime covariant derivative $u^\mu{}_{;\sigma} = \partial_\sigma u^\mu + \left\{ {\mu \atop \rho\sigma} \right\} u^\rho$. Since the action (5) is conformally invariant, these equations of motion are covariant under conformal transformations, but are not manifestly so. If one uses local scale invariance to impose the Einstein gauge $\varphi = \varphi_0$ (where, if desired, one can set φ_0 to unity without loss of generality), then (10) reduces to

$$u^\sigma u^\mu{}_{;\sigma} \doteq 0, \tag{11}$$

where $\doteq$ denotes that the equality holds only in a specific gauge. Thus, in the Einstein gauge, a particle moving only under gravity follows a geodesic of the metric $g^{\mathrm{E}}_{\mu\nu}$ in this frame, as we already noted above for the special case of a static, spherically-symmetric system. As described in [25], however, it is unnecessary to break the scale symmetry by adopting a particular gauge, since one may instead work in terms of scale-invariant variables. Suppose in some arbitrary gauge, the metric and scalar field are related to those in the Einstein gauge by $g_{\mu\nu} = \Omega^2 g^{\mathrm{E}}_{\mu\nu}$ and $\varphi = \Omega^{-1}\varphi_0$. As mentioned above, one should identify $d\tau = (\varphi/\varphi_0)\,d\xi = \Omega^{-1}\,d\xi$ as the interval of particle proper time along its worldline. This leads one to define the scale-invariant 4-velocity

$$\hat{u}^\mu \equiv \frac{dx^\mu}{d\tau} = \frac{d\xi}{d\tau}\frac{dx^\mu}{d\xi} = \left(\frac{\varphi}{\varphi_0}\right)^{-1} u^\mu = \Omega u^\mu, \tag{12}$$

which clearly has Weyl weight $w = 0$. One may also define the scale-invariant metric $\hat{g}_{\mu\nu} \equiv (\varphi/\varphi_0)^2 g_{\mu\nu} = \Omega^{-2} g_{\mu\nu}$ and its associated Christoffel connection

$$\begin{aligned}\widehat{\left\{ {\mu \atop \rho\sigma} \right\}} &= \tfrac{1}{2}\hat{g}^{\mu\nu}(\partial_\rho \hat{g}_{\nu\sigma} + \partial_\sigma \hat{g}_{\rho\nu} - \partial_\nu \hat{g}_{\rho\sigma}),\\ &= \left\{ {\mu \atop \rho\sigma} \right\} + \varphi^{-1}(2\delta^\mu_{(\rho}\partial_{\sigma)}\varphi - g_{\rho\sigma}g^{\mu\nu}\partial_\nu\varphi).\end{aligned} \tag{13}$$

It is then straightforward to show that (10) may be written in terms of the above scale-invariant variables as

$$\hat{u}^\sigma \hat{u}^\mu_{\widehat{;\sigma}} = 0, \tag{14}$$

where we have defined $\hat{u}^\mu_{\widehat{;\sigma}} \equiv \partial_\sigma \hat{u}^\mu + \widehat{\left\{ {\mu \atop \rho\sigma} \right\}}\hat{u}^\rho$. Thus, irrespective of the gauge and without breaking the scale symmetry, the scale-invariant 4-velocity $\hat{u}^\mu$, which is appropriately defined in terms of the particle proper time, satisfies the geodesic equation of the scale-invariant metric $\hat{g}_{\mu\nu} = \Omega^{-2} g_{\mu\nu} = g^{\mathrm{E}}_{\mu\nu}$, which is equal merely to the metric in the Einstein gauge. Thus, we arrive at the conclusion that, quite generally, the particle dynamics is independent of the choice of conformal frame, as expected, and moreover satisfies the weak equivalence principle.

Finally, we conclude by noting that the above conclusion applies not only to conformally-invariant gravity theories defined on Riemannian spacetimes, but also to those defined on more general Weyl–Cartan spacetimes, which include Weyl, Riemann–Cartan and Riemannian spacetimes as special cases. In a Weyl–Cartan (Y_4) spacetime, the covariant derivative $\nabla_\mu = \partial_\mu + \Gamma^\sigma{}_{\rho\mu}\mathsf{X}^\rho{}_\sigma$, where $\Gamma^\sigma{}_{\rho\mu}$ is an affine connection and $\mathsf{X}^\rho{}_\sigma$ are the $GL(4, R)$ generator matrices appropriate to the tensor character of the quantity to which ∇_μ is applied. In particular, in a Y_4 spacetime, ∇_μ satisfies

[2] We discuss the wider implications of the gauge choice $A(r) = B(r)$ in [29], and in particular describe how, in fact, it is not only unnecessary, but also distorts the scaling properties of variables, thereby making it extremely difficult to identify 'intrinsic' φ-independent quantities that may be used for performing all calculations, including the derivation of the geodesic equations.

the semi-metricity condition

$$\nabla_\sigma g_{\mu\nu} = -2B_\sigma g_{\mu\nu}, \tag{15}$$

where B_μ is the Weyl potential (and we have included a factor of -2 for later convenience). On performing the simultaneous conformal (gauge) transformations $g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}$, $B_\mu \to B_\mu - \partial_\mu \ln\Omega(x)$, the condition (15) is preserved [43]. From (15), the connection is given by

$$\Gamma^\lambda{}_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\} + \delta^\lambda_\nu B_\mu + \delta^\lambda_\mu B_\nu - g_{\mu\nu} B^\lambda + K^{*\lambda}{}_{\mu\nu}, \tag{16}$$

where $K^{*\lambda}{}_{\mu\nu}$ is the Y_4 contortion tensor, which is given in terms of (minus) the Y_4 torsion $T^{*\lambda}{}_{\mu\nu} = 2\Gamma^\lambda{}_{[\nu\mu]}$ by $K^{*\lambda}{}_{\mu\nu} = -\frac{1}{2}(T^{*\lambda}{}_{\mu\nu} - T^*{}_\nu{}^\lambda{}_\mu + T^*{}_{\mu\nu}{}^\lambda)$ (the asterisks and the sign of the torsion are consistent with the usual notation adopted in Weyl gauge theory [25,44]). The matter action adopted in Y_4 spacetime typically has the same form as that in (4) *et seq.*, but with the replacements $\left\{ {\lambda \atop \mu\nu} \right\} \to \Gamma^\lambda{}_{\mu\nu}$ and $\partial_\mu \to \partial^*_\mu = \partial_\mu + wB_\mu$, where w is the Weyl weight of the field being differentiated. As shown in [30], however, the corresponding action for a spin-$\frac{1}{2}$ point particle is again equivalent to (5), which thus yields the equations of motion (10). Equivalently, as shown in [25], the equations of motion derived from the point particle action in Y_4 spacetime may be rewritten directly as (10). In either case, one thus arrives at the same conclusions as reached above for Riemannian spacetimes.

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