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Resonance contributions in $B^- \to K^+ K^- \pi^-$ within the light-cone sum rule approach

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Abstract In this work, we study the resonance contributions to the decay $B^- \to K^+ K^- \pi^-$, which is dominated by the scalar $f_0(980)$, $K_0^*(1430)$, vector $\rho(1450)$, $\phi(1020)$, $K^*(892)$ and tensor $f_2(1270)$ resonances. The three-body decay is reduced to various quasi two-body decays, where the B meson firstly decays into a resonance and a pion or a kaon, subsequently the resonance decays into the other two final-state mesons. The quasi two-body decays are calculated within the light-cone sum rule approach utilizing the leading twist B meson light-cone distribution amplitudes. Finally, the decay branching fraction contributions from each resonance is calculated. Some of them are compatible with experiment or previous theoretical works. Including the effects of non-resonant and final-state interactions, we also evaluate the total branching fraction.

1 Introduction

The non-leptonic three-body decays of B mesons are an ideal platform for the study of direct CP violation. Nowadays, using the Dalitz plot analysis, the LHCb collaboration has measured direct CP violation in charmless three-body decays of B mesons [1–3], where evidence for inclusively integrated CP asymmetries has been found. Generally, the decay amplitude is a crucial quantity to obtain the CP asymmetries. The decay amplitude of $B^- \to K^+ K^- \pi^-$ consists mainly of three components, which involve resonant and non-resonant (NR) effects as well as final-state interaction (FSI) effects. Recently, LHCb and BaBar have studied the decays $B^+ \to \pi^+ \pi^- \pi^+$, $B^+ \to K^+ K^- \pi^+$ and $B^\pm \to \pi^\pm K^+ K^-$ [4–6]. For $B^\pm \to \pi^\pm K^+ K^-$, LHCb

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measured the contribution of the resonance $K^*(892)$ in the $\pi^\pm K^\mp$ final state, and the contribution of the resonances $\rho(1450)$, $\phi(1020)$, $f_2(1270)$ in the $K^\pm K^\mp$ channel. In this work, we will focus on the resonant contribution in the $B^- \to K^+ K^- \pi^-$ decay.

Using the narrow-width approximation, the resonant contribution of the three-body B decay can be reduced to various quasi two-body decays, where the B meson firstly decays into a resonance R and a pion or a kaon M_1, $B \to R + M_1$, and subsequently the resonance R decays into the other two final-state mesons, $R \to M_2 + M_3$. The latter process simply depends on the strong couplings of the resonances to the two mesons, while in the first process one has to deal with nontrivial matrix elements like $\langle R, M_1 | \mathcal{O} | B \rangle$, with $\mathcal{O}$ being some four-quark operator. In the literature, there are some phenomenological studies of the three-body B decays [7–9], where these matrix elements are handled by naive factorization. On the other hand, a more theoretical and widely used method for the quasi two-body decays of heavy mesons is perturbative QCD (PQCD), for which we refer e.g. to Refs. [10–17]. In the PQCD approach, the two mesons produced by the resonance are usually described by the two-meson light-cone distribution amplitudes (LCDAs). However, in a decay process involving a number of resonances, one has to search for all the LCDAs corresponding to them. Therefore, an obvious question is whether these LCDAs are all available or reliable, especially for the excited states like $\rho(1450)$ or $K_0^*(1430)$. To avoid this problem, one can search for a method which only depends on the familiar LCDAs like those of B meson or the ground state pseudoscalar mesons.

In this work, we will use light-cone sum rules (LCSR) to calculate the matrix elements for the $B \to R + M_1$ decays. Generally, LCSR are used for the calculation of the form factors of the semi-leptonic B or D decays. However,

recently the use of LCSR for two-body non-leptonic decays $B/D \to \pi\pi$ has been developed in Refs. [18–20]. For the case of $B \to R + M_1$, one can follow a similar idea to perform the LCSR calculation, where all the non-perturbative inputs are the LCDAs. The only difference is that here we use the LCDAs of the B meson instead of the final pseudo-scalar meson. The advantage of LCSR is that in its framework the final states can be created by two interpolating currents, which have a definite form no matter the state it creates is the ground state or an excited state. The way to extract the pole contribution of the excited states is to apply a two-step sum rules, which will be illustrated in the following sections.

This article is organized as follows: In Sect. 2, we present the form of all the relevant decay amplitudes in the narrow-width approximation. In Sect. 3, we introduce the LCSR calculation at the hadron level, which includes the case of scalar, vector and tensor resonances. In Sect. 4 we introduce the LCSR calculation at the quark-gluon level, where we take the case of vector resonance as an example to present the calculational details. In Sect. 5, we present the numerical results including the decay amplitudes and the branching fractions. Section 6 gives the conclusions of this work.

2 $B^- \to K^+ K^- \pi^-$ decay amplitude

In this section, we firstly present the form of the decay amplitude together with the relevant effective Hamiltonian. From Ref. [8], the decay amplitude of $B^- \to K^+ K^- \pi^-$ reads

$$\langle K^+ K^- \pi^- \,|\mathcal{H}_{\text{eff}}|\, B^- \rangle$$
$$= \frac{G_F}{\sqrt{2}} \sum_{p=u,c} \lambda_p \langle K^+ K^- \pi^- \,|H_p|\, B^- \rangle, \tag{1}$$

where $\lambda_p \equiv V_{pb} V_{pd}^*$. The expression of the Hamiltonian H_p is

$$H_p = a_1 \delta_{pu} (\bar{u}b)_{V-A} \otimes (\bar{d}u)_{V-A}$$
$$+ a_2 \delta_{pu} (\bar{d}b)_{V-A} \otimes (\bar{u}u)_{V-A}$$
$$+ a_3 (\bar{d}b)_{V-A} \otimes \sum_q (\bar{q}q)_{V-A}$$
$$+ a_4^p \sum_q (\bar{q}b)_{V-A} \otimes (\bar{d}q)_{V-A}$$
$$+ a_5 (\bar{d}b)_{V-A} \otimes \sum_q (\bar{q}q)_{V+A}$$
$$- 2a_6^p \sum_q (\bar{q}b)_{S-P} \otimes (\bar{d}q)_{S+P}$$
$$+ a_7 (\bar{d}b)_{V-A} \otimes \sum_q \frac{3}{2} e_q (\bar{q}q)_{V+A}$$
$$- 2a_8^p \sum_q (\bar{q}b)_{S-P} \otimes \frac{3}{2} e_q (\bar{d}q)_{S+P}$$

$$+ a_9 (\bar{d}b)_{V-A} \otimes \sum_q \frac{3}{2} e_q (\bar{q}q)_{V-A}$$
$$+ a_{10}^p \sum_q (\bar{q}b)_{V-A} \otimes \frac{3}{2} e_q (\bar{d}q)_{V-A}. \tag{2}$$

The notations used here are: $(\bar{q}q')_{V\pm A} \equiv \bar{q}\gamma_\mu (1 \pm \gamma_5) q'$, $(\bar{q}q')_{S\pm P} \equiv \bar{q} (1 \pm \gamma_5) q'$, and the summation on q runs over u, d, s. The pertinent Wilson coefficients are taken from Ref. [9]:

$$a_1 \approx 0.988 \pm 0.102i, \quad a_2 \approx 0.183 - 0.348i,$$
$$a_3 \approx -0.0023 + 0.0174i, \quad a_5 \approx 0.00644 - 0.0231i,$$
$$a_4^u \approx -0.025 - 0.021i, \quad a_4^c \approx -0.030 - 0.012i,$$
$$a_6^u \approx -0.042 - 0.014i, \quad a_6^c \approx -0.045 - 0.005i$$
$$a_7 \approx (-0.5 + 2.7i) \times 10^{-4}, \quad a_8^u \approx (5.2 - 1.0i) \times 10^{-4},$$
$$a_8^c \approx (5.0 - 0.5i) \times 10^{-4}, \quad a_9 \approx (-8.9 - 0.9i) \times 10^{-3},$$
$$a_{10}^u \approx (-1.45 + 3.12i) \times 10^{-3}, \quad a_{10}^c \approx (-1.51 + 3.17i) \times 10^{-3} \tag{3}$$

at the renormalization scale $\mu = 4.18$ GeV. The matrix element of T_p in Eq. (1) can be arranged as a sum of the following matrix elements [8]:

$$\langle \pi^- K^+ K^- \,|H_p|\, B^- \rangle$$
$$= \langle K^+ K^- \pi^- \,|\mathcal{O}_1|\, B^- \rangle$$
$$\times \left[a_1 \delta_{pu} + a_4^p + a_{10}^p - \left(a_6^p + a_8^p \right) r_\chi^\pi \right]$$
$$+ \langle K^+ K^- \pi^- \,|\mathcal{O}_2|\, B^- \rangle \left(a_2 \delta_{pu} + a_3 + a_5 + a_7 + a_9 \right)$$
$$+ \langle K^+ K^- \pi^- \,|\mathcal{O}_3|\, B^- \rangle$$
$$\times \left[a_3 + a_4^p + a_5 - \frac{1}{2} \left(a_7 + a_9 + a_{10}^p \right) \right]$$
$$+ \langle K^+ K^- \pi^- \,|\mathcal{O}_4|\, B^- \rangle \left[a_3 + a_5 - \frac{1}{2} (a_7 + a_9) \right]$$
$$+ \langle K^+ K^- \pi^- \,|\mathcal{O}_5|\, B^- \rangle \left(-2a_6^p + a_8^p \right)$$
$$+ \langle K^+ K^- \pi^- \,|\mathcal{O}_6|\, B^- \rangle \left(a_4^p - \frac{1}{2} a_{10}^p \right)$$
$$+ \langle K^+ K^- \pi^- \,|\mathcal{O}_7|\, B^- \rangle \left(-2a_6^p + a_8^p \right), \tag{4}$$

where $r_\chi^\pi (\mu) = 2m_\pi^2 / [m_b(\mu) (m_d(\mu) - m_u(\mu))]$, with $\mu = 2.1$ GeV, $m_b(\mu) = 4.94$ GeV, $m_d(\mu) = 5$ MeV and $m_u(\mu) = 2.2$ MeV. The four-quark operators $\mathcal{O}_i$ are

$$\mathcal{O}_1 = (\bar{u}b)_{V-A}(\bar{d}u)_{V-A}, \quad \mathcal{O}_2 = (\bar{d}b)_{V-A}(\bar{u}u)_{V-A},$$
$$\mathcal{O}_3 = (\bar{d}b)_{V-A}(\bar{d}d)_{V-A}, \quad \mathcal{O}_4 = (\bar{d}b)_{V-A}(\bar{s}s)_{V-A},$$
$$\mathcal{O}_5 = (\bar{d}b)(\bar{d}d), \quad \mathcal{O}_6 = (\bar{s}b)_{V-A}(\bar{d}s)_{V-A},$$
$$\mathcal{O}_7 = (\bar{s}b)(\bar{d}s). \tag{5}$$

It should be mentioned that in principle Eq. (4) also contain two three-particle terms:

$$
\begin{aligned}
&\langle K^+ K^- \pi^- \left|(\bar{d}u)_{V-A}\right| 0\rangle \langle 0 \left|(\bar{u}b)_{V-A}\right| B^-\rangle \\
&\quad \times \left(a_1 \delta_{pu} + a_4^p + a_{10}^p\right) + \langle K^+ K^- \pi^- \left|\bar{d}\left(1+\gamma_5\right)u\right| 0\rangle \\
&\quad \times \langle 0 \left|\bar{u}\gamma_5 b\right| B^-\rangle \left(2a_6^p + 2a_8^p\right).
\end{aligned} \tag{6}
$$

However, since these three-particle matrix elements are suppressed in the chiral limit [7], we will not consider them in this work.

In general, most of the resonant contribution to the matrix element in Eq. (4) can be described by the quasi-two-body decay process with the use of the Breit–Wigner (BW) formalism. However, for the scalar meson $f_0(980)$ with its mass near the $K\bar{K}$ threshold, using the BW formalism is inappropriate. Instead one has to use the Flatté approach [21,22] for such resonances. For the use of Flatté approach in the resonance dominated heavy meson decays, and related discussions, see e.g. Refs. [23–26]. Particularly, for the $f_0(980)$ contribution, one has to replace the $\Gamma_{f_0(980)}$ term in the standard BW formalism by $g_{\pi\pi}\rho_{\pi\pi} + g_{KK}F_{KK}^2\rho_{KK}$ [23,24,27], where

$$
\begin{aligned}
\rho_{\pi\pi} &= \frac{2}{3}\sqrt{1 - \frac{4m_{\pi^\pm}^2}{m_{\pi\pi}^2}} + \frac{1}{3}\sqrt{1 - \frac{4m_{\pi^0}^2}{m_{\pi\pi}^2}}, \\
\rho_{KK} &= \frac{1}{2}\sqrt{1 - \frac{4m_{K^\pm}^2}{m_{\pi\pi}^2}} + \frac{1}{2}\sqrt{1 - \frac{4m_{K^0}^2}{m_{\pi\pi}^2}}
\end{aligned} \tag{7}
$$

with $g_{\pi\pi} = 167$ MeV and $g_{KK}/g_{\pi\pi} = 3.05$ [28], and $m_{\pi\pi}^2$ is the invariant mass squared of the two-pion system. $F_{KK} = \exp\left(-\alpha k^2\right)$ is an correction introduced by the LHCb fit above the $K\bar{K}$ threshold with $\alpha = 2\,\text{GeV}^{-2}$, and k is the magnitude of the kaon three-momentum in the $K\bar{K}$ rest frame [28]. Here, we will simply choose $m_{K^\pm} = m_{K^0} \equiv m_K$ and $m_{\pi^\pm} = m_{\pi^0} \equiv m_\pi$. For further discussion, see e.g. Ref. [29].

The B meson firstly decays into a resonance R and a meson M_1, and then the resonance R decays into the rest two mesons M_2, M_3. The first process is described by the matrix elements $\langle R, M_1 |\mathcal{O}_i| B^-\rangle$, while the second process depends on the strong coupling of R to M_2, M_3. It should be mentioned that in Ref. [8], naive factorization is used to factorize each matrix element in Eq. (4) into two current induced terms. For example:

$$
\begin{aligned}
&\langle K^+ K^- \pi^- |\mathcal{O}_1| B^-\rangle \\
&\quad = \langle K^+ K^- |(\bar{u}b)_{V-A}| B^-\rangle \langle \pi^- \left|(\bar{d}u)_{V-A}\right| 0\rangle.
\end{aligned} \tag{8}
$$

This means that if the $K^+ K^-$ are produced by a resonance R, what one actually calculates is a factorized form of $\langle R, \pi^- |\mathcal{O}_1| B^-\rangle$

$$
\langle R, \pi^- |\mathcal{O}_1| B^-\rangle = \langle R |(\bar{u}b)_{V-A}| B^-\rangle \langle \pi^- \left|(\bar{d}u)_{V-A}\right| 0\rangle. \tag{9}
$$

The first matrix element on the right-hand side is parameterized by transition form factors, while the second one is described by the decay constant of the resonance. In this work, with the use of LCSR, we do not have to assume this factorization, instead we are able to calculate the matrix element $\langle R, \pi^- |\mathcal{O}_i| B\rangle$ as a whole.

We follow Ref. [9] to introduce the resonances appearing in the $B^- \to K^+ K^- \pi^-$ decay. In terms of the resonant contribution to the $\mathcal{O}_1$ matrix element, we include a scalar resonance $f_0(980)$, a vector resonance $\rho(1450)$ and a tensor resonance $f_2(1270)$:

$$
\begin{aligned}
&\langle \pi^-(p_1)K^+(p_2)K^-(p_3)|\mathcal{O}_1|B^-(p)\rangle_R \\
&= \frac{g^{f_0(980)\to K^+K^-}}{s_{23} - m_{f_0(980)}^2 + i\,m_{f_0(980)}\left(g_{\pi\pi}\rho_{\pi\pi} + g_{KK}F_{KK}^2\rho_{KK}\right)} \\
&\quad \times \langle f_0(980)(p-p_1)\pi^-(p_1)|\mathcal{O}_1|B^-(p)\rangle \\
&\quad + \frac{g^{\rho(1450)\to K^+K^-}}{s_{23} - m_{\rho(1450)}^2 + i\,m_{\rho(1450)}\Gamma_{\rho(1450)}} \times \sum_\lambda \epsilon(\lambda)\cdot(p_2 - p_3) \\
&\quad \langle \rho(1450)(p-p_1,\lambda)\pi^-(p_1)| \times \mathcal{O}_1|B^-(p)\rangle \\
&\quad + \frac{g^{f_2(1270)\to K^+K^-}}{s_{23} - m_{f_2(1270)}^2 + i\,m_{f_2(1270)}\Gamma_{f_2(1270)}} \sum_\lambda \epsilon_{\mu\nu}(\lambda) \\
&\quad \times p_2^\mu p_3^\nu \langle f_2(1270)(p-p_1,\lambda)\times\pi^-(p_1)|\mathcal{O}_1|B^-(p)\rangle.
\end{aligned} \tag{10}
$$

The strong coupling constants are defined as in Ref. [9]. Further, $s_{23} = (p_2 + p_3)^2$, $\epsilon_\mu(\lambda)$ denotes the polarization vector of the $\rho(1450)$ and $\epsilon_{\mu\nu}(\lambda)$ denotes the polarization tensor of the $f_2(1270)$. In terms of the resonant contribution to the $\mathcal{O}_{2,3}$ matrix element, we include the vector resonance $\rho(1450)$:

$$
\begin{aligned}
&\langle \pi^-(p_1)K^+(p_2)K^-(p_3)|\mathcal{O}_{2,3}|B^-(p)\rangle_R \\
&= \frac{g^{\rho(1450)\to K^+K^-}}{s_{23} - m_{\rho(1450)}^2 + i\,m_{\rho(1450)}\Gamma_{\rho(1450)}} \times \sum_\lambda \epsilon(\lambda) \\
&\quad \cdot (p_2 - p_3)\langle \rho(1450)(p-p_1,\lambda)\pi^-(p_1)\times|\mathcal{O}_2|B^-(p)\rangle.
\end{aligned} \tag{11}
$$

For the $\mathcal{O}_4$ matrix element, we include the vector resonance $\phi(1020)$:

$$
\begin{aligned}
&\langle \pi^-(p_1)K^+(p_2)K^-(p_3)|\mathcal{O}_4|B^-(p)\rangle_R \\
&= \frac{g^{\phi(1020)\to K^+K^-}}{s_{23} - m_{\phi(1020)}^2 + i\,m_{\phi(1020)}\Gamma_{\phi(1020)}} \\
&\quad \times \sum_\lambda \epsilon(\lambda)\cdot(p_2 - p_3)
\end{aligned}
$$

$$\langle \phi(1020)(p - p_1, \lambda)\pi^-(p_1) \times |\mathcal{O}_4|B^-(p)\rangle. \quad (12)$$

For the $\mathcal{O}_5$ matrix element, we include the scalar resonance $f_0(980)$:

$$
\begin{aligned}
&\langle \pi^-(p_1)K^+(p_2)K^-(p_3)|\mathcal{O}_5|B^-(p)\rangle_R \\
&= \frac{g^{f_0(980)\to K^+K^-}}{s_{23} - m_{f_0(980)}^2 + i\, m_{f_0(980)}\left(g_{\pi\pi}\rho_{\pi\pi} + g_{KK}F_{KK}^2\rho_{KK}\right)} \\
&\quad \times \langle f_0(980)(p - p_1)\pi^-(p_1)|\mathcal{O}_5|B^-(p)\rangle. \quad (13)
\end{aligned}
$$

For the $\mathcal{O}_6$ matrix element, we include a scalar resonance $K_0^*(1430)$ and a vector resonance $K^*(892)$:

$$
\begin{aligned}
&\langle \pi^-(p_1)K^+(p_2)K^-(p_3)|\mathcal{O}_6|B^-(p)\rangle_R \\
&= \frac{g^{K^*(892)\to K^+\pi^-}}{s_{12} - m_{K^*(892)}^2 + i\, m_{K^*(892)}\Gamma_{K^*(892)}} \\
&\quad \times \sum_\lambda \epsilon(\lambda)\cdot(p_1 - p_2) \\
&\quad \langle K^*(892)(p - p_3, \lambda)K^-(p_3)|\mathcal{O}_6|B^-(p)\rangle \\
&\quad + \frac{g^{K_0^*(1430)\to K^+\pi^-}}{s_{12} - m_{K_0^*(1430)}^2 + i\, m_{K_0^*(1430)}\Gamma_{K_0^*(1430)}} \\
&\quad \times \langle K_0^*(1430)(p - p_3)K^-(p_3)|\mathcal{O}_6|B^-(p)\rangle, \quad (14)
\end{aligned}
$$

where $s_{12} = (p_1 + p_2)^2$. For the $\mathcal{O}_7$ matrix element, we include the scalar resonance $K_0^*(1430)$:

$$
\begin{aligned}
&\langle \pi^-(p_1)K^+(p_2)K^-(p_3)|\mathcal{O}_7|B^-(p)\rangle_R \\
&= \frac{g^{K_0(1430)\to K^+\pi^-}}{s_{12} - m_{K_0(1430)}^2 + i\, m_{K_0(1430)}\Gamma_{K_0(1430)}} \\
&\quad \times \langle K_0(1430)(p - p_3)K^-(p_3)|\mathcal{O}_7|B^-(p)\rangle. \quad (15)
\end{aligned}
$$

The matrix elements $\langle R, M|\mathcal{O}_i|B^-\rangle$ appearing in Eqs. (10)-(15) can be parameterized as

$$
\begin{aligned}
&\langle \rho(1450)(p - p_1, \lambda)\pi^-(p_1)|\mathcal{O}_1|B^-(p)\rangle \\
&\quad = T_{\mathcal{O}_1}^{\rho(1450)}\epsilon^*(p - p_1, \lambda)\cdot p_1, \\
&\langle f_0(980)(p - p_1)\pi^-(p_1)|\mathcal{O}_1|B^-(p)\rangle \\
&\quad = T_{\mathcal{O}_1}^{f_0(980)}, \\
&\langle f_2(1270)(p - p_1, \lambda)\pi^-(p_1)|\mathcal{O}_1|B^-(p)\rangle \\
&\quad = T_{\mathcal{O}_1}^{\rho(1270)}\epsilon_{\mu\nu}^*(p - p_1, \lambda)p_1^\mu p_1^\nu, \\
&\langle \rho(1450)(p - p_1, \lambda)\pi^-(p_1)|\mathcal{O}_{2,3}|B^-(p)\rangle \\
&\quad = T_{\mathcal{O}_{2,3}}^{\rho(1450)}\epsilon^*(p - p_1, \lambda)\cdot p_1, \\
&\langle \phi(1020)(p - p_1, \lambda)\pi^-(p_1)|\mathcal{O}_4|B^-(p)\rangle \\
&\quad = T_{\mathcal{O}_4}^{\phi(1020)}\epsilon^*(p - p_1, \lambda)\cdot p_1,
\end{aligned}
$$

$$
\begin{aligned}
&\langle f_0(980)(p - p_1)\pi^-(p_1)|\mathcal{O}_5|B^-(p)\rangle \\
&\quad = T_{\mathcal{O}_5}^{f_0(980)}, \\
&\langle K^*(892)(p - p_3, \lambda)K^-(p_3)|\mathcal{O}_6|B^-(p)\rangle \\
&\quad = T_{\mathcal{O}_6}^{K^*(892)}\epsilon^*(p - p_3, \lambda)\cdot p_3, \\
&\langle K_0^*(1430)(p - p_3)K^-(p_3)|\mathcal{O}_{6,7}|B^-(p)\rangle \\
&\quad = T_{\mathcal{O}_{6,7}}^{K_0^*(1430)}. \quad (16)
\end{aligned}
$$

The main task of this work is to obtain these $T_{\mathcal{O}_i}^R$ using the LCSR approach.

3 Hadron level calculation in LCSR

In this and the next section, we will give an introduction to the calculation of the $B^- \to R, M$ decays within the LCSR approach. In the spirit of LCSR, one should define an appropriate correlation function corresponding to the process to be studied. Due to the quark-hadron duality, one has to calculate the correlation function both at the hadron and quark-gluon level. In this section, we will take the decays induced by $\mathcal{O}_1$ as an example to introduce the calculation at the hadron level, while the decays induced by $\mathcal{O}_{2,\dots 7}$ can be derived similarly.

3.1 $B^- \to f_0(980)\pi^-$ induced by $\mathcal{O}_1$

In the framework of LCSR and with respect to the decay process $B^- \to f_0(980)\pi^-$, we firstly define a three-point correlation function as

$$
\begin{aligned}
\Pi_\alpha^{S\mathcal{O}_1}(p, q, k) &= i^2 \int d^4x d^4y\, e^{i(p-q)\cdot y}e^{i(q+k)\cdot x} \\
&\quad \times \langle 0|T\left\{j_{5\alpha}^\pi(y)\mathcal{O}_1(0)j_{f_0}(x)\right\}|B(p)\rangle, \quad (17)
\end{aligned}
$$

where the final pion as well as the S-wave resonance $f_0(980)$ are created by the interpolating currents $j_{5\alpha}^\pi$ and j_{f_0}, respectively:

$$j_{5\alpha}^\pi = \bar{u}\gamma_\alpha\gamma_5 d, \quad j_{f_0} = \bar{s}s\cos\alpha + (\bar{u}u + \bar{d}d)\frac{\sin\alpha}{\sqrt{2}}. \quad (18)$$

Here, α is the mixing angle between the $f_0(980)$ and $\sigma(500)$, which is chosen as $\alpha = 20°$ [9]. In terms of the flavor wave function, such mixing is given by [30]

$$
\begin{aligned}
\sigma &= \frac{\cos\alpha}{\sqrt{2}}(\bar{u}u + \bar{d}d) - \sin\alpha\,\bar{s}s, \\
f_0 &= \frac{\sin\alpha}{\sqrt{2}}(\bar{u}u + \bar{d}d) + \cos\alpha\,\bar{s}s. \quad (19)
\end{aligned}
$$

In Eq. (17), k is an auxiliary momentum which flows into $\mathcal{O}_1(0)$. This auxiliary momentum was introduced in

Ref. [18], where the non-leptonic $B \to \pi\pi$ decay was studied using LCSR. In general, the correlation function in Eq. (17) can be parameterized as

$$\Pi_\alpha^{S\mathcal{O}_1}(p,q,k) = (p-q)_\alpha F_S^{\mathcal{O}_1} + q_\alpha G_S^{\mathcal{O}_1} + k_\alpha H_S^{\mathcal{O}_1}$$
$$+ \epsilon_{\alpha\beta\rho\sigma} p^\beta q^\rho k^\sigma I_S^{\mathcal{O}_1}, \quad (20)$$

where the $F_S^{\mathcal{O}_1} \sim I_S^{\mathcal{O}_1}$ are functions of the Lorentz invariants $p^2 = m_B^2$, $(p-q)^2$, $(q+k)^2$, k^2, q^2 and $\bar{P}^2 = (p-q-k)^2$. Since the correlation function will be calculated via the Operator-Product-Expansion (OPE), we must require that the spacetime intervals are small enough: $x^2 \sim y^2 \sim (x-y)^2 \sim 0$. This means that the external momenta must be extensively spacelike: $(p-q)^2 \sim (q+k)^2 \sim \bar{P}^2 \ll 0$. On the other hand, as proposed in Ref. [18], to simplify the calculation, we can set the remaining squared momenta k^2 and q^2 to zero.

At the hadronic level, one can firstly insert a complete state with the same quantum number as j_{f_0} into the correlation function, which then becomes

$$\Pi_\alpha^{S\mathcal{O}_1}(p,q,k)_{\text{Hadron}}$$
$$= -i\frac{m_{f_0} f_{f_0}}{(q+k)^2 - m_{f_0}^2}$$
$$\times \int d^4 y e^{i(p-q)\cdot y}\langle f_0(q+k)|T\{j_{5\alpha}^\pi(y)\mathcal{O}_1(0)\}|B(p)\rangle + \cdots, \quad (21)$$

where f_{f_0} is the decay constant of the $f_0(980)$ defined via $\langle 0|j_{f_0}(0)|f_0(p_f)\rangle = m_{f_0} f_{f_0}$. We have explicitly kept the contribution from the lowest $f_0(980)$ state, and attributed the higher multi-particle states and continuous spectrum into the ellipsis. On the other hand, the same correlation function can be calculated by OPE at the quark-gluon level which is denoted as $\Pi_\alpha^{\mathcal{O}_1}(p,q,k)_{\text{QCD}}$. It should match with that in Eq. (21), which leads to

$$-i\frac{m_{f_0} f_{f_0}}{(q+k)^2 - m_{f_0}^2}$$
$$\times \int d^4 y e^{i(p-q)\cdot y}\langle f_0(q+k)|T\{j_{5\alpha}^\pi(y)\mathcal{O}_1(0)\}|B(p)\rangle$$
$$= \frac{1}{\pi}\int_0^{s_{f_0}} ds_1 \frac{\text{Im}\Pi_\alpha^{S\mathcal{O}_1}\left[(p-q)^2, s_1, \bar{P}^2\right]_{\text{QCD}}}{s_1 - (q+k)^2}, \quad (22)$$

where s_{f_0} is the threshold parameter of the sum rule, which is chosen as the mass squared of the first excited state above the $f_0(980)$ with $J^{PC} = 0^{++}$, namely the $f_0(1370)$. On the right-hand side, the quark-gluon level correlation function is expressed by a dispersion integral. According to the quark-hadron duality, its integration from s_{f_0} to infinity is canceled by the contribution of higher excited states on the left-hand side.

The matrix element on the left-hand side of Eq. (22) can be understood as the emission of a hard momentum $\bar{P}$ from the B meson which then leaves a $f_0(980)$ as the final state. According to Eq. (22) and the fact that the right-hand side is an analytic function of $\bar{P}^2$, one can perform an analytic continuation to transform the extensively spacelike $\bar{P}^2$ to the timelike region. Note that k as an auxiliary momentum will be set to zero so that $\bar{P} \to p-q$ at last. Since $p-q$ corresponds to the momentum flow out off the pion vertex, it is natural to choose $\bar{P}^2 = m_\pi^2$. At this physical region, the outgoing f_0 can be moved to the initial state with an inverse momentum

$$-i\frac{m_{f_0} f_{f_0}}{(q+k)^2 - m_{f_0}^2}$$
$$\times \int d^4 y e^{i(p-q)\cdot y}\langle 0|T\{j_{5\alpha}^\pi(y)\mathcal{O}_1(0)\}|B(p) f_0(-q-k)\rangle$$
$$= \frac{1}{\pi}\int_0^{s_{f_0}} ds_1 \frac{\text{Im}\Pi_\alpha^{S\mathcal{O}_1}\left[(p-q)^2, s_1, \bar{P}^2 = m_\pi^2\right]_{\text{QCD}}}{s_1 - (q+k)^2}. \quad (23)$$

Then, by inserting a complete state with the same quantum number as $j_{5\alpha}^\pi$ on the left-hand side, using the quark-hadron duality to cancel out its higher excited contributions, and using crossing symmetry to move the f_0 to the final-state again, one arrives at

$$i\frac{m_{f_0} f_{f_0} f_\pi}{((q+k)^2 - m_{f_0}^2)((p-q)^2 - m_\pi^2)}$$
$$\times (p-q)_\alpha \langle f_0(q+k)\pi(p-q)|\mathcal{O}_1(0)|B(p)\rangle$$
$$= \frac{1}{\pi^2}\int_0^{s_{f_0}} ds_1 \int_0^{s_\pi} ds_2 \frac{\text{Im}^2\Pi_\alpha^{S\mathcal{O}_1}\left[s_2, s_1, \bar{P}^2 = m_\pi^2\right]_{\text{QCD}}}{(s_1 - (q+k)^2)(s_2 - (p-q)^2)}, \quad (24)$$

where we have used the definition of the pion decay constant, $\langle 0|j_{5\alpha}^\pi(0)|\pi(p)\rangle = if_\pi p_\alpha$, and s_π is chosen as the squared mass of the $\pi(1300)$. The symbol Im^2 means extracting the double imaginary part corresponding to the discontinuities across s_1 and s_2. Using the Borel transformation formula:

$$\mathcal{B}_{p^2}\left[\frac{1}{m^2 - p^2}\right] = e^{-m^2/M^2} \quad (25)$$

for the external momenta squared $(q+k)^2$ and $(p-q)^2$ in Eq. (24), and denoting the corresponding Borel parameters as M_1 and M_2, one arrives at

$$\langle f_0(q+k)\pi(p-q)|\mathcal{O}_1(0)|B(p)\rangle$$
$$= -\frac{i}{\pi^2 m_{f_0} f_{f_0} f_\pi} e^{m_\pi^2/M_2^2} e^{m_{f_0}^2/M_1^2}$$
$$\times \int_0^{s_{f_0}} ds_1 \int_0^{s_\pi} ds_2 e^{-s_2/M_2^2} e^{-s_1/M_1^2} \text{Im}^2 F_S^{\mathcal{O}_1}$$
$$\times \left[s_2, s_1, \bar{P}^2 = m_\pi^2\right]_{\text{QCD}}. \quad (26)$$

Note that on the left-hand side of Eq. (24), only the Lorentz structure of $(p-q)_\alpha$ appears. Thus only the $F_S^{\mathcal{O}_1}$ in Eq. (20) contributes.

3.2 $B^- \to \rho(1450)\pi^-$ induced by $\mathcal{O}_1$

In this subsection we consider the process $B^- \to \rho(1450) \pi^-$, where the resonance is a vector particle. The definition of the correlation function is similar except that the interpolating current for the resonance is changed to a vector current, which is given by

$$\Pi_{\alpha\beta}^{V\mathcal{O}_1}(p,q,k) = i^2 \int d^4x \, d^4y \, e^{i(p-q)\cdot y} e^{i(q+k)\cdot x}$$
$$\times \langle 0|T\left\{ j_{5\alpha}^\pi(y)\mathcal{O}_1(0)j_\beta^V(x)\right\}|B(p)\rangle, \quad (27)$$

with

$$j_\beta^V = \frac{1}{\sqrt{2}}\left(\bar{u}\gamma_\beta u - \bar{d}\gamma_\beta d\right). \quad (28)$$

The procedure of inserting the states with the same quantum numbers of π^- and $\rho(1450)$ is similar to that of $B^- \to f_0(980)\pi^-$. However, it should be noted that the $\rho(1450)$ is the second excited state with $J^{PC} = 1^{--}$, while the lowest one is in fact the $\rho(770)$. The strategy to extract the contribution from the $\rho(1450)$ is to construct a two-step sum rule. First, we just keep the lowest state $\rho(770)$ at the hadron level and attribute the $\rho(1450)$ to the higher excited spectrum. Then we can calculate the matrix element $\langle \rho(770)\pi^-|\mathcal{O}_1|B^-\rangle$ with the threshold parameter chosen as $s_{\rho(770)} = m_{\rho(1450)}^2$. At the second step, both $\rho(770)$ and $\rho(1450)$ are kept and the higher excited spectrum begins from the squared mass of the $\rho(1570)$, in other words $s_{\rho(1450)} = m_{\rho(1570)}^2$.

In the first step, we obtain the sum rule equation for the $\rho(770)$ as

$$i\frac{m_{\rho(770)}f_{\rho(770)}f_\pi(p-q)_\alpha}{\left((q+k)^2 - m_{\rho(770)}^2\right)\left((p-q)^2 - m_\pi^2\right)} \times \sum_\lambda \epsilon_\beta(q+k,\lambda)$$
$$\langle\rho(770)(q+k,\lambda)\pi(p-q)|\mathcal{O}_1(0)| \times B(p)\rangle$$
$$= \frac{1}{\pi^2}\int_0^{s_{\rho(770)}} ds_1 \times \int_0^{s_\pi} ds_2$$
$$\frac{\mathrm{Im}^2\Pi_{\alpha\beta}^{V\mathcal{O}_1}\left[s_2,s_1,\bar{P}^2 = m_\pi^2\right]_{\mathrm{QCD}}}{(s_1 - (q+k)^2)(s_2 - (p-q)^2)}, \quad (29)$$

where $f_{\rho(770)}$ is defined via

$$\langle 0\left| j_\beta^V(0)\right| \rho(770)(p,\lambda)\rangle = m_{\rho(770)}f_{\rho(770)}\epsilon_\beta(p,\lambda), \quad (30)$$

and similarly for the $f_{\rho(1450)}$. On the left-hand side of Eq. (29), the matrix element not only depends on p,q, but

also on the auxiliary momentum k. Thus defining $p_\pi = p - q$, $p_V = q + k$ and using the constraint $\epsilon \cdot p_\pi = 0$, it should be parameterized by two terms:

$$\langle\rho(770)(p_V,\lambda)\pi(p_\pi)|\mathcal{O}_1(0)|B(p)\rangle$$
$$= T_{\mathcal{O}_1}^{\rho(770)}\epsilon^*(p_V,\lambda)\cdot p_\pi + T_{\mathcal{O}_1}^{\rho(770)'}\epsilon^*(p_V,\lambda)\cdot k. \quad (31)$$

Although the second term vanishes when we set $k \to 0$ at the end of the calculation, both of them should be kept in the intermediate steps. Now we need two equations to solve for $T_{\mathcal{O}_1}^{\rho(770)}$ and $T_{\mathcal{O}_1}^{\rho(770)'}$. To do this, we can contract with two independent vectors, $\Gamma_i^\beta = \{m_B v^\beta, q^\beta\}$ ($i = 1,2$), respectively on the both sides of Eq. (29). This leaves only one Lorentz index for the correlation function on the right-hand side, which again has the same structures as Eq. (20):

$$\Gamma_i^\beta\Pi_{\alpha\beta}^{V\mathcal{O}_1}(p,q,k) = (p-q)_\alpha F_{V(i)}^{\mathcal{O}_1} + q_\alpha G_{V(i)}^{\mathcal{O}_1} + k_\alpha H_{V(i)}^{\mathcal{O}_1}$$
$$+ \epsilon_{\alpha\beta\rho\sigma}p^\beta q^\rho k^\sigma I_{V(i)}^{\mathcal{O}_1}. \quad (32)$$

As a result, the sum rule equation becomes

$$i\frac{m_{\rho(770)}f_{\rho(770)}f_\pi}{\left(p_V^2 - m_{\rho(770)}^2\right)\left(p_\pi^2 - m_\pi^2\right)}\Gamma_{i\mu}\left[-g^{\mu\nu} + \frac{p_V^\mu p_V^\nu}{m_{\rho(770)}^2}\right]$$
$$\times\left(T_{\mathcal{O}_1}^{\rho(770)}p_{\pi\nu} + T_{\mathcal{O}_1}^{\rho(770)'}k_\nu\right)$$
$$= \frac{1}{\pi^2}\int_0^{s_{\rho(770)}} ds_1 \times \int_0^{s_\pi} ds_2$$
$$\frac{\mathrm{Im}^2 F_{V(i)}^{\mathcal{O}_1}\left[s_2,s_1,\bar{P}^2 = m_\pi^2\right]_{\mathrm{QCD}}}{(s_1 - p_V^2)(s_2 - p_\pi^2)}, \quad (33)$$

where have used the polarization summation formula for $\epsilon_\mu(p_V,\lambda)$

$$\sum_\lambda \epsilon^\mu(p_V,\lambda)\epsilon^{*\nu}(p_V,\lambda) = -g^{\mu\nu} + \frac{p_V^\mu p_V^\nu}{m_{\rho(770)}^2}. \quad (34)$$

All the momentum contractions on the left-hand side of Eq. (33) should be expressed by the squared momenta, namely $p_V^2, p_\pi^2, q^2 = 0, k^2 = 0$ and $\bar{P}^2 = m_\pi^2$. Note that the Borel transformation for any polynomial of p_V^2 and p_π^2 is zero. Thus we have to use the trick $p_V^2 = -(m_{\rho(770)}^2 - p_V^2) + m_{\rho(770)}^2$ and $p_\pi^2 = -(m_\pi^2 - p_V^2) + m_\pi^2$ to express the left-hand side of Eq. (33) as

$$\frac{C_1}{\left(m_\pi^2 - p_\pi^2\right)\left(m_{\rho(770)}^2 - p_V^2\right)} + \frac{C_2}{\left(m_\pi^2 - p_\pi^2\right)}$$
$$+ \frac{C_3}{\left(m_{\rho(770)}^2 - p_V^2\right)} + C_4, \quad (35)$$

where the C_i are independent of p_π^2 and p_V^2. After the double Borel transformation in terms of p_V^2 and p_π^2, the $C_{2,3,4}$ terms

vanish. Then we obtain

$$
T_{\mathcal{O}_1}^{\rho(770)} = -\int_0^{s_{\rho(770)}} ds_1 \int_0^{s_\pi} ds_2
$$

$$
\times \frac{2i\, m_V e^{(m_V^2 - s_1)/M_1^2} e^{(m_\pi^2 - s_2)/M_2^2}}{\pi^2 f_\pi f_V (m_B^4 - 2m_B^2 m_\pi^2 - 2m_B^2 m_V^2 + m_\pi^4 + m_V^4)}
$$

$$
\times \left\{ \mathrm{Im}^2 F_{V(1)}^{\mathcal{O}_1} \left[s_2, s_1, m_\pi^2 \right]_{\mathrm{QCD}} \right.
$$

$$
\left. + \left(\frac{m_B^2}{m_V^2} - \frac{m_\pi^2}{m_V^2} - 1 \right) \mathrm{Im}^2 F_{V(2)}^{\mathcal{O}_1} \left[s_2, s_1, m_\pi^2 \right]_{\mathrm{QCD}} \right\},
$$

$$
\tag{36}
$$

$$
T_{\mathcal{O}_1}^{\rho(770)'} = -\int_0^{s_{\rho(770)}} ds_1 \int_0^{s_\pi} ds_2
$$

$$
\times \frac{2i\, m_V e^{(m_{\rho(770)}^2 - s_1)/M_1^2} e^{(m_\pi^2 - s_2)/M_2^2}}{\pi^2 f_\pi f_V (m_B^4 - 2m_B^2 m_\pi^2 - 2m_B^2 m_V^2 + m_\pi^4 + m_V^4)}
$$

$$
\times \left\{ \left(-\frac{m_B^2}{m_V^2} + \frac{m_\pi^2}{m_V^2} + 2 \right) \mathrm{Im}^2 F_{V(1)}^{\mathcal{O}_1} \left[s_2, s_1, m_\pi^2 \right]_{\mathrm{QCD}} \right.
$$

$$
+ \left(\frac{m_B^4}{m_V^4} - 2\frac{m_B^2 m_\pi^2}{m_V^4} - \frac{m_B^2}{m_V^2} + \frac{m_\pi^4}{m_V^4} - 3\frac{m_\pi^2}{m_V^2} \right)
$$

$$
\left. \times \mathrm{Im}^2 F_{V(2)}^{\mathcal{O}_1} \left[s_2, s_1, m_\pi^2 \right]_{\mathrm{QCD}} \right\}.
$$

$$
\tag{37}
$$

We have denoted the mass and decay constant of the $\rho(770)$ by m_V and f_V, in order.

In the second step, we keep both the $\rho(770)$ and the $\rho(1450)$, then similarly to Eq. (29), the sum rule equation becomes

$$
i\, \frac{m_{\rho(770)} f_{\rho(770)} f_\pi (p-q)_\alpha}{\left((q+k)^2 - m_{\rho(770)}^2 \right) \left((p-q)^2 - m_\pi^2 \right)}
$$

$$
\times \sum_\lambda \epsilon_\beta (q+k, \lambda) \langle \rho(770)(q+k, \lambda)\pi(p-q)|\mathcal{O}_1(0)|B(p)\rangle
$$

$$
+ i\, \frac{m_{\rho(1450)} f_{\rho(1450)} f_\pi (p-q)_\alpha}{\left((q+k)^2 - m_{\rho(1450)}^2 \right) \left((p-q)^2 - m_\pi^2 \right)}
$$

$$
\times \sum_\lambda \epsilon_\beta' (q+k, \lambda) \langle \rho(1450)(q+k, \lambda)\pi(p-q)|\mathcal{O}_1(0)|B(p)\rangle
$$

$$
= \frac{1}{\pi^2} \int_0^{s_{\rho(1450)}} ds_1
$$

$$
\times \int_0^{s_\pi} ds_2 \frac{\mathrm{Im}^2 \Pi_{\alpha\beta}^{V\mathcal{O}_1} \left[s_2, s_1, \bar{P}^2 = m_\pi^2 \right]_{\mathrm{QCD}}}{(s_1 - (q+k)^2)(s_2 - (p-q)^2)}.
$$

$$
\tag{38}
$$

Note that now the upper limit of the s_1 integration is $s_{\rho(1450)} = m_{\rho(1570)}^2$. The parameterization for the second matrix element above is similar to Eq. (31). Using the results from Eqs. (36) and (37), we can obtain the expression for $T_{\mathcal{O}_1}^{\rho(1450)}$:

$$
T_{\mathcal{O}_1}^{\rho(1450)}
$$

$$
= -\frac{f_V e^{(m_V'^2 - m_V^2)/M_1^2}\, T_{\mathcal{O}_1}^{\rho(770)}}{2 f_V' m_V m_V' (m_B^4 - 2m_B^2 (m_\pi^2 + m_V'^2) + m_\pi^4 + m_V'^4)}
$$

$$
\times \left[m_B^4 (m_V^2 + m_V'^2) - 2m_B^2 (m_\pi^2 + m_V^2)(m_V^2 + m_V'^2) \right.
$$

$$
+ m_\pi^4 (m_V^2 + m_V'^2)
$$

$$
\left. + 2m_\pi^2 (m_V^4 - m_V'^2 m_V'^2) + 2m_V^4 m_V'^2 \right]
$$

$$
+ \frac{f_V m_V e^{(m_V'^2 - m_V^2)/M_1^2} (m_B^2 - m_\pi^2)(m_V^2 - m_V'^2)}{2 f_V' m_V' (m_B^4 - 2m_B^2 (m_\pi^2 + m_V'^2) + m_\pi^4 + m_V'^4)} T_{\mathcal{O}_1}^{\rho(770)'}
$$

$$
- \int_0^{s_{\rho(1450)}} ds_1 \int_0^{s_\pi} ds_2
$$

$$
\times \frac{2i\, m_V' e^{(m_V'^2 - s_1)/M_1^2} e^{(m_\pi^2 - s_2)/M_2^2}}{\pi^2 f_\pi f_V' (m_B^4 - 2m_B^2 m_\pi^2 - 2m_B^2 m_V'^2 + m_\pi^4 + m_V'^4)}
$$

$$
\times \left\{ \mathrm{Im}^2 F_{V(1)}^{\mathcal{O}_1} \left[s_2, s_1, m_\pi^2 \right]_{\mathrm{QCD}} + \left(\frac{m_B^2}{m_V'^2} - \frac{m_\pi^2}{m_V'^2} - 1 \right) \right.
$$

$$
\left. \times \mathrm{Im}^2 F_{V(2)}^{\mathcal{O}_1} \left[s_2, s_1, m_\pi^2 \right]_{\mathrm{QCD}} \right\},
$$

$$
\tag{39}
$$

where m_V' and f_V' denote the mass and decay constant of the $\rho(1450)$, respectively. We do not give the expression for $T_{\mathcal{O}_1}^{\rho(1450)'}$ since it is irrelevant when $k \to 0$ at the end.

3.3 $B^- \to f_2(1270)\pi^-$ induced by $\mathcal{O}_1$

In this subsection we consider the process of $B^- \to f_2(1270)\pi^-$, where the resonance is a tensor particle. Note that after introducing the auxiliary momentum k, the $B^- \to f_2(1270)\pi^-$ matrix element should be parameterized by three terms:

$$
\langle f_2(1270)(p_T, \lambda)\pi(p_\pi)|\mathcal{O}_1(0)|B(p)\rangle = \epsilon_{\mu\nu}^*(p_T, \lambda)
$$

$$
\times \left\{ T_{\mathcal{O}_1}^{f_2(1270)} p_\pi^\mu p_\pi^\nu + \frac{1}{2} T_{\mathcal{O}_1}^{f_2(1270)'} \right.
$$

$$
\left. \times \left(p_\pi^\mu k^\nu + p_\pi^\nu k^\mu \right) + T_{\mathcal{O}_1}^{f_2(1270)''} k^\mu k^\nu \right\},
$$

$$
\tag{40}
$$

where $p_T = q + k$ and $p_\pi = p - q$, and the constraints $\epsilon_{\mu\nu} = \epsilon_{\nu\mu}, \epsilon_{\mu\nu} p_T^\mu = 0$ are used. Accordingly, when defining the correlation function, we have to introduce three projection tensors:

$$
\Gamma_i^{\mu\nu} = \left\{ m_B^2 v^\mu v^\nu,\ \frac{1}{2} m_B (v^\mu q^\nu + v^\nu q^\mu),\ q^\mu q^\nu \right\}
$$

$$
(i = 1, 2, 3)
$$

$$
\tag{41}
$$

to contract with the tensor current. The projected correlation function then reads

$$
\Gamma_i^{\beta\sigma} \Pi_{\alpha\beta\sigma}^{T\mathcal{O}_1}(p, q, k)
$$

$$
= i^2 \int d^4x\, d^4y\, e^{i(p-q)\cdot y} e^{i(q+k)\cdot x}
$$

$$
\times \Gamma_i^{\beta\sigma} \langle 0|T \left\{ j_{5\alpha}^\pi(y)\mathcal{O}_1(0) j_{\beta\sigma}^T(x) \right\} |B(p)\rangle,
$$

$$
\tag{42}
$$

where

$$j_{\beta\sigma}^T = \frac{i}{4\sqrt{2}} \left(\bar{u}\gamma_\beta \overset{\leftrightarrow}{\partial}_\sigma u + \bar{u}\gamma_\sigma \overset{\leftrightarrow}{\partial}_\beta u \right) - (u \leftrightarrow d) \qquad (43)$$

with $\overset{\leftrightarrow}{\partial} = \overset{\rightarrow}{\partial} - \overset{\leftarrow}{\partial}$ [31]. Its parameterization is the same as that of Eq. (20) and Eq. (32) :

$$\Gamma_i^{\beta\sigma} \Pi_{\alpha\beta\sigma}^{T\mathcal{O}_1}(p, q, k)$$
$$= (p - q)_\alpha F_{T(i)}^{\mathcal{O}_1} + q_\alpha G_{T(i)}^{\mathcal{O}_1} + k_\alpha H_{T(i)}^{\mathcal{O}_1}$$
$$+ \epsilon_{\alpha\beta\rho\sigma} p^\beta q^\rho k^\sigma I_{T(i)}^{\mathcal{O}_1}. \qquad (44)$$

Then we follow the similar procedure to insert a state with the same quantum numbers of the $f_2(1270)$. Using the definition of the $f_2(1270)$ decay constant:

$$\langle 0 | j_{\mu\nu}^V(0) | f_2(1270)(p, \lambda) \rangle = m_{f_2(1270)}^2 f_{f_2(1270)} \epsilon_{\mu\nu}(p, \lambda), \qquad (45)$$

as well as the summation formula of the polarization tensor:

$$\sum_\lambda \epsilon_{\mu\nu}(p_T, \lambda) \epsilon_{\rho\sigma}^*(p_T, \lambda)$$
$$= \frac{1}{2} P_{\mu\rho}^T P_{\nu\sigma}^T + \frac{1}{2} P_{\mu\sigma}^T P_{\nu\rho}^T - \frac{1}{3} P_{\mu\nu}^T P_{\rho\sigma}^T \qquad (46)$$

with $P_{\mu\nu}^T = g_{\mu\nu} - p_{T\mu} p_{T\nu}/m_{f_2(1270)}^2$, we obtain

$$T_{\mathcal{O}_1}^{f_2(1270)}$$
$$= \frac{4i}{\pi^2 f_\pi f_T m_B^6} e^{m_V^2/M_1^2} e^{m_\pi^2/M_2^2} \int_0^{s_{f_2(1270)}} ds_1 \int_0^{s_\pi} ds_2$$
$$\times \sum_{i=1}^3 \frac{1}{\xi_i(\lambda_\pi, \lambda_T)} e^{-s_1/M_1^2} e^{-s_2/M_2^2}$$
$$\times \text{Im}^2 F_{T(i)}^{\mathcal{O}_1} \left[s_2, s_1, m_\pi^2 \right]_{\text{QCD}} \qquad (47)$$

where $\lambda_\pi = m_\pi^2/m_B^2$, $\lambda_T = m_T^2/m_B^2$. The expression for $\xi_i(\lambda_\pi, \lambda_T)$ can be found in Appendix A. The threshold is chosen as $s_{f_2(1270)} = m_{f_2'(1520)}^2$. The $T_{\mathcal{O}_1}^{f_2(1270)'}$ and $T_{\mathcal{O}_1}^{f_2(1270)''}$ are irrelevant when taking $k \to 0$ so that they are not shown here.

4 Quark-gluon level calculation in LCSR

In this section we will present the calculation of the correlation functions given in Eqs. (17), (27) and (42) on the quark-gluon level. Since the calculation procedure of these three cases is similar, we only take Eq. (27) as an example to give a detailed derivation.

There are two diagrams contributing to the correlation function. As shown in Fig. 1, (a) is the W-exchange diagram, and (b) is the W-emission diagram. Note that the factorization given in Eq. (9) only involves the contribution from W-emission, while it misses the W-exchange effect. We will give a detailed calculation for the diagram (a) within the light-cone expansion, the calculation for the diagram (b) is similar and will more be given in any detail here. The definition of the leading twist light-cone distribution amplitude (LCDA) is [32–35]:

$$\langle 0 | \bar{u}_\alpha(x)[x, 0] b_{\nu\beta}(0) | \bar{B}_\nu^0 \rangle$$
$$= -\frac{i f_B m_B}{4} \int_0^\infty d\omega e^{-i\omega\nu\cdot x}$$
$$\times \left\{ (1 + \slashed{\nu}) \left(\phi_+^B(\omega) + \frac{\phi_-^B(\omega) - \phi_+^B(\omega)}{2\nu\cdot x} \slashed{x} \right) \gamma_5 \right\}_{\beta\alpha} . \qquad (48)$$

According to the experience from Ref. [36], the contribution from the next-to-leading order twist LCDAs is expected to be one order smaller than that of the leading twist LCDAs, so we do not consider them in this work. Note that the first term above has no terms $\sim \slashed{x}$ and $\sim \nu\cdot x$, thus its contribution to the correlation function can be derived directly as

$$\Gamma_i^\beta \Pi_{\alpha\beta}^{\mathcal{O}_1}(p, q, k)_{(a)x^0}$$
$$= \frac{f_B m_B}{4\sqrt{2}} \int_0^\infty d\omega \, \phi_+^B(\omega) \frac{1}{(p_\pi - \omega\nu)^2 - m^2}$$
$$\times \int \frac{d^4 k_l}{(2\pi)^4} \frac{1}{k_l^2 - m^2} \frac{1}{(k_l + p_V)^2 - m^2}$$
$$\times \text{tr} \left[\gamma_\alpha \gamma_5 (\slashed{p}_\pi - \omega\slashed{\nu} + m) \gamma^\mu (1 - \gamma_5)(\slashed{k}_l + m) \right.$$
$$\times \Gamma_i \cdot \gamma (\slashed{p}_V + \slashed{k}_l + m) \gamma_\mu (1 - \gamma_5)(1 + \slashed{\nu}) \gamma_5 \right], \qquad (49)$$

where m denotes the light (u, d) quark mass which will be approximated as zero in this work. The double imaginary part $\text{Im}^2 \equiv \text{Im}_{p_\pi^2} \text{Im}_{p_V^2}$ of the correlation function is proportional to the discontinuity in terms of the invariants p_V^2 and p_π^2, and the discontinuity only comes from the denominators of the integrand in Eq. (49). It is found that p_V^2 and p_π^2 do not appear in the same denominator, so we can extract the two imaginary parts independently. The imaginary part in terms of p_V^2 can be obtained by using the cutting-rule for the bubble in the lower right corner of diagram (a). Thus the loop integration of $d^4 k_l$ in this bubble is replaced by a two-body phase space integration after using the cutting-rule. The tensor basis used for the two-body phase space integration can be found in Appendix B. On the other hand, the imaginary part in terms
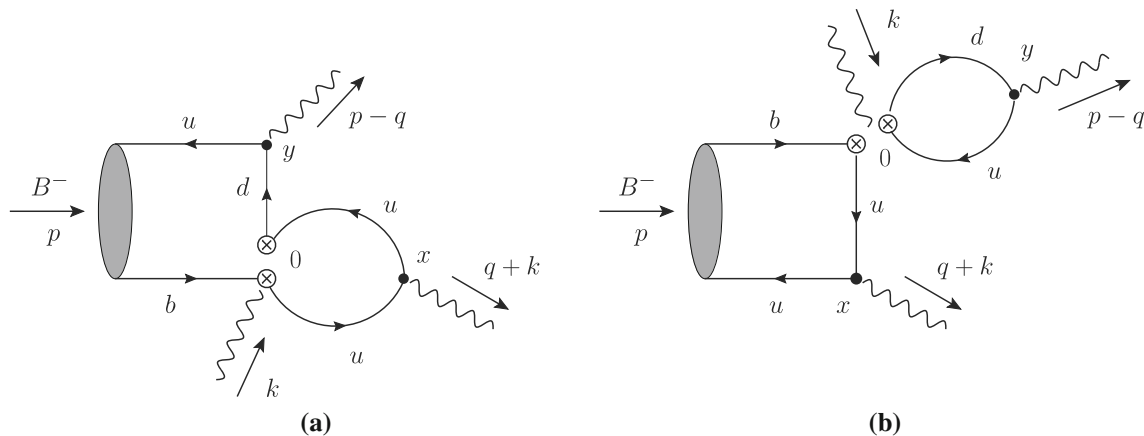
Fig. 1 Feynman diagrams of the correlation function given in Eq. (27). **a** is the W-exchange diagram, and **b** is the W-emission diagram. The gray bubble denotes the B meson, the pairs of crossed circles represent the four-quark operators $\mathcal{O}_1$, the black dots denote the interpolating currents for final particles and the wiggly lines denote the incoming or outgoing momentum flows

of p_π^2 only comes from the first denominator:

$$\text{Im}_{p_\pi^2} \frac{1}{(p_\pi - \omega v)^2 - m^2}$$
$$= (-\pi)\delta\left[\left(1 - \frac{\omega}{m_B}\right) p_\pi^2 - \omega m_B + \omega^2 - m^2\right]. \quad (50)$$

After calculating the double imaginary part of Eq. (49), and extracting the coefficient of $p_{\pi\alpha}$, we obtain the corresponding contribution to the imaginary part of $F_{V(i)}^{\mathcal{O}_1}$:

$$\text{Im}^2 F_{V(i)}^{\mathcal{O}_1}(p_\pi^2, p_V^2)_{(a)x^0}$$
$$= -i\sqrt{2}\pi^5 f_B m_B \int_0^\infty d\omega\, \phi_+^B(\omega)\delta$$
$$\times \left[\left(1 - \frac{\omega}{m_B}\right) p_\pi^2 - \omega m_B + \omega^2 - m^2\right]$$
$$\times \int d\Phi_2(p_V^2, k_2, k_3)\, \mathcal{M}_{(a)}^{V(i)}(p_\pi, k_2, k_3), \quad (51)$$

where we have used the requirements $k^2 = q^2 = 0$ and $\bar{P}^2 = m_\pi^2$ as argued in the last section. Further, k_2, k_3 are the momenta of the two light quarks in the bubble. The expression of $\mathcal{M}_{(a)x^0}^{(i)}(p_\pi, k_2, k_3)$ can be found in Appendix C. The two-body phase space integration is defined as

$$\int d\Phi_2(p_V^2, k_2, k_3) = \int \frac{d^3\vec{k}_2}{(2\pi)^3} \frac{1}{2E_{k_2}} \frac{d^3\vec{k}_3}{(2\pi)^3} \frac{1}{2E_{k_3}}$$
$$\delta^{(4)} \times (p_V - k_2 - k_3)\delta(k_2^2 - m^2)\delta(k_3^2 - m^2). \quad (52)$$

The result of the two-body phase space integration in Eq. (58) is given in Appendix B.

For the term proportional to $\not{x}/v \cdot x$ in Eq. (48), we use the following trick to remove the troublesome denominator

$v \cdot x$. We define:

$$\tilde{\phi}_\pm^B(\omega) = \int_0^\omega d\tau\, \phi_\pm^B(\tau) \quad \longrightarrow \quad \phi_\pm^B(\omega) = \frac{d}{d\omega}\tilde{\phi}_\pm^B(\omega) \quad (53)$$

so that the integration on the spacetime coordinate can be simplified as

$$\int_0^\infty d\omega\, e^{-i\omega v \cdot x} \frac{\phi_\pm^B(\omega)}{v \cdot x} \cdots = i \int_0^\infty d\omega\, e^{-i\omega v \cdot x} \tilde{\phi}_\pm^B(\omega) \cdots. \quad (54)$$

Then, the only x dependence comes from the term $\sim \exp[i(q+k)\cdot x]$ in the definition of the correlation function in Eq. (27). Thus in the momentum space the term $\sim \not{x}$ can be replaced by $-i\gamma^\rho \partial/\partial k_\rho$. However, a direct result from such an operation is the appearance of a higher power denominator, $1/\left[(p_V + k_l)^2 - m^2\right]^2$, when the extra derivative operates on the momentum k. Thus we have to use the following trick

$$\text{Disc}_{p_V^2}\left[\int \frac{1}{\left[(p_V + k_l)^2 - m^2\right]^2}(\cdots)\right]$$
$$= (-2\pi i)\frac{\partial}{\partial\Omega}\int \delta\left[(p_V + k_l)^2 - \Omega\right](\cdots)\Big|_{\Omega=m^2}, \quad (55)$$

to extract the discontinuity. Note that the auxiliary parameter Ω should be set to m at the end of all the integrations. The imaginary part of the $F_{V(i)}^{\mathcal{O}_1}$ from the $\not{x}$ term of Eq. (48) then is

$$\text{Im}^2 F_{V(i)}^{\mathcal{O}_1}(p_\pi^2, p_V^2)_{(a)\not{x}}$$
$$= \frac{-i\pi^5 f_B m_B}{\sqrt{2}} \int_0^\infty d\omega \left(\tilde{\phi}_-^B(\omega) - \tilde{\phi}_+^B(\omega)\right)$$

$$\delta \times \left[\left(1 - \frac{\omega}{m_B} \right) p_\pi^2 - \omega m_B + \omega^2 - m^2 \right]$$

$$\times \left\{ \int d\Phi_2(p_V^2, k_2, k_3) \, \mathcal{N}_{(a)}^{V(i)}(p_\pi, k_2, k_3) \right.$$

$$\left. + \frac{\partial}{\partial \Omega} \int d\Phi_2(p_V^2, k_2, k_3)_\Omega \, \mathcal{L}_{(a)}^{V(i)}(p_\pi, k_2, k_3) \right\} \Big|_{\Omega = m^2}, \tag{56}$$

where

$$\int d\Phi_2(p_V^2, k_2, k_3)_\Omega$$

$$= \int \frac{d^3 \vec{k}_2}{(2\pi)^3} \frac{1}{2E_{k_2}} \frac{d^3 \vec{k}_3}{(2\pi)^3} \frac{1}{2E_{k_3}}$$

$$\times \delta^{(4)}(p_V - k_2 - k_3) \delta(k_2^2 - m^2) \delta(k_3^2 - \Omega). \tag{57}$$

The calculation for the contribution from diagram (b) in Fig. 1 is similar except that the loop bubble is at the corner of pion the vertex. Therefore, now the imaginary part in terms of p_π^2 comes from the loop integration, while the imaginary part in terms of p_V^2 comes from the rest of the propagator. The imaginary part from the diagram (b) follows as

$$\text{Im}^2 F_{V(i)}^{\mathcal{O}_1}(p_\pi^2, p_V^2)_{(b)}$$

$$= 3i\sqrt{2}\pi^5 f_B m_B \int_0^\infty d\omega \, \phi_+^B(\omega)$$

$$\delta \times \left[\left(1 - \frac{\omega}{m_B} \right) p_V^2 - \omega m_B + \frac{\omega}{m_B} m_\pi^2 + \omega^2 - m^2 \right]$$

$$\times \int d\Phi_2(p_V^2, k_2, k_3) \, \mathcal{M}_{(b)}^{V(i)}(p_\pi, k_2, k_3)$$

$$+ \frac{3i\pi^5 f_B m_B}{\sqrt{2}} \int_0^\infty d\omega \left(\tilde{\phi}_-^B(\omega) - \tilde{\phi}_+^B(\omega) \right)$$

$$\delta \times \left[\left(1 - \frac{\omega}{m_B} \right) p_V^2 - \omega m_B + \frac{\omega}{m_B} m_\pi^2 + \omega^2 - m^2 \right]$$

$$\times \int d\Phi_2(p_V^2, k_2, k_3) \, \mathcal{N}_{(b)}^{V(i)}(p_\pi, k_2, k_3).$$

$$+ \frac{3i\pi^5 f_B m_B}{\sqrt{2}} \int_0^\infty d\omega \left(\tilde{\phi}_-^B(\omega) - \tilde{\phi}_+^B(\omega) \right)$$

$$\times \frac{\partial}{\partial \Omega} \left\{ \delta \left[\left(1 - \frac{\omega}{m_B} \right) p_V^2 - \omega m_B + \frac{\omega}{m_B} m_\pi^2 + \omega^2 - \Omega \right] \right.$$

$$\left. \times \int d\Phi_2(p_V^2, k_2, k_3)_\Omega \, \mathcal{L}_{(b)}^{V(i)}(p_\pi, k_2, k_3) \right\} \Big|_{\Omega = m^2}. \tag{58}$$

For the cases of scalar and tensor resonances, the imaginary part of the corresponding correlation function can be derived similarly. For simplicity, we only present the results of the corresponding calculations. For the scalar resonance, only the diagram (b) contributes in the approximation $m = 0$, and the expression of $\text{Im}^2 F_S^{\mathcal{O}_1}$ reads

$$\text{Im}^2 F_S^{\mathcal{O}_1}(p_\pi^2, p_S^2)$$

$$= 3\sqrt{2}i \sin\alpha \, \pi^5 f_B m_B \int_0^\infty d\omega \, \phi_+^B(\omega)$$

$$\delta \times \left[\left(1 - \frac{\omega}{m_B} \right) p_T^2 - \omega m_B + \frac{\omega}{m_B} m_\pi^2 + \omega^2 - m^2 \right]$$

$$\times \int d\Phi_2(p_S^2, k_2, k_3) \, \mathcal{M}_{(b)}^S(p_\pi, k_2, k_3)$$

$$+ \frac{3i \sin\alpha \, \pi^5 f_B m_B}{\sqrt{2}} \int_0^\infty d\omega \left(\tilde{\phi}_-^B(\omega) - \tilde{\phi}_+^B(\omega) \right)$$

$$\delta \times \left[\left(1 - \frac{\omega}{m_B} \right) p_S^2 - \omega m_B + \frac{\omega}{m_B} m_\pi^2 + \omega^2 - m^2 \right]$$

$$\times \int d\Phi_2(p_S^2, k_2, k_3) \, \mathcal{N}_{(b)}^S(p_\pi, k_2, k_3).$$

$$+ \frac{3i \sin\alpha \, \pi^5 f_B m_B}{\sqrt{2}} \int_0^\infty d\omega \left(\tilde{\phi}_-^B(\omega) - \tilde{\phi}_+^B(\omega) \right)$$

$$\times \frac{\partial}{\partial \Omega} \left\{ \delta \left[\left(1 - \frac{\omega}{m_B} \right) p_S^2 - \omega m_B + \frac{\omega}{m_B} m_\pi^2 + \omega^2 - \Omega \right] \right.$$

$$\left. \times \int d\Phi_2(p_T^2, k_2, k_3)_\Omega \, \mathcal{L}_{(b)}^S(p_\pi, k_2, k_3) \right\} \Big|_{\Omega = m^2}. \tag{59}$$

For the tensor resonance, like the case of vector resonance, both the diagram (a) and (b) contribute to the correlation function. The expression of $\text{Im}^2 F_{T(i)}^{\mathcal{O}_1}$ is given by

$$\text{Im}^2 F_{T(i)}^{\mathcal{O}_1}(p_\pi^2, p_T^2)$$

$$= \text{Im}^2 F_{T(i)}^{\mathcal{O}_1}(p_\pi^2, p_T^2)_{(a)} + \text{Im}^2 F_{T(i)}^{\mathcal{O}_1}(p_\pi^2, p_T^2)_{(b)}, \tag{60}$$

$$\text{Im}^2 F_{T(i)}^{\mathcal{O}_1}(p_\pi^2, p_T^2)_{(a)}$$

$$= \frac{-i\pi^5 f_B m_B}{\sqrt{2}} \int_0^\infty d\omega \, \phi_+^B(\omega) \delta$$

$$\times \left[\left(1 - \frac{\omega}{m_B} \right) p_\pi^2 - \omega m_B + \omega^2 - m^2 \right]$$

$$\times \int d\Phi_2(p_T^2, k_2, k_3) \, \mathcal{M}_{(a)}^{T(i)}(p_\pi, k_2, k_3)$$

$$- \frac{i\pi^5 f_B m_B}{2\sqrt{2}} \int_0^\infty d\omega \left(\tilde{\phi}_-^B(\omega) - \tilde{\phi}_+^B(\omega) \right)$$

$$\delta \times \left[\left(1 - \frac{\omega}{m_B} \right) p_\pi^2 - \omega m_B + \omega^2 - m^2 \right]$$

$$\times \int d\Phi_2(p_T^2, k_2, k_3) \, \mathcal{N}_{(a)}^{T(i)}(p_\pi, k_2, k_3).$$

$$- \frac{i\pi^5 f_B m_B}{2\sqrt{2}} \int_0^\infty d\omega \left(\tilde{\phi}_-^B(\omega) - \tilde{\phi}_+^B(\omega) \right)$$

$$\delta \times \left[\left(1 - \frac{\omega}{m_B} \right) p_\pi^2 - \omega m_B + \omega^2 - m^2 \right]$$

$$\times \frac{\partial}{\partial \Omega} \left\{ \int d\Phi_2(p_T^2, k_2, k_3)_\Omega \, \mathcal{L}_{(a)}^{T(i)}(p_\pi, k_2, k_3) \right\} \Big|_{\Omega = m^2}, \tag{61}$$

$$\text{Im}^2 F_{T(i)}^{\mathcal{O}_1}(p_\pi^2, p_T^2)_{(b)}$$

$$= \frac{3i\pi^5 f_B m_B}{\sqrt{2}} \int_0^\infty d\omega \, \phi_+^B(\omega) \delta$$

$$\times \left[\left(1 - \frac{\omega}{m_B} \right) p_T^2 - \omega m_B + \frac{\omega}{m_B} m_\pi^2 + \omega^2 - m^2 \right]$$

$$\times \int d\Phi_2(p_T^2, k_2, k_3) \, \mathcal{M}_{(b)}^{T(i)}(p_\pi, k_2, k_3)$$

$$+ \frac{3i\pi^5 f_B m_B}{2\sqrt{2}} \int_0^\infty d\omega \left(\tilde{\phi}^B_-(\omega) - \tilde{\phi}^B_+(\omega) \right)$$

$$\delta \times \left[\left(1 - \frac{\omega}{m_B} \right) p_T^2 - \omega m_B + \frac{\omega}{m_B} m_\pi^2 + \omega^2 - m^2 \right]$$

$$\times \int d\Phi_2(p_T^2, k_2, k_3) \, \mathcal{N}^{T(i)}_{(b)}(p_\pi, k_2, k_3).$$

$$+ \frac{3i\pi^5 f_B m_B}{2\sqrt{2}} \int_0^\infty d\omega \left(\tilde{\phi}^B_-(\omega) - \tilde{\phi}^B_+(\omega) \right)$$

$$\times \frac{\partial}{\partial \Omega} \left\{ \delta \left[\left(1 - \frac{\omega}{m_B} \right) p_T^2 - \omega m_B + \frac{\omega}{m_B} m_\pi^2 + \omega^2 - \Omega \right] \right.$$

$$\left. \times \int d\Phi_2(p_T^2, k_2, k_3)_\Omega \, \mathcal{L}^{T(i)}_{(b)}(p_\pi, k_2, k_3) \right\} \Big|_{\Omega = m^2}. \quad (62)$$

The explicit expressions for the $\mathcal{M}, \mathcal{N}$ and $\mathcal{L}$ functions are given in Appendix C. The LCSR calculation for the $\mathcal{O}_{2\ldots7}$ matrix elements can be done similarly, so we will not explicitly present them here.

5 Phenomenological results

5.1 Numerical results for $\langle R, M \, |\mathcal{O}_i| \, B^- \rangle$

First, we list the values of all the parameters used in this work. All the mass parameters are: $m_u = m_d = m = 0$, $m_s = 93$ MeV$(\mu = 2$ GeV$)$, $m_\pi = 0.139$ GeV, $m_{K^\pm} = 0.496$ GeV, $m_B = 5.28$ GeV, $m_{f_0(980)} = 0.99$ GeV, $m_{\rho(770)} = 0.775$ GeV, $m_{\rho(1450)} = 1.465$ GeV, $m_{f_2(1270)} = 1.275$ GeV, $m_{\phi(1020)} = 1.02$ GeV, $m_{K_0^*(1430)} = 1.43$ GeV, $m_{K^*(892)} = 0.892$ GeV, $m_{f_0(1370)} = 1.37$ GeV, $m_{\rho(1570)} = 1.57$ GeV, $m_{f_2'(1525)} = 1.52$ GeV, $m_{\phi(1680)} = 1.68$ GeV, $m_{K^*(1410)} = 1.41$ GeV, $m_{K_0(1950)} = 1.95$ GeV [37].

The value of all the decay constants used here are: $f_B = 0.207$ GeV [38], $f_\pi = 0.13$ GeV [39], $f_{\rho(770)} = 0.21$ GeV [40], $f_{\rho(1450)} = 0.186$ GeV [11], $f_{f_2(1270)} = 0.102$ GeV [41], $f_{\phi(1020)} = 0.241$ GeV [42], $f_K = 0.11$ GeV [43], $f_{K^*(892)} = 0.204$ GeV [40], $f_{K_0^*(1430)} = 0.427$ GeV [44].

For the leading twist LCDAs of the B meson, their expressions as well as the associated coefficients are taken from [35]

$$\phi_B^+(\omega) = \frac{\omega}{\omega_0^2} e^{-\omega/\omega_0},$$

$$\phi_B^-(\omega) = \frac{1}{\omega_0} e^{-\omega/\omega_0} - \frac{\lambda_E^2 - \lambda_H^2}{9\,\omega_0^3}$$

$$\times \left[1 - 2\left(\frac{\omega}{\omega_0}\right) + \frac{1}{2}\left(\frac{\omega}{\omega_0}\right)^2 \right] e^{-\omega/\omega_0}, \quad (63)$$

where $\omega_0 = (2/3)\bar{\Lambda}$, $\lambda_H^2 = 2\lambda_E^2$ and $\lambda_H = \bar{\Lambda}$, with $\bar{\Lambda} = m_B - m_b = 0.45$ GeV in the heavy quark limit [45]. The numerical result for the $T^R_{\mathcal{O}_i}$ are listed in Table 1, where s_R is the threshold parameter for the resonance, while s_π and s_K are the threshold parameters for the non-resonant particle, namely the final pion or kaon. The Borel parameters are

chosen in the region where the numerical values of $T^R_{\mathcal{O}_i}$ are stable. The errors of the $T^R_{\mathcal{O}_i}$ come from this uncertainties of the Borel parameters.

5.2 Branching fractions for the $B^- \to K^+ K^- \pi^-$ decay

In a first step, we give a general formula for the calculation of three-body decays. We define three dimensionless Lorentz invariant variables:

$$x_1 = \frac{2p_B \cdot p_{\pi^-}}{m_B^2}, \quad x_2 = \frac{2p_B \cdot p_{K^+}}{m_B^2}, \quad x_3 = \frac{2p_B \cdot p_{K^-}}{m_B^2}. \quad (64)$$

Thus, the three Mandelstam variables can be expressed by the x_i as

$$s = (p_{K^+} + p_{K^-})^2 = m_B^2(1 + \lambda_\pi - x_1),$$

$$t = (p_{\pi^-} + p_{K^+})^2 = m_B^2(1 + \lambda_K - x_3),$$

$$u = (p_{\pi^-} + p_{K^-})^2 = m_B^2(1 + \lambda_K - x_2), \quad (65)$$

where $\lambda_K = m_K^2/m_B^2$ and the x_i satisfy $x_1 + x_2 + x_3 = 2$. Each Lorentz invariant quantity in the decay amplitude can be expressed by the x_i. In the rest-frame of the B meson, the formula for the three-body decay width is

$$\Gamma\left[B^- \to K^+ K^- \pi^-\right] = \frac{m_B}{256\pi^3} \int_S dx_2 dx_3 \, \mathcal{A}(x_1, x_2, x_3), \quad (66)$$

where $\mathcal{A}(x_1, x_2, x_3)$ is the total decay amplitude. The integration region S is:

$$-1 < \frac{(x_2 x_3 - 2x_2 - 2x_3 + 2) + 4\lambda_K - 2\lambda_\pi}{\sqrt{x_2^2 - 4\lambda_K}\sqrt{x_3^2 - 4\lambda_K}} < 1, \quad (67)$$

as well as $2\sqrt{\lambda_K} < x_2 < 2(1 - \sqrt{\lambda_K} - \sqrt{\lambda_\pi})$ and $2\sqrt{\lambda_K} < x_3 < 2(1 - \sqrt{\lambda_\pi}) - x_2$. The boundary values ± 1 in Eq. (67) are reached when the direction of the K^+ three-momentum is parallel or anti-parallel with that of the K^-,

All the strong coupling constants are taken from the Eq. (2.40) of Ref. [9]. The decays widths of the resonances are [37]:

$$\Gamma_{K_0(1430)} = 0.27 \text{ GeV}, \quad \Gamma_{K^*(892)} = 0.051 \text{ GeV},$$

$$\Gamma_{\rho(1450)} = 0.4 \text{ GeV},$$

$$\Gamma_{f_2(1270)} = 0.187 \text{ GeV}, \quad \Gamma_{f_0(980)} = 0.05 \text{ GeV},$$

$$\Gamma_{\phi(1020)} = 0.0042 \text{ GeV}. \quad (68)$$

The decay branching fraction stemming from the various resonances are listed in Table 2. We also compare our

Table 1 Numerical results of the parameters for the matrix element $\langle R, M \,|\mathcal{O}_i|\, B^- \rangle$, where the threshold parameter for the final pion or kaon are $s_\pi = m^2_{\pi(1300)}$ or $s_K = m^2_{K(1460)}$ respectively

$T^R_{\mathcal{O}_i}$	Value	s_R	s_π or s_K	M_1 (GeV)	M_2 (GeV)
$T^{f_0(980)}_{\mathcal{O}_1}$	-0.23 ± 0.05 GeV^3	$m^2_{f_0(1370)}$	s_π	10 ± 1	0.4 ± 0.05
$T^{\rho(770)}_{\mathcal{O}_1}$	0.043 ± 0.013 GeV^2	$m^2_{\rho(1450)}$	s_π	5 ± 1	0.15 ± 0.05
$T^{\rho(770)}_{\mathcal{O}_2}$	0.015 ± 0.003 GeV^2	$m^2_{\rho(1450)}$	s_π	6 ± 1	0.15 ± 0.05
$T^{\rho(770)}_{\mathcal{O}_3}$	$(3 \pm 1) \times 10^{-4}$ GeV^2	$m^2_{\rho(1450)}$	s_π	6 ± 1	0.15 ± 0.05
$T^{\rho(1450)}_{\mathcal{O}_1}$	0.014 ± 0.008 GeV^2	$m^2_{\rho(1570)}$	s_π	10 ± 1	0.2 ± 0.05
$T^{f_2(1270)}_{\mathcal{O}_1}$	0.0021 ± 0.0006 GeV	$m_{f'_2(1525)}$	s_π	10 ± 1	0.15 ± 0.05
$T^{\rho(1450)}_{\mathcal{O}_2}$	0.0055 ± 0.002 GeV^2	$m^2_{\rho(1570)}$	s_π	8 ± 1	0.2 ± 0.05
$T^{\rho(1450)}_{\mathcal{O}_3}$	$(1.3 \pm 1.0) \times 10^{-4}$ GeV^2	$m^2_{\rho(1570)}$	s_π	10 ± 1	0.1 ± 0.05
$T^{\phi(1020)}_{\mathcal{O}_4}$	0.003 ± 0.0016 GeV^2	$m^2_{\phi(1680)}$	s_π	8 ± 1	0.3 ± 0.05
$T^{f_0(980)}_{\mathcal{O}_5}$	$(1.5 \pm 1.0) \times 10^{-4}$ GeV^3	$m^2_{f_0(1370)}$	s_π	5 ± 1	0.1 ± 0.05
$T^{K^*(892)}_{\mathcal{O}_6}$	0.176 ± 0.009 GeV^2	$m^2_{K^*(1410)}$	s_K	8 ± 1	3 ± 1
$T^{K_0(700)}_{\mathcal{O}_6}$	-0.093 ± 0.004 GeV^3	$m^2_{K^*_0(1430)}$	s_K	8 ± 1	0.36 ± 0.05
$T^{K_0(700)}_{\mathcal{O}_7}$	0.222 ± 0.02 GeV^3	$m^2_{K^*_0(1430)}$	s_K	6 ± 1	0.4 ± 0.05
$T^{K^*_0(1430)}_{\mathcal{O}_6}$	-0.018 ± 0.004 GeV^3	$m^2_{K_0(1950)}$	s_K	8 ± 1	0.4 ± 0.05
$T^{K_0(1430)}_{\mathcal{O}_7}$	0.14 ± 0.017 GeV^3	$m^2_{K_0(1950)}$	s_K	6 ± 1	0.45 ± 0.05

Table 2 Decay branching fraction from the various resonances (in units of 10^{-6}). We also compare our results with those from experiments $\mathcal{B}_{\text{expt}}$ and those from Ref. [9]. The $\mathcal{B}_{\text{expt}}$ are inferred from the measured branching fraction of each resonance [4] and the world averaged branching fraction of the entire decay process $\mathcal{B}_{\text{averaged}} \left(B^\pm \to \pi^\pm K^+ K^- \right) = (5.24 \pm 0.42) \times 10^{-6}$ [46]

Contribution	$\mathcal{B}_{\text{expt}}$	Ref.[9]	This work
$K^*(890)$	0.39 ± 0.05	$0.23^{+0.04}_{-0.04}$	0.014 ± 0.0015
$K^*_0(1430)$	0.23 ± 0.08	$0.71^{+0.13}_{-0.12}$	0.11 ± 0.025
$\rho(1450)$	1.61 ± 0.15	–	0.035 ± 0.04
$f_2(1270)$	0.39 ± 0.06	$0.05^{+0.01}_{-0.01}$	0.44 ± 0.24
$\phi(1020)$	0.016 ± 0.008	$0.0079^{+0.0019}_{-0.0017}$	$(0.97 \pm 0.76) \times 10^{-5}$
$f_0(980)$	–	$0.19^{+0.03}_{-0.03}$	0.12 ± 0.04

results with those from experiments $\mathcal{B}_{\text{expt}}$ and those from Ref. [9]. The $\mathcal{B}_{\text{expt}}$ are inferred from the measured branching fraction of each resonance [4] as well as the branching fraction of the entire decay process $\mathcal{B} \left(B^\pm \to \pi^\pm K^+ K^- \right) = (5.24 \pm 0.42) \times 10^{-6}$ [46]. We note that the $f_2(1270)$ and $K^*_0(1430)$ contributions obtained in this work is consistent with those from experiment. Our result for the $f_0(980)$ is consistent with that from Ref. [9], which is still waiting for future experimental tests. The $\phi(1020)$ contribution is much smaller than the other determinations, which is due to the tiny value of the Wilson coefficients a_3, a_5, a_7 and a_9 as shown in Eq. (3). The other fractions are about one order smaller than the experimental values. A possible reason for this difference is due to the uncertainty of the strong couplings,

which require further precise measurements in the future. The contribution of all the resonances considered above to the branching fraction is

$$\mathcal{B}_{\text{resonant}} \left[B^- \to K^+ K^- \pi^- \right] = (0.54 \pm 0.23) \times 10^{-6}. \quad (69)$$

Thus, the resonant contribution obtained above is only a small part of the total branching fraction $\mathcal{B} \left(B^\pm \to \pi^\pm K^+ K^- \right) = (5.24 \pm 0.42) \times 10^{-6}$ [46]. Besides the resonances, the non-resonant (NR) contributions also contribute to the decay process. In our case, it comes from the matrix element of $\mathcal{O}_1$ and $\mathcal{O}_7$. The $\mathcal{O}_1$ matrix element can be generally parameterized by

$$\langle \pi^-(p_1) K^+(p_2) K^-(p_3)|\mathcal{O}_1|B^-(p)\rangle_{NR}$$

$$= -\frac{f_\pi}{2} \left[2m_\pi^2 r + \left(m_B^2 - s_{23} - m_\pi^2 \right) \omega_+ + (s_{12} - s_{13}) \, \omega_- \right]. \tag{70}$$

The form factors r, $\omega_\pm$ are calculated within heavy meson chiral perturbation theory (HMChPT) [47]. However, since HMChPT is only reliable in the low-energy region, in the case of B decays, the final-state energy is high enough so that a direct use of HMChPT will make the amplitude blow up. There is a practical method to solve this problem. As proposed by Ref. [48], one can introduce an exponential term to suppress the amplitude at high energy, and thus the $\mathcal{O}_1$ matrix element is modified as

$$\begin{aligned}
&\langle \pi^-(p_1) K^+(p_2) K^-(p_3) | \mathcal{O}_1 | B^-(p) \rangle \\
&= \langle \pi^-(p_1) K^+(p_2) K^-(p_3) | \mathcal{O}_1 | B^-(p) \rangle_R \\
&\quad + e^{-\alpha_{NR} p_B \cdot (p_2 + p_3)} \\
&\quad \times \langle \pi^-(p_1) K^+(p_2) K^-(p_3) | \mathcal{O}_1 | B^-(p) \rangle_{NR}, \tag{71}
\end{aligned}$$

where $\alpha_{NR} = 0.16 \, \text{GeV}^{-2}$ [9]. On the other hand, we will not include the NR contribution of $\mathcal{O}_7$. The first reason is that as shown by the Eqs. (2.22)–(2.24) of Ref. [48], such NR contribution depends on the NR component of a scalar matrix element $\langle \pi^-(p_1) K^+(p_2) | \bar{d}s | 0 \rangle^{NR}$. This scalar matrix element itself has been obtained in Ref. [49]. However, its exact NR component is still unknown. Although Refs. [8,50] proposed a formula to characterize its NR component, there still exists an unknown phase parameter. Another reason is that the Wilson coefficients a_6^p and a_8^p corresponding to $\mathcal{O}_7$ are suppressed compared with that of $\mathcal{O}_1$. Thus, in practice we only consider the NR effect in the $\mathcal{O}_1$ matrix element.

The final-state interaction (FSI) of $\pi^+ \pi^- \leftrightarrow K^+ K^-$ also contributes to the decay branching fraction. The rescattering amplitude reads [9]

$$\begin{aligned}
&\mathcal{A} \left(B^- \to K^+ K^- \pi^- \right)_{FRS} \\
&= e^{i\delta_{\pi\pi}} \left[\cos(\phi/2) \mathcal{A} \left(B^- \to (f_0(980) \to K^+ K^-) \pi^- \right) \right. \\
&\quad \left. + i \sin(\phi/2) \mathcal{A} \left(B^- \to (\sigma(500) \to \pi^+ \pi^-) \pi^- \right) \right] \tag{72}
\end{aligned}$$

where the re-scattering mixes the S-wave components of $B^- \to K^+ K^- \pi^-$ and $B^- \to \pi^+ \pi^- \pi^-$. The mixture coefficients $\delta_{\pi\pi}$ and ϕ are functions of the S-wave invariant mass square s_{23} which are given in Ref. [50]. The calculation of $\langle \sigma(500)(p - p_1) \pi^-(p_1) | \mathcal{O}_{1,5} | B^-(p) \rangle$ is similar to that of $f_0(980)$, and the results are

$$\begin{aligned}
T_{\mathcal{O}_1}^{\sigma(500)} &= -0.033 \pm 0.008 \, \text{GeV}^3, \\
T_{\mathcal{O}_5}^{\sigma(500)} &= (1.2 \pm 0.08) \times 10^{-4} \, \text{GeV}^3. \tag{73}
\end{aligned}$$

The mass and decay constant used here is $m_{\sigma(500)} = 0.5 \, \text{GeV}$ [37] and $f_{\sigma(500)} = 0.89 \, \text{GeV}$ [51]. The decay width of the $\sigma(500)$ is $\Gamma_{\sigma(500)} = 0.3 \, \text{GeV}$ [37]. Finally, combining the

resonant, NR and FSI contributions, we obtain the total decay branching fraction:

$$\mathcal{B}_{\text{total}} \left[B^- \to K^+ K^- \pi^- \right] = (0.87 \pm 0.24) \times 10^{-6}, \tag{74}$$

which is of the same order as the world averaged value, but is still sizeably smaller than that. The possible reasons for this discrepancy are:

- In this work, we do not consider the contributions from the higher twist LCDAs of the B meson since they are expected to be one order suppressed according to our previous work [36]. However, to remedy this part, in the future we will perform a more in-depth study on the contribution of higher twist LCDAs.
- The NR contribution to the $\mathcal{O}_7$ matrix element is not introduced to the total branching fraction. We have argued that it is suppressed by its Wilson coefficients a_6^p, a_8^p. However, in principle its exact contribution should also depend on the model parameters introduced in Ref. [48], but they are not determined unambiguously.
- Generally, the matrix elements $\langle R, M | \mathcal{O}_i | B^- \rangle$ contains a strong phase which cannot be obtained by the approach used in this work. The interference between these strong phases will affect the amount of the total branching fraction. In principle, such strong phase can be produced by the charming or strange penguin loop which was studied by Ref. [19] for the case of $B \to \pi\pi$ decay. Similarly, for the $B \to M + R$ decay, the penguin loop can also produce a strong phase, which is expected to be studied in the future.

6 Conclusion

In this work, we have studied the resonant contribution to the decay amplitude of $B^- \to K^+ K^- \pi^-$, which is dominated by the scalar $f_0(980)$, $K_0^*(1430)$, vector $\rho(1450)$, $\phi(1020)$, $K^*(892)$ and tensor $f_2(1270)$ resonances. The three-body decay is reduced to various quasi two-body decays, where the B meson first decays into a resonance and a pion or a kaon, and subsequently the resonance decays into the other two final-state mesons. The quasi two-body decays are calculated within the LCSR approach using the leading twist B meson LCDAs. Then, we calculated the decay branching fraction for $B^- \to K^+ K^- \pi^-$ from each of the resonances considered here. Some of them are consistent with experiment while the others are smaller than the measured values. One possible reason for this discrepancy are the uncertainties of the strong couplings between the corresponding resonance with pseudoscalar mesons. Including the effects of the non-resonant and final state rescattering effects, we also evaluate the total branching fraction and the result is smaller than the

world-averaged value. We have listed three possible reasons for this discrepancy, which requires further researches in the future.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: There are no external data associated with the manuscript.]

Appendix A: Expression of $\xi_i(\lambda_\pi, \lambda_T)$

The expressions of $\xi_i(\lambda_\pi, \lambda_T)$ given in the end of Sect. 3 are

$$\xi_1(\lambda_\pi, \lambda_T) = -\frac{1}{\lambda_T}$$
$$\times \left[\lambda_\pi^4 + 2\lambda_\pi^3(\lambda_T - 2) + \lambda_\pi^2\left(4\lambda_T^2 - 8\lambda_T + 6\right)\right.$$
$$\left. + 2\lambda_\pi(\lambda_T - 2)(\lambda_T - 1)^2 + (\lambda_T - 1)^4\right],$$

$$\xi_2(\lambda_\pi, \lambda_T) = \frac{1}{2(\lambda_\pi + \lambda_T - 1)^3}$$
$$\times \left[\lambda_\pi^6 + 2\lambda_\pi^5(\lambda_T - 3) + \lambda_\pi^4\left(5\lambda_T^2 - 14\lambda_T + 15\right)\right.$$
$$+ 4\lambda_\pi^3\left(\lambda_T^3 - 6\lambda_T^2 + 9\lambda_T - 5\right)$$
$$+ \lambda_\pi^2(\lambda_T - 1)^2\left(5\lambda_T^2 - 14\lambda_T + 15\right)$$
$$\left. + 2\lambda_\pi(\lambda_T - 3)(\lambda_T - 1)^4 + (\lambda_T - 1)^6\right],$$

$$\xi_3(\lambda_\pi, \lambda_T)$$
$$= -\left[\lambda_\pi^6 + 2\lambda_\pi^5(\lambda_T - 3) + \lambda_\pi^4\left(5\lambda_T^2 - 14\lambda_T + 15\right)\right.$$
$$+ 4\lambda_\pi^3\left(\lambda_T^3 - 6\lambda_T^2 + 9\lambda_T - 5\right)$$

$$+ \lambda_\pi^2(\lambda_T - 1)^2\left(5\lambda_T^2 - 14\lambda_T + 15\right)$$
$$+ 2\lambda_\pi(\lambda_T - 3)(\lambda_T - 1)^4 + (\lambda_T - 1)^6\right] \Big/$$
$$\times \left[\lambda_\pi^4 + 4\lambda_\pi^3(\lambda_T - 1) + 2\lambda_\pi^2\left(5\lambda_T^2 - 6\lambda_T + 3\right)\right.$$
$$\left. + 4\lambda_\pi(\lambda_T - 1)^3 + (\lambda_T - 1)^4\right]. \tag{A1}$$

Appendix B: Two-body phase space integration

In this appendix we give the tensor bases used in this work for the two-body phase space integration. The rank-0 integration is defined as

$$\Phi_2^{(0)}(s, m_1, m_2)$$
$$\equiv \int \frac{d^3\vec{k}_1}{(2\pi)^3} \frac{1}{2E_{k_2}} \frac{d^3\vec{k}_2}{(2\pi)^3} \frac{1}{2E_{k_2}}$$
$$\times \delta^{(4)}(p - k_1 - k_2)\delta(k_1^2 - m^1)\delta(k_2^2 - m_2^2)$$
$$\equiv \int d\Phi_2 = \frac{\pi\sqrt{\left(s - (m_1 + m_2)^2\right)\left(s - (m_1 - m_2)^2\right)}}{(2\pi)^6 2s}$$
$$\tag{B1}$$

with $s = p^2$. The higher rank integrations are defined as

$$\int d\phi_2 \, k_1^\mu = \Phi_2^{(0)}(s, m_1, m_2)A_1 p^\mu, \tag{B2}$$

$$\int d\phi_2 \, k_1^\mu k_2^\nu = \Phi_2^{(0)}(s, m_1, m_2)\left[A_2 p^\mu p^\nu + B_2 p^2 g^{\mu\nu}\right], \tag{B3}$$

$$\int d\phi_2 \, k_1^\mu k_2^\nu k_2^\rho = \Phi_2^{(0)}(s, m_1, m_2)$$
$$\times \left[A_3 p^\mu p^\nu p^\rho + B_3 p^2 p^\mu g^{\nu\rho} + C_3 p^2\left(g^{\mu\nu} p^\rho + g^{\mu\rho} p^\nu\right)\right], \tag{B4}$$

where A_i, B_i and C_i are functions of s, m_1 and m_2. Their expressions are

$$A_1 = \frac{m_1^2 - m_2^2 + s}{2s}, \tag{B5}$$

$$A_2 = -\frac{s\left(-m_1^2 - m_2^2 + s\right) - 2\left(m_1^2 - m_2^2 + s\right)\left(-m_1^2 + m_2^2 + s\right)}{6s^2}, \tag{B6}$$

$$B_2 = -\frac{\left(m_1^2 - m_2^2 + s\right)\left(-m_1^2 + m_2^2 + s\right) - 2s\left(-m_1^2 - m_2^2 + s\right)}{12s^2}, \tag{B7}$$

$$A_3 = -\frac{1}{3s^3}\left[\frac{1}{2}m_1^2 s\left(m_1^2 - m_2^2 + s\right)\right.$$
$$- \frac{3}{4}\left(m_1^2 - m_2^2 + s\right)\left(-m_1^2 + m_2^2 + s\right)^2$$
$$\left. + \frac{1}{2}s\left(-m_1^2 - m_2^2 + s\right)\left(-m_1^2 + m_2^2 + s\right)\right], \tag{B8}$$

$$B_3 = \frac{\left(m_1^2 - m_2^2 + s\right)\left(4m_1^2 s - \left(-m_1^2 + m_2^2 + s\right)^2\right)}{24s^3}, \tag{B9}$$

$$C_3 = -\frac{1}{3s^3}\left[\frac{1}{8}\left(m_1^2 - m_2^2 + s\right)\left(-m_1^2 + m_2^2 + s\right)^2\right.$$
$$\left. -\frac{1}{4}s\left(-m_1^2 - m_2^2 + s\right)\left(-m_1^2 + m_2^2 + s\right)\right]. \tag{B10}$$

Appendix C: Expression of the $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{L}$ functions

In this appendix, we list all the $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{L}$ functions appearing in the imaginary part of the correlation function for $\langle \pi^-(S, V, T)|\mathcal{O}_1|B^-\rangle$. For the case of scalar resonance, we have

$$\int d\Phi_2(p_S^2, k_2, k_3)\, \mathcal{M}^S(p_\pi, k_2, k_3)$$
$$= -\frac{1}{m_B}\left[8\left(2A_2\left(-m_B^3 + m_B^2\omega + m_B m_\pi^2 + p_\pi^2\omega\right)\right.\right.$$
$$\left.\left. + (4B_2 - 1)p_\pi^2\omega\right)\right], \tag{C1}$$

$$\int d\Phi_2(p_S^2, k_2, k_3)\, \mathcal{N}^S(p_\pi, k_2, k_3)$$
$$= \frac{1}{m_B}\left[32\left(2A_2\left(m_B^2 + p_\pi^2\right) + (4B_2 - 1)p_\pi^2\right)\right], \tag{C2}$$

$$\int d\Phi_2(p_S^2, k_2, k_3)\, \mathcal{L}^S(p_\pi, k_2, k_3)$$
$$= -\frac{1}{m_B^2}\left[16\left(2A_2\left(m_B^2 + p_\pi^2\right) + (4B_2 - 1)p_\pi^2\right)\right.$$
$$\left. \times \left(m_B^2\omega - m_B\left(\text{pVs} + \omega^2\right) + \omega\left(\text{pVs} - m_\pi^2\right)\right)\right]. \tag{C3}$$

For the case of vector resonance, we have

$$\int d\Phi_2(p_V^2, k_2, k_3)\, \mathcal{M}_{(a)}^{V(1)}(p_\pi, k_2, k_3)$$
$$= -\frac{8}{m_B^2}\left[A_2\left(m_B^2 - m_\pi^2 + p_V^2\right)\right.$$
$$\times \left(2m_B^3 - 2m_B^2\omega + m_B\left(p_V^2 - 2m_\pi^2\right) + 2\omega\left(m_\pi^2 - p_V^2\right)\right)$$
$$\left. + (4B_2 - 1)m_B^2 p_V^2(m_B - \omega)\right], \tag{C4}$$

$$\int d\Phi_2(p_V^2, k_2, k_3)\, \mathcal{M}_{(a)}^{V(2)}(p_\pi, k_2, k_3)$$
$$= -\frac{1}{m_B^2}\left[4p_V^2\left(m_B^3(4A_2 + 4B_2 - 1)\right.\right.$$
$$- 2m_B^2\omega(2A_2 + 4B_2 - 1)$$
$$+ m_B\left(A_2\left(2p_V^2 - 4m_\pi^2\right) + (4B_2 - 1)(p_V^2 - p_\pi^2)\right)$$
$$\left.\left. + 2\omega\left(2A_2\left(m_\pi^2 - p_V^2\right) + (4B_2 - 1)p_\pi^2\right)\right)\right], \tag{C5}$$

$$\int d\Phi_2(p_V^2, k_2, k_3)\, \mathcal{N}_{(a)}^{V(1)}(p_\pi, k_2, k_3)$$
$$= -\frac{32A_1(m_B - \omega)\left(m_B^2 - m_\pi^2 + p_V^2\right)}{m_B}, \tag{C6}$$

$$\int d\Phi_2(p_V^2, k_2, k_3)\, \mathcal{N}_{(a)}^{V(2)}(p_\pi, k_2, k_3) = \frac{32A_1 p_V^2(\omega - m_B)}{m_B}, \tag{C7}$$

$$\int d\Phi_2(p_V^2, k_2, k_3)_\Omega\, \mathcal{L}_{(a)}^{V(1)}(p_\pi, k_2, k_3)$$
$$= -\frac{1}{m_B}\left[16\left(m_B^3(p_V^2(A_1 - 2B_3 + 2C_3 - 1) - A_1\Omega_2 + \Omega_2)\right.\right.$$
$$+ m_B^2\omega((A_1 - 1)\Omega_2 - p_V^2(A_1 - 2B_3 + 2C_3 - 1))$$
$$- m_B\left(m_\pi^2(p_V^2(A_1 - 2B_3 + 2C_3 - 1) - A_1\Omega_2 + \Omega_2)\right.$$
$$+ A_1 p_V^2 \Omega_2\right) + \omega\left(m_\pi^2 - p_V^2\right)(p_V^2(A_1 - 2B_3 + 2C_3 - 1)$$
$$\left.\left. - A_1\Omega_2 + \Omega_2))\right], \tag{C8}$$

$$\int d\Phi_2(p_V^2, k_2, k_3)_\Omega\, \mathcal{L}_{(a)}^{V(2)}(p_\pi, k_2, k_3) = 16A_1 p_V^2\Omega\left(1 - \frac{\omega}{m_B}\right), \tag{C9}$$

$$\int d\Phi_2(p_V^2, k_2, k_3)\, \mathcal{M}_{(b)}^{V(1)}(p_\pi, k_2, k_3)$$
$$= 8\left(2A_2\left(m_B^3 - m_B^2\omega - m_B m_\pi^2 - p_\pi^2\omega\right) + (1 - 4B_2)p_\pi^2\omega\right)\omega, \tag{C10}$$

$$\int d\Phi_2(p_V^2, k_2, k_3)\, \mathcal{M}_{(b)}^{V(2)}(p_\pi, k_2, k_3)$$
$$= \frac{4}{m_B^2}\left(2A_2\left(m_B^5 - 2m_B^4\omega - m_B^3\left(m_\pi^2 + p_\pi^2\right) + 2m_B^2 p_V^2\omega\right.\right.$$
$$+ m_B\left(m_\pi^2(p_\pi^2 + p_V^2) + p_V^2(p_\pi^2 - p_V^2)\right) + 2p_\pi^4\omega\right)$$
$$\left. + (4B_2 - 1)p_\pi^2\left(-2m_B^2\omega + m_B p_V^2 + 2p_\pi^2\omega\right)\right), \tag{C11}$$

$$\int d\Phi_2(p_V^2, k_2, k_3)\, \mathcal{N}_{(b)}^{V(1)}(p_\pi, k_2, k_3)$$
$$= -32A_2 m_B^2 - 32A_2 p_\pi^2 - 64B_2 p_\pi^2 + 16p_\pi^2, \tag{C12}$$

$$\int d\Phi_2(p_V^2, k_2, k_3)\, \mathcal{N}_{(b)}^{V(2)}(p_\pi, k_2, k_3) = -32A_2 p_V^2, \tag{C13}$$

$$\int d\Phi_2(p_V^2, k_2, k_3)_\Omega\, \mathcal{L}_{(b)}^{V(1)}(p_\pi, k_2, k_3)$$
$$= -16\left(2A_2\left(m_B^4 - 2m_B^3\omega + m_B^2\left(\omega^2 - 2m_\pi^2\right)\right.\right.$$
$$\left. + 2m_B m_\pi^2\omega + m_\pi^4 - m_\pi^2 p_V^2 + p_\pi^2\omega^2 - p_\pi^2 p_V^2\right)$$
$$\left. - (4B_2 - 1)p_\pi^2\left(p_V^2 - \omega^2\right)\right), \tag{C14}$$

$$\int d\Phi_2(p_V^2, k_2, k_3)_\Omega\, \mathcal{L}_{(b)}^{V(2)}(p_\pi, k_2, k_3)$$
$$= \frac{1}{m_B^2}\left[16\left(2A_2\left(m_B^5\omega - m_B^4\left(p_V^2 + \omega^2\right)\right.\right.\right.$$
$$- m_B^3\omega\left(m_\pi^2 + p_\pi^2\right) + m_B^2 p_V^2\left(m_\pi^2 + p_V^2 + \omega^2\right)$$
$$+ m_B\omega\left(m_\pi^2(p_\pi^2 + p_V^2) + p_V^2(p_\pi^2 - p_V^2)\right) + p_\pi^4\omega^2\right)$$
$$\left.\left. + (4B_2 - 1)p_\pi^2\omega\left(m_B^2(-\omega) + m_B p_V^2 + p_\pi^2\omega\right)\right)\right]. \tag{C15}$$

For the case of tensor resonance, we have

$$\int d\Phi_2(p_V^2, k_2, k_3)\, \mathcal{M}_{(a)}^{T(1)}(p_\pi, k_2, k_3)$$
$$= \frac{1}{m_B^2}\left[4\left(m_B^2 p_T^2\left(m_B^3(-2A_1 + 4B_2 - 12B_3 - 20C_3 + 1)\right.\right.\right.$$
$$+ m_B^2\omega(2A_1 - 4B_2 + 12B_3 + 20C_3 - 1)$$
$$+ m_B\left(m_\pi^2(2A_1 - 4B_2 + 12B_3 + 20C_3 - 1)\right.$$
$$\left. + p_T^2(-2A_1 + 4B_2 - 8B_3 - 16C_3 + 1)\right)$$

$$-\omega\left(m_\pi^2 - p_T^2\right)(2A_1 - 4B_2 + 12B_3 + 20C_3 - 1)\Big)$$
$$+ A_2\left(m_B^2 - m_\pi^2 + p_T^2\right)^2\left(2m_B^3 - 2m_B^2\omega + m_B\left(p_T^2 - 2m_\pi^2\right)\right.$$
$$+2\omega\left(m_\pi^2 - p_T^2\right)\Big)$$
$$- 2A_3\left(m_B^2 - m_\pi^2 + p_T^2\right)^2$$
$$\times\left(2m_B^3 - 2m_B^2\omega + m_B\left(p_T^2 - 2m_\pi^2\right) + 2\omega\left(m_\pi^2 - p_T^2\right)\right)\Big)\Big], \tag{C16}$$

$$\int d\Phi_2(p_V^2, k_2, k_3)\, \mathcal{M}_{(a)}^{T(2)}(p_\pi, k_2, k_3)$$
$$= \frac{1}{m_B^2}\Big[p_T^2\Big(m_B^5(-2A_1 + 8A_2 - 16A_3$$
$$+ 8B_2 - 16B_3 - 32C_3 + 1)$$
$$+2m_B^4\omega(2A_1 - 4A_2 + 8A_3 - 8B_2 + 8B_3 + 24C_3 - 1)$$
$$+ m_B^3\Big(m_\pi^2(2A_1 - 16A_2 + 32A_3 - 8B_2 + 16B_3 + 32C_3 - 1)$$
$$- 4p_T^2(2A_1 - 3A_2 + 6A_3 - 4B_2 + 6B_3 + 14C_3 - 1)$$
$$+ p_\pi^2(2A_1 - 8B_2 + 16B_3 + 32C_3 - 1)\Big)$$
$$-2m_B^2\omega\Big(m_\pi^2(2A_1 - 8A_2 + 16A_3 - 8B_2 + 8B_3 + 24C_3 - 1)$$
$$- 2p_T^2(2A_1 - 4A_2 + 8A_3 - 3B_2 + 9B_3 + 15C_3 - 1)$$
$$+ p_\pi^2(2A_1 - 8B_2 + 8B_3 + 24C_3 - 1)\Big)$$
$$+ m_B\Big(-2A_1 m_\pi^2 p_\pi^2 + 2A_1 m_\pi^2 p_T^2 - 2A_1 p_T^4$$
$$+ 2A_1 p_\pi^2 p_T^2 + 4A_2\left(2m_\pi^4 - 3m_\pi^2 p_T^2 + p_T^4\right)$$
$$- 8A_3\left(2m_\pi^4 - 3m_\pi^2 p_T^2 + p_T^4\right) + 8B_2 m_\pi^2 p_\pi^2$$
$$- 8B_2 m_\pi^2 p_T^2 + 8B_2 p_T^4 - 8B_2 p_\pi^2 p_T^2$$
$$- 16B_3 m_\pi^2 p_\pi^2 + 8B_3 p_\pi^2 p_T^2$$
$$- 32C_3 m_\pi^2 p_\pi^2 + 16C_3 m_\pi^2 p_T^2 - 16C_3 p_T^4$$
$$+24C_3 p_\pi^2 p_T^2 + m_\pi^2 p_\pi^2 - m_\pi^2 p_T^2 + p_T^4 - p_\pi^2 p_T^2\Big)$$
$$+2\omega\left(m_\pi^2 - p_T^2\right)\Big(p_\pi^2(2A_1 - 8B_2 + 8B_3 + 24C_3 - 1)$$
$$-4A_2\left(m_\pi^2 - p_T^2\right) + 8A_3\left(m_\pi^2 - p_T^2\right)\Big)\Big)\Big], \tag{C17}$$

$$\int d\Phi_2(p_V^2, k_2, k_3)\, \mathcal{M}_{(a)}^{T(3)}(p_\pi, k_2, k_3)$$
$$= -\frac{1}{m_B^2}\Big[2p_T^4\Big(m_B^3(2A_1 - 4A_2 + 8A_3 - 2B_2 + 6B_3 + 10C_3 - 1)$$
$$- 2m_B^2\omega(2A_1 - 2A_2 + 4A_3 - 2B_2 + 6B_3 + 10C_3 - 1)$$
$$+ m_B\Big(-2A_1 p_\pi^2 + 2A_1 p_T^2 + 4A_2 m_\pi^2 - 2A_2 p_T^2$$
$$- 8A_3 m_\pi^2 + 4A_3 p_T^2 + 2B_2 p_\pi^2$$
$$- 6B_2 p_T^2 - 6B_3 p_\pi^2 + 2B_3 p_T^2 - 10C_3 p_\pi^2 + 14C_3 p_T^2 + p_\pi^2 - p_T^2\Big)$$
$$+2\omega\Big(p_\pi^2(2A_1 - 2B_2 + 6B_3 + 10C_3 - 1)$$
$$-2A_2\left(m_\pi^2 - p_T^2\right) + 4A_3\left(m_\pi^2 - p_T^2\right)\Big)\Big)\Big], \tag{C18}$$

$$\int d\Phi_2(p_V^2, k_2, k_3)\, \mathcal{N}_{(a)}^{T(1)}(p_\pi, k_2, k_3)$$
$$= -\frac{1}{m_B}\Big[16(m_B - \omega)\Big(A_1\left(m_B^2 - m_\pi^2 + p_T^2\right)^2$$

$$- 2\left(A_2\left(m_B^2 - m_\pi^2 + p_T^2\right)^2 + 4B_2 m_B^2 p_T^2\right)\Big)\Big], \tag{C19}$$

$$\int d\Phi_2(p_V^2, k_2, k_3)\, \mathcal{N}_{(a)}^{T(2)}(p_\pi, k_2, k_3)$$
$$= \frac{1}{m_B}\Big[16p_T^2(m_B - \omega)\Big(A_1\left(m_B^2 - m_\pi^2 + p_T^2\right)$$
$$- 2\Big(A_2\left(m_B^2 - m_\pi^2 + p_T^2\right) + 2B_2\left(m_B^2 - p_\pi^2\right)\Big)\Big)\Big], \tag{C20}$$

$$\int d\Phi_2(p_V^2, k_2, k_3)\, \mathcal{N}_{(a)}^{T(3)}(p_\pi, k_2, k_3)$$
$$= \frac{1}{m_B}\Big[16p_T^4(A_1 - 2A_2)(m_B - \omega)\Big], \tag{C21}$$

$$\int d\Phi_2(p_V^2, k_2, k_3)\Omega\, \mathcal{L}_{(a)}^{T(1)}(p_\pi, k_2, k_3)$$
$$= \frac{1}{m_B}\Big[8p_T^2\left(m_B^2 - m_\pi^2 + p_T^2\right)$$
$$\times\left(-m_B^3 + m_B^2\omega + m_B m_\pi^2 + \omega\left(p_T^2 - m_\pi^2\right)\right)$$
$$\times(A_1 + 2A_2 + 2B_2 + 4B_3 + 4C_3 - 1)\Big], \tag{C22}$$

$$\int d\Phi_2(p_V^2, k_2, k_3)\Omega\, \mathcal{L}_{(a)}^{T(2)}(p_\pi, k_2, k_3)$$
$$= \frac{1}{m_B}\Big[4p_T^4\Big(m_B^3(-(A_1 + 2A_2 - 4B_2 - 2B_3 + 10C_3 - 1))$$
$$+ m_B\Big(m_\pi^2(A_1 + 2A_2 - 4B_2 - 2B_3 + 10C_3 - 1)$$
$$+4p_T^2(B_2 + B_3 - C_3)\Big)$$
$$+ m_B^2\omega(A_1 + 2A_2 + 2B_3 + 6C_3 - 1)$$
$$-\omega\left(m_\pi^2 - p_T^2\right)(A_1 + 2A_2 + 2B_3 + 6C_3 - 1)\Big)\Big], \tag{C23}$$

$$\int d\Phi_2(p_V^2, k_2, k_3)\Omega\, \mathcal{L}_{(a)}^{T(3)}(p_\pi, k_2, k_3) = 0, \tag{C24}$$

$$\int d\Phi_2(p_V^2, k_2, k_3)\, \mathcal{M}_{(b)}^{T(1)}(p_\pi, k_2, k_3)$$
$$= 4\left(m_B^2 - 2m_B\omega - m_\pi^2 + p_T^2\right)$$
$$\times\left(2A_2\left(m_B^3 - m_B^2\omega - m_B m_\pi^2 - p_T^2\omega\right) + (1 - 4B_2)p_T^2\omega\right), \tag{C25}$$

$$\int d\Phi_2(p_V^2, k_2, k_3)\, \mathcal{M}_{(b)}^{T(2)}(p_\pi, k_2, k_3)$$
$$= \frac{1}{m_B^2}\Big[2A_2\left(m_B^7 - 6m_B^6\omega - m_B^5\left(2m_\pi^2 + p_\pi^2 - 3p_T^2 - 6\omega^2\right)\right.$$
$$+ 2m_B^4\omega\left(3m_\pi^2 + 2p_\pi^2 - p_T^2\right)$$
$$+ m_B^3\left(m_\pi^4 - 2m_\pi^2(p_T^2 - p_\pi^2) - p_T^2\left(p_T^2 + 4\omega^2\right)\right)$$
$$+ 2m_B^2\omega\left(-2m_\pi^2(p_\pi^2 + p_T^2) + p_\pi^4 + 2p_T^4 - 2p_\pi^2 p_T^2\right)$$
$$- m_B\left(m_\pi^4(p_\pi^2 + p_T^2) - 2m_\pi^2 p_T^4 + 6p_T^4\omega^2 + p_T^6 - p_\pi^2 p_T^4\right)$$
$$+2p_T^4\omega\left(p_T^2 - m_\pi^2\right)\Big)$$
$$- (4B_2 - 1)p_\pi^2\left(2m_B^4\omega - m_B^3\left(p_T^2 + 6\omega^2\right)\right.$$
$$- 2m_B^2\omega\left(m_\pi^2 + p_\pi^2 - 3p_T^2\right)$$
$$+ m_B\left(m_\pi^2 p_T^2 + 6p_\pi^2\omega^2 - p_T^4\right) + 2p_\pi^2\omega\left(m_\pi^2 - p_T^2\right)\Big)\Big], \tag{C26}$$

$$\int d\Phi_2(p_V^2, k_2, k_3)\, \mathcal{M}_{(b)}^{T(3)}(p_\pi, k_2, k_3)$$

$$
= \frac{1}{m_B^3} \Big[2 \left(m_B^2 \omega - m_B p_T^2 - p_\pi^2 \omega \right)
$$

$$
\times \Big(2A_2 \left(m_B^5 - 2m_B^4 \omega - m_B^3 \left(m_\pi^2 + p_\pi^2 \right) \right.
$$

$$
+ 2m_B^2 p_T^2 \omega + m_B \left(m_\pi^2 (p_\pi^2 + p_T^2) + p_T^2 (p_\pi^2 - p_T^2) \right)
$$

$$
+ 2p_\pi^4 \omega \Big) + (4B_2 - 1) p_\pi^2 \left(-2m_B^2 \omega + m_B p_T^2 + 2p_\pi^2 \omega \right) \Big) \Big],
$$

(C27)

$$
\int d\Phi_2(p_V^2, k_2, k_3)\, \mathcal{N}_{(b)}^{T(1)}(p_\pi, k_2, k_3)
$$

$$
= -8 \left(2A_2 \left(m_B^2 + p_\pi^2 \right) + (4B_2 - 1) p_\pi^2 \right)
$$

$$
\times \left(m_B^2 - 2m_B \omega - m_\pi^2 + p_T^2 \right),
$$

(C28)

$$
\int d\Phi_2(p_V^2, k_2, k_3)\, \mathcal{N}_{(b)}^{T(2)}(p_\pi, k_2, k_3)
$$

$$
= \frac{1}{m_B} \Big[4 \left(2A_2 \left(m_B^4 \omega - 2m_B^3 p_T^2 + 2m_B^2 p_T^2 \omega \right.\right.
$$

$$
- m_B p_T^2 \left(-m_\pi^2 + p_\pi^2 + p_T^2 \right) - p_\pi^4 \omega \Big)
$$

$$
- (4B_2 - 1) p_\pi^2 \left(-\omega m_B^2 + m_B p_T^2 + p_\pi^2 \omega \right) \Big) \Big],
$$

(C29)

$$
\int d\Phi_2(p_V^2, k_2, k_3)\, \mathcal{N}_{(b)}^{T(3)}(p_\pi, k_2, k_3)
$$

$$
= -\frac{1}{m_B} \Big[16A_2 p_T^2 \left(-\omega m_B^2 + m_B p_T^2 + p_\pi^2 \omega \right) \Big],
$$

(C30)

$$
\int d\Phi_2(p_V^2, k_2, k_3)_\Omega\, \mathcal{L}_{(b)}^{T(1)}(p_\pi, k_2, k_3)
$$

$$
= 4 \left(m_B^2 - 2m_B \omega - m_\pi^2 + p_T^2 \right)
$$

$$
\times \Big(2A_2 \left(m_B^4 - 2m_B^3 \omega + m_B^2 \left(\omega^2 - 2m_\pi^2 \right) \right.
$$

$$
+ 2m_B m_\pi^2 \omega + m_\pi^4 - m_\pi^2 p_T^2 + p_\pi^2 \omega^2 - p_\pi^2 p_T^2 \Big)
$$

$$
- (4B_2 - 1) p_\pi^2 \left(p_T^2 - \omega^2 \right) \Big),
$$

(C31)

$$
\int d\Phi_2(p_V^2, k_2, k_3)_\Omega\, \mathcal{L}_{(b)}^{T(2)}(p_\pi, k_2, k_3)
$$

$$
= \frac{1}{m_B^2} \Big[2 \left(2A_2 \left(2m_B^7 \omega - m_B^6 \left(2p_T^2 + 5\omega^2 \right) \right.\right.
$$

$$
+ m_B^5 \omega \left(-4m_\pi^2 - 2p_\pi^2 + 5p_T^2 + 3\omega^2 \right)
$$

$$
+ m_B^4 \left(m_\pi^2 \left(4p_T^2 + 5\omega^2 \right) - \omega^2 (p_T^2 - 4p_\pi^2) \right)
$$

$$
- m_B^3 \omega \left(-2m_\pi^4 + m_\pi^2 (5p_\pi^2 - 4p_T^2) + p_T^2 \left(p_\pi^2 + 3p_T^2 + 2\omega^2 \right) \right)
$$

$$
+ m_B^2 \left(-2m_\pi^4 p_T^2 + m_\pi^2 \left(-4p_\pi^2 \omega^2 + p_T^4 - 3p_T^2 \omega^2 \right) \right.
$$

$$
+ p_\pi^4 \omega^2 + p_T^6 + p_T^4 \left(p_\pi^2 + 3\omega^2 \right) - 3p_\pi^2 p_T^2 \omega^2 \Big)
$$

$$
+ m_B \omega \left(m_\pi^4 (-(2p_\pi^2 + p_T^2)) + m_\pi^2 p_T^2 (p_\pi^2 + 2p_T^2) \right.
$$

$$
- 3p_\pi^4 \omega^2 - p_T^6 + p_\pi^2 p_T^4 + p_\pi^4 p_T^2 \Big) + p_\pi^4 \omega^2 \left(p_T^2 - m_\pi^2 \right) \Big)
$$

$$
+ (4B_2 - 1) p_\pi^2 \left(m_B^2 \omega + m_B \left(p_T^2 - 3\omega^2 \right) \right.
$$

$$
+ \omega \left(p_T^2 - m_\pi^2 \right) \Big) \left(m_B^2 (-\omega) + m_B p_T^2 + p_\pi^2 \omega \right) \Big) \Big],
$$

(C32)

$$
\int d\Phi_2(p_V^2, k_2, k_3)_\Omega\, \mathcal{L}_{(b)}^{T(3)}(p_\pi, k_2, k_3)
$$

$$
= \frac{1}{m_B^3} \Big[4 \left(m_B^2 \omega - m_B p_T^2 - p_\pi^2 \omega \right)
$$

$$
\times \Big(2A_2 \left(m_B^5 \omega - m_B^4 \left(p_T^2 + \omega^2 \right) \right.
$$

$$
- m_B^3 \omega \left(m_\pi^2 + p_\pi^2 \right) + m_B^2 p_T^2 \left(m_\pi^2 + p_T^2 + \omega^2 \right)
$$

$$
+ m_B \omega \left(m_\pi^2 (p_\pi^2 + p_T^2) + p_T^2 (p_\pi^2 - p_T^2) \right) + p_\pi^4 \omega^2 \Big)
$$

$$
+ (4B_2 - 1) p_\pi^2 \omega \left(-m_B^2 \omega + m_B p_T^2 + p_\pi^2 \omega \right) \Big) \Big].
$$

(C33)

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