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Letter

Spherically symmetric 't Hooft–Polyakov monopoles

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Abstract A general analytic spherically symmetric solution of the Bogomol'nyi equations is found. It depends on two constants and one arbitrary function on radius and contains the Bogomol'nyi–Prasad–Sommerfield and Singleton solutions as particular cases. Thus all spherically symmetric 't Hooft–Polyakov monopoles with massless scalar field and minimal energy are derived.

1 Introduction

The 't Hooft–Polyakov monopole solutions are exact static spherically symmetric solutions with finite energy of the field equations of the $\mathbb{SU}(2)$ gauge model with the triplet of scalar fields φ in the adjoint representation and $\lambda\varphi^4$ type interaction [1,2]. There are many other related solutions of the equations of motion without spherical symmetry and different boundary conditions. All solutions are divided into homotopically inequivalent classes parameterized by the degree of the map $\mathbb{S}^2 \to \mathbb{S}^2$ (topological charge Q taking integer values) defined by the boundary conditions (see, e.g. [3–6]). Solutions with spherically symmetric boundary conditions belong to the class with $Q = 1$.

Monopole-type solutions have many point particle properties: finite energy, stability and localisation in space, and are interesting both from mathematical and physical point of view. So far they are not observed in nature. Recently, the 't Hooft–Polyakov monopoles were given new physical interpretation in solid state physics [7–9] describing elastic media with continuous distribution of disclinations and dislocations.

In each Q-sector of the monopole-type solutions, there are field configurations with minimal energy. They are defined by the Bogomol'nyi equations [11]. The solutions with minimal energy satisfy also the original equations of motion of the model for massless scalar fields without self interaction. Therefore solutions of the Bogomol'nyi equations are important and interesting.

Bogomol'nyi equations reduce to the system of nonlinear ordinary differential equations in the spherically symmetric case. The author was aware only of two exact analytic solutions of this system of equations: the Bogomol'nyi–Prasad–Sommerfield [10,11] and Singleton [12] solutions. In the present paper, we have found a general analytic spherically symmetric solution of the Bogomol'nyi equations. It is parameterized by one arbitrary function of radius and two constants. There is also one degenerate solution parameterized by one arbitrary constant. In particular cases, a general solution yields the Bogomol'nyi–Prasad–Sommerfield and Singleton solutions. Thus we have found all spherically symmetric 't Hooft–Polyakov monopoles for massless scalar fields which minimize the energy in the $Q = 1$ sector.

1.1 A general solution

We consider the Euclidean space $\mathbb{R}^3$ with Cartesian coordinates x^μ and Euclidean metric $\delta_{\mu\nu} := \mathsf{diag}\,(+\,+\,+)$, $\mu, \nu = 1, 2, 3$. Let there be the $\mathbb{SU}(2)$ local connection form $A_\mu{}^i(x)$ (the Yang–Mills fields) and the triplet of scalar fields $\varphi^i(x)$, $i = 1, 2, 3$, in the adjoint representation of $\mathbb{SU}(2)$. The totally antisymmetric tensor is denoted by ε_{ijk}, $\varepsilon_{123} = 1$, and raising and lowering of Latin indices is performed by the Euclidean metric δ_{ij} (the Killing–Cartan form of $\mathbb{SU}(2)$).

We are looking for spherically symmetric solutions of the Bogomol'nyi equations [11]

$$F_{\mu\nu}{}^i = \varepsilon_{\mu\nu\rho}\nabla^\rho\varphi^i, \tag{1}$$

where

$$F_{\mu\nu}{}^i := \partial_\mu A_\nu{}^i - \partial_\nu A_\mu{}^i + A_\mu{}^j A_\nu{}^k \varepsilon_{jk}{}^i$$

is the local curvature form (the Yang–Mills field strength) and

$$\nabla_\mu\varphi^i := \partial_\mu\varphi^i + A_\mu{}^j\varphi^k\varepsilon_{jk}{}^i \tag{2}$$

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is the covariant derivative of scalar fields.

Any solution of the Bogomol'nyi equations satisfies also the field equations of $\mathbb{SU}(2)$ gauge model in Minkowskian space-time $\mathbb{R}^{1,3}$ with massless scalar fields without self interaction in the time gauge $A_0{}^i = 0$. The inverse statement is not true. Smooth solutions of the Bogomol'nyi equations have minimal energy in each sector of topologically different (nonhomotopic) solutions of the model (see, e.g. [3–6]).

The boundary conditions are supposed to be spherically symmetric

$$\lim_{r\to\infty} A_\mu{}^i \to 0, \quad \lim_{r\to\infty} \varphi^i \to \frac{x^i}{r}a, \quad a \neq 0, \tag{3}$$

where $r := \sqrt{x^\mu x_\mu}$ is the usual radius in the spherical coordinate system.

The Bogomol'nyi equations are the system of 9 first order nonlinear partial differential equations for 12 unknown functions $A_\mu{}^i$ and φ^i. They are simpler then the original field equations of the $\mathbb{SU}(2)$ gauge model.

Now we find a general static spherically symmetric solution of the Bogomol'nyi equations. We assume that the global rotation group $\mathbb{SO}(3)$ acts simultaneously both on the base $\mathbb{R}^3$, and on the Lie algebra $\mathfrak{so}(3)$, which, as a vector space, is also a three-dimensional Euclidean space $\mathbb{R}^3$. It means that if $S \in \mathbb{SO}(3)$ is an orthogonal matrix, then the transformation has the form

$$A_\mu{}^i \mapsto S^{-1}{}_\mu{}^\nu A_\nu{}^j S_j{}^i, \quad S \in \mathbb{SO}(3).$$

Under this assumption, the difference between Greek and Latin indices disappears, but we shall, as far as possible, distinguish them for clarity.

The most general spherically symmetric components of the connection have the form

$$A_\mu{}^i(x) := \varepsilon_\mu{}^{ij}\frac{x_j}{r}W(r) + \delta^i_\mu V(r) + \frac{x_\mu x^i}{r^2}U(r), \tag{4}$$

where W, V, U are arbitrary sufficiently smooth functions of radius.

If we include reflections into the rotation group, then $A_\mu{}^k$ are components of the second rank pseudo-tensor with respect to the action of the full rotation group $\mathbb{O}(3)$, due to the presence of the third rank pseudo-tensor ε_{ijk} in Eq. (2). Under the action of the full rotation group $\mathbb{O}(3)$ the function W is a scalar, and V and U are pseudoscalars.

The famous 't Hooft–Polyakov monopole solution [1,2] corresponds to ansatz (4) with $V \equiv U \equiv 0$.

A general spherically symmetric ansatz for the scalar fields is

$$\varphi^i := \frac{x^i}{r}F(r),$$

where F is an arbitrary function.

To simplify equations, we introduce dimensionless functions $K(r)$, $L(r)$, $M(r)$, and $H(r)$:

$$W := \frac{K-1}{r}, \quad U := \frac{L}{r}, \quad V := \frac{M}{r}, \quad F := \frac{H}{r}. \tag{5}$$

Then the full system of Bogomol'nyi equations becomes

$$rK' + M(L+M) = KH, \tag{6}$$
$$-rK' + K^2 - 1 - LM = rH' - H - KH, \tag{7}$$
$$rM' - K(L+M) = MH. \tag{8}$$

A general solution of this system of equations for $H \equiv 0$ was found in [13], where it was given physical interpretation in solid state physics as describing media with disclinations. Therefore we assume that $H \neq 0$ in what follows.

Now we introduce new independent variable

$$r \mapsto \xi := \ln r, \quad r > 0, \tag{9}$$

and the index will denote differentiation with respect to ξ, e.g.

$$M_\xi := \frac{dM}{d\xi} = rM', \quad M_{\xi\xi} = r^2M'' + rM'.$$

Theorem 1.1 *A general solution of the system of Eqs. (6)–(8) is*

$$M(\xi) = \pm\sqrt{1 - K^2 + H_\xi - H}, \tag{10}$$

$$L(\xi) = \mp\frac{K_\xi - K^2 + 1 + H_\xi - (K+1)H}{\sqrt{1 - K^2 + H_\xi - H}}, \tag{11}$$

where $H(\xi)$ is a solution of the Riccati equation

$$H_\xi + H - H^2 = C\,\mathrm{e}^{2\xi} \tag{12}$$

with arbitrary constant $C \in \mathbb{R}$ and K is an arbitrary function satisfying inequality

$$1 - K^2 + H_\xi - H \geq 0. \tag{13}$$

The upper and lower signs in Eqs. (10), (11) must be chosen simultaneously.

Proof Add Eqs. (6) and (7):

$$M^2 + K^2 - 1 = rH' - H.$$

It implies Eq. (10). Substitution of this solution into Eq. (7) yields

$$-K_\xi + K^2 - 1 \mp L\sqrt{1 - K^2 + H_\xi - H^2} = H_\xi - H - KH.$$

It gives Eq. (11).

After substitution of M (10) and L (11) into Eq. (8) and changing of coordinate (9) all terms with K cancel, and we get the equation for $H(\xi)$:

$$H_{\xi\xi} - H_\xi - 2H_\xi H - 2H + 2H^2 = 0. \tag{14}$$

It is rewritten as

$$(H_\xi + H - H^2)_\xi - 2(H_\xi + H - H^2) = 0.$$

This equation can be easily integrated yielding the Riccati Eq. (12) with constant of integration C.

We are looking for real valued solutions, therefore Eq. (13) must hold. □

Thus we reduced the whole problem to solution of the Riccati Eq. (12), functions M and L are expressed through H and K, the function K being arbitrary.

Now we consider two special cases. Let arbitrary function K satisfy equation

$$K^2 = 1 + rH' - H. \tag{15}$$

It implies $M = 0$. Then Eq. (8) yields $KL = 0$, and we have two subcases: $K = 0$ and $L = 0$.

Subcase $M = 0$, $K = 0$. Then Eq. (6) is satisfied, and Eq. (7) yields

$$rH' - H + 1 = 0.$$

Its general solution is

$$H = 1 + C_1 r, \quad C_1 = \mathsf{const}. \tag{16}$$

Thus we get

Proposition 1.1 *If Eq. (15) holds and $K = 0$, then a general solution of the Bogomol'nyi Eqs. (6)–(8) is*

$$M = 0, \quad H = 1 + C_1 r, \tag{17}$$

the function L being arbitrary.

The gauge and scalar fields for this solution are

$$\begin{aligned} \varphi^i &= \frac{x^i}{r}\left(\frac{1}{r} + C_1\right), \\ A_\mu{}^i &= -\varepsilon_\mu{}^{ij}\frac{x_j}{r^2} + \frac{x_\mu x^i}{r^2}U(r), \end{aligned} \tag{18}$$

the function $U := L/r$ being arbitrary. To satisfy the boundary conditions (3) we must assume that $U(\infty) = 0$ and $C_1 \neq 0$. This solution seems to be new.

Subcase $M = 0$, $L = 0$. Then the full system of the Bogomol'nyi equations reduces to

$$\begin{aligned} rK' &= KH, \\ rH' &= K^2 - 1 + H. \end{aligned} \tag{19}$$

This subcase corresponds to the 't Hooft–Polyakov ansatz. If we solve the second Eq. (19) for K and substitute the solution into the first equation, then we obtain Eq. (14). Thus the original 't Hooft–Polyakov monopoles correspond to the special case of general solution given by Theorem 1.1 when arbitrary function is given by Eq. (15).

A general case. The Riccati Eq. (12) in old coordinate r is

$$rH' + H - H^2 = Cr^2.$$

Substitution

$$H(r) := \frac{r}{Z(r)} + 1$$

results in the special Riccati equation (see, e.g. [14, Part III, Chapter I, Eq. 1.99])

$$Z' + CZ^2 = -1. \tag{20}$$

Its solution going through the point $Z(0) = Z_0$ is

$$Z(r) = \begin{cases} \dfrac{Z_0\sqrt{-C} - \tanh(\sqrt{-C}r)}{\sqrt{-C} + CZ_0\tanh(\sqrt{-C}r)}, & C < 0, \\ Z_0 - r, & C = 0, \\ \dfrac{Z_0\sqrt{C} - \tan(\sqrt{C}r)}{\sqrt{C} + CZ_0\tan(\sqrt{C}r)}, & C > 0. \end{cases} \tag{21}$$

The constant of integration $C \neq 0$ can be absorbed by rescaling the field and radius

$$Z \mapsto \sqrt{|C|}\,Z, \quad r \mapsto \sqrt{|C|}\,r.$$

Then the solution is

$$Z(r) = \begin{cases} \dfrac{Z_0 - \tanh r}{1 - Z_0\tanh r}, & C < 0, \\ Z_0 - r, & C = 0, \\ \dfrac{Z_0 - \tan r}{1 + Z_0\tan r}, & C > 0. \end{cases} \tag{22}$$

For $C = 0$ the solution remains the same (21). Thus the constant C in general solution (12) takes, in fact, only three different values: $C = -1, 0, 1$.

The scalar fields for solution (22) are

$$\varphi^i(r) = \begin{cases} \dfrac{x^i}{r}\left(\dfrac{1 - Z_0\tanh r}{Z_0 - \tanh r} + \dfrac{1}{r}\right), & C < 0, \\ \dfrac{x^i}{r}\left(\dfrac{1}{Z_0 - r} + \dfrac{1}{r}\right), & C = 0, \\ \dfrac{x^i}{r}\left(\dfrac{1 + Z_0\tan r}{Z_0 - \tan r} + \dfrac{1}{r}\right), & C > 0. \end{cases} \tag{23}$$

At infinity the limit is

$$\varphi^i(\infty) = \begin{cases} -\dfrac{x^i}{r}, & C < 0, \\ 0, & C = 0, \\ ?, & C > 0. \end{cases}$$

Thus only solutions with $C < 0$ satisfy boundary condition (3). In the case $C > 0$ the scalar field has periodic singularities and does not have the limit as $r \to \infty$. Therefore its physical meaning is obscure.

A general solution for the gauge field is more complicated and depends on arbitrary function $K(r)$. We return to its analysis in future.

Up to now only a few spherically symmetric solutions of the Bogomol'nyi equations are known.

If $C < 0$, $Z_0 = 0$, and Eq. (15) holds, then

$$Z = -\tanh r, \quad H = 1 - \frac{r}{\tanh r}, \quad K = \pm\frac{r}{\sinh r}. \qquad (24)$$

This is precisely the famous Bogomol'nyi–Prasad–Sommerfield solution [10,11]. For $Z_0 \neq 0$ and arbitrary function $K(r)$ we have infinitely many new solutions which differ, for example, by the tensorial structure of the gauge field (4).

If $C = 0$ and Eq. (15) holds, then

$$Z = Z_0 - r, \quad H = \frac{Z_0}{Z_0 - r}, \quad K = \pm\frac{r}{Z_0 - r}. \qquad (25)$$

This is the solution found in [12].

2 Conclusion

We considered the most general spherically symmetric ansatz for the gauge and scalar fields in the $\mathbb{SU}(2)$ gauge model. A general analytic solution of the Bogomol'nyi equations is found. It includes the Bogomol'nyi–Prasad–Sommerfield and Singleton solutions as particular cases. A general solution describes also infinitely many new solutions for different values of constant Z_0 and arbitrary function $K(r)$. Thus we obtained all spherically symmetric 't Hooft–Polyakov monopoles minimizing the energy in the $Q = 1$ sector. All smooth solutions have the same minimal energy.

Scalar functions for $C > 0$ are not smooth. They have periodic singularities when r ranges from 0 to ∞ and do not have the limit as $r \to \infty$. Therefore physical meaning of these solutions of the Bogomol'nyi equations is obscure. Anyway, we have proved that there are no other spherically symmetric solutions.

The Lie algebra $\mathfrak{su}(2)$ is isomorphic to $\mathfrak{so}(3)$. Therefore the 't Hooft–Polyakov monopole solutions may be given physical interpretation in solid state physics assuming that the rotational group $\mathbb{SO}(3)$ acts in the tangent space. They describe media with point disclinations [7–9]. Probably, the obtained spherically symmetric solutions may be observed in solids.

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